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OPTIMIZATION WITH RESPECT TO GENERAL PREFERENCE MAPPINGS

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Abstract. We present a theorem on necessary optimality conditions in set-valued optimization in which the ordering structure is given by a general preference mapping.

1. Introduction

In some economic models (e.g. in equilibrium theory and in qualitative game theory), the preferences of economic agents or players are often described by gereral preference mappings whose values are not necessarily cones. The condition that one assumes in this case is that the deciding agent in state x is able to specify those states $P(x)$ which he prefers to x. This implies the condition $x \notin P(x)$ which is called *irreflexivity* (see [2, p. 7]).

When we speak of preferences of consumers, their possibilities to act on the market are usually restricted by a *budget set* which is defined for a given *price vector* and is assumed to be compact (see [2, p. 11]). However, there are also restrictions not directly connected with the budget. For example, no responsible person would buy 10 laptops for private use, even if he or she has enough money to do it. Therefore, it is reasonable to assume that the set of preferences of a consumer is not necessarily unbounded.

In [1, Theorem 3.4], the authors have obtained some first-order necessary optimality conditions for a set-valued optimization problem in which the order structure is given by a general preference mapping. However, these results are proved under asymptotic closedness property which is a rather restrictive assumption because it excludes bounded sets as possible preference sets. The aim of this short note is to prove necessary conditions for weak minimizers in a set-valued optimization problem, avoiding the asymptotic closedness assumption.

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2. Directional derivatives of set-valued mappings

Let X be a real normed space, and let $f : X \to \mathbb{R}$. The generalized directional derivatives $\underline{d}^m f$ and $\overline{d}^m f$ have been defined in [5] as follows:

$$\underline{d}^m f(\bar{x}; y) := \liminf_{(t,v)\to(0^+,y)} \frac{f(\bar{x}+tv)-f(\bar{x})}{t^m}, \tag{1}$$

$$\overline{d}^m f(\bar{x}; y) := \limsup_{(t,v)\to(0^+,y)} \frac{f(\bar{x}+tv)-f(\bar{x})}{t^m}, \tag{2}$$

These derivatives have been extended in [3] to the case of a set-valued mapping $F : X \rightrightarrows Y$, where Y is another normed space:

(3)

$$\underline{d}^m F(\bar{x}, \bar{y})(u) := \{v \in Y : \forall h_n \to 0^+, \forall u_n \to u, \exists v_n \to v \text{ s.t. } \bar{y} + h_n^m v_n \in F(\bar{x} + h_n u_n)\},$$

(4)

$$\overline{d}^m F(\bar{x}, \bar{y})(u) := \{v \in Y : \exists h_n \to 0^+, \exists u_n \to u, \exists v_n \to v \text{ s.t. } \bar{y} + h_n^m v_n \in F(\bar{x} + h_n u_n)\}.$$

The authors of [3] claim (see Remark 2.13(2)) that definitions (3)–(4) reduce to (1)–(2) for a single-valued mapping F. In fact, the relation between these generalized derivatives is a little more complicated. Let us observe that the derivatives (1)–(2) may take on infinite values even if the values of f are finite. On the contrary, for a set-valued mapping $F : X \rightrightarrows \mathbb{R}$, the generalized derivatives $\underline{d}^m F$ and $\overline{d}^m F$ introduced in [3] are always subsets of $\mathbb{R}$, which, however, in some cases may be empty. In particular, for $F = \{f\}$, we have the following relations:

(5)

$$\begin{aligned} &\underline{d}^m F(\bar{x}, f(\bar{x}))(u) \\ = \ & \left\{v \in \mathbb{R} : \forall h_n \to 0^+, \forall u_n \to u, \exists v_n \to v \text{ s.t. } f(\bar{x}) + h_n^m v_n = f(\bar{x} + h_n u_n)\right\} \\ = \ & \left\{v \in \mathbb{R} : \forall h_n \to 0^+, \forall u_n \to u \text{ we have } \frac{f(\bar{x}+h_n u_n)-f(\bar{x})}{h_n^m} \to v\right\} \\ = \ & \begin{cases} \{\underline{d}^m f(\bar{x}; u)\}, \text{ if } \underline{d}^m f(\bar{x}; u) = \overline{d}^m f(\bar{x}; u) \in \mathbb{R}, \\ \emptyset, \text{ in the opposite case,} \end{cases} \end{aligned}$$

and

$$
\begin{aligned}
&(6)\overline{d}^m F(\bar{x}, f(\bar{x}))(u)\\
= &\ \left\{v \in \mathbb{R} : \exists h_n \to 0^+, \exists (u_n, v_n) \to (u, v) \text{ s.t. } f(\bar{x}) + h_n^m v_n = f(\bar{x} + h_n u_n)\right\}\\
= &\ \left\{v \in \mathbb{R} : \exists h_n \to 0^+, \exists u_n \to u \text{ s.t. } \frac{f(\bar{x} + h_n u_n) - f(\bar{x})}{h_n^m} \to v\right\}\\
\subset &\ \left\{v \in \mathbb{R} : \underline{d}^m f(\bar{x}; u) \leq v \leq \overline{d}^m f(\bar{x}; u)\right\}.
\end{aligned}
$$

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 1 & \textit{if } x \textit{ is irrational,} \\ 0 & \textit{if } x \textit{ is rational.} \end{cases}$$

We have $\bar{d}^1 F(0,0)(1) = \{0, +\infty\}$, $\underline{d}^1 f(0;1) = 0$ and $\overline{d}^1 f(0;1) = +\infty$, so the last set in (6) is $[0, +\infty)$, and the inclusion is strict.

Example 2. *If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$, then $\underline{d}^2 f(0;1) = \overline{d}^2 f(0;1) = +\infty$, and $\bar{d}^2 F(0,0)(1) = \emptyset$ because the last set in (6) is empty.*

3. NECESSARY CONDITIONS IN SET-VALUED OPTIMIZATION

In this section we consider two normed spaces X, Y, two set-valued mappings $F : X \rightrightarrows Y$ and $P : Y \rightrightarrows Y$, and a nonempty set $S \subset X$.

Definition 1. *P is called a* preference mapping *if $y \notin P(y)$ for all $y \in Y$.*

Definition 2. *A preference mapping P is called*

(a) star-shaped at y *if $(y, z) \subset P(y)$ for all $z \in P(y)$, where (y, z) is the open line segment joining y and z,*

(b) solid *at y if* $\mathrm{int}P(y) \neq \emptyset$.

Remark 1. *The condition that P is star-shaped at y means that, for each $\lambda \in (0, 1)$, we have*

$$\lambda P(y) + (1 - \lambda)y \subset P(y),$$

which is equivalent to

$$\lambda(P(y) - y) \subset P(y) - y. \tag{7}$$

Definition 3. *The* contingent cone *to S at $\bar{x} \in \mathrm{cl}S$ is defined as follows:*

$$K(S, \bar{x}) := \left\{v \in X : \exists h_n \to 0^+, \exists v_n \to v \textit{ s.t. } \bar{x} + h_n v_n \in S, \forall n\right\}.$$

Definition 4. *Let $\bar{x} \in S$. We say that a pair $(\bar{x}, \bar{y}) \in \mathrm{graph}F$ is a* weak minimizer *for F over S with respect to preference mapping $P : Y \rightrightarrows Y$ if*

$$F(S) \cap \mathrm{int}P(\bar{y}) = \emptyset, \tag{8}$$

where

$$F(S) := \bigcup_{x \in S} F(x). \tag{9}$$

Here we consider only the simplest set-valued optimization problem with an abstract set constraint, which is formulated as follows:

$$\text{Minimize } F(x) \text{ subject to } x \in S, \tag{10}$$

where the minimization is understood with respect to P. More results for problems with functional constraints and for other kind of minima will be presented in [4].

Theorem 1. *We consider problem* (10). *Let* $\bar{x} \in S$, $(\bar{x}, \bar{y}) \in \text{graph}F$ *and let* m *be a positive integer. Suppose that the preference mapping* $P : Y \rightrightarrows Y$ *is star-shaped and solid at* $\bar{y}$. *If* $(\bar{x}, \bar{y})$ *is a weak minimizer for* F *over* S *with respect to* P, *then*

$$\underline{d}^m F(\bar{x}, \bar{y})(u) \cap (\text{int}P(\bar{y}) - \bar{y}) = \emptyset, \tag{11}$$

for all $u \in K(S, \bar{x})$.

Proof. Since $(\bar{x}, \bar{y})$ is a weak minimizer for F over S with respect to P, we have by (8)–(9)

$$F(x) \cap \text{int}P(\bar{y}) = \emptyset, \text{ for all } x \in S. \tag{12}$$

Assume that the conclusion does not hold. Then there exists $\bar{u} \in K(S, \bar{x})$ such that

$$\underline{d}^m F(\bar{x}, \bar{y})(\bar{u}) \cap (\text{int}P(\bar{y}) - \bar{y}) \neq \emptyset. \tag{13}$$

Since $\bar{u} \in K(S, \bar{x})$, there exist sequences $\{h_n\} \subset (0, \infty)$ and $\{u_n\} \subset X$ with $h_n \to 0$ and $u_n \to \bar{u}$ such that

$$\bar{x} + h_n u_n \in S. \tag{14}$$

On the other hand, it follows from (13) that there exists $\bar{v} \in \underline{d}^m F(\bar{x}, \bar{y})(\bar{u})$ such that $\bar{v} \in \text{int}(P(\bar{y}) - \bar{y})$. Since $\bar{v} \in \underline{d}^m F(\bar{x}, \bar{y})(\bar{u})$, for the preceding sequences $\{h_n\}$ and $\{u_n\}$, there exists a sequence $\{v_n\} \subset Y$ with $v_n \to \bar{v}$ such that

$$\bar{y} + h_n^m v_n \in F(\bar{x} + h_n u_n). \tag{15}$$

Since $\bar{v} \in \text{int}P(\bar{y}) - \bar{y}$, we have $v_n \in \text{int}P(\bar{y}) - \bar{y}$ for sufficiently large n. We may assume that $h_n^m \in (0, 1)$, which gives, in view of (7), $h_n^m v_n \in \text{int}P(\bar{y}) - \bar{y}$. This is equivalent to

$$\bar{y} + h_n^m v_n \in \text{int}P(\bar{y}). \tag{16}$$

By (15)–(16), we have

$$\bar{y} + h_n^m v_n \in F(\bar{x} + h_n u_n) \cap \mathrm{int} P(\bar{y}),$$

which, in view of (14), contradicts (12). The proof is complete. □

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