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IDEAL CONVERGENCE OF SEQUENCES AND SOME OF ITS APPLICATIONS

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Abstract. We give a short survey of results on ideal convergence with some applications. In particular, we present a contribution of mathematicians from Łódź to these investigations during the recent 16 years.

Families of small sets have been important objects of investigations in the Chair of Real Functions at Łódź University in the recent period. This direction of research was indicated by the boss of the Chair, Władysław Wilczyński many years ago. A tradition of such interests has a source in a meaningful influence of the book "Measure and category" by J. C. Oxtoby [26] which shows several similarities and differences between the Lebesgue measure and the Baire category. Also, the structures of ideals and σ-ideals, as families of small sets, play a significant role here. An ideal of subsets of positive integers $\mathbb{N}$ seems to be a simple notion. However, there is a big variety of such ideals. Moreover, the associated notion of a generalized convergence of sequences has many interesting properties and applications.

Definition 1. *A family* $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ *is called an* ideal on $\mathbb{N}$ *if it is stable under operations of taking subsets and finite unions, and such that* $\mathbb{N} \notin \mathcal{I}$ *and* $\mathrm{Fin} \subset \mathcal{I}$ *where* Fin *stands for the family of finite subsets of* $\mathbb{N}$.

Let us mention a few examples of ideals on $\mathbb{N}$:

(a) $\mathcal{I} = \mathrm{Fin}$;

(b) $\mathcal{I}_d$ – the ideal of sets of *density zero* given by

$$A \in \mathcal{I}_d \iff \limsup_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{n} = 0;$$

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(c) a *summable ideal* $\mathcal{I}_{(a_n)}$ (where $\sum_{n\in\mathbb{N}} a_n = \infty$ is a series with $a_n \geq 0$) given by

$$A \in \mathcal{I}_{(a_n)} \iff \sum_{n\in A} a_n < \infty;$$

(d) given a partition $\{A_n\colon n \in \mathbb{N}\}$ of $\mathbb{N}$ into infinite sets, we let $\mathcal{I} := \{A \subset \mathbb{N}\colon \{n \in \mathbb{N}\colon A \cap A_n \neq \emptyset\} \in \mathrm{Fin}\}$.

For more examples, see [18] and [5].

A filter is a notion dual to an ideal. A family $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ is called a *filter on* $\mathbb{N}$ if it is stable under operations of taking supersets and finite intersections, and such that $\emptyset \notin \mathcal{F}$ and $\{\mathbb{N} \setminus A\colon A \in \mathrm{Fin}\} \subset \mathcal{F}$.

If $\mathcal{I}$ is an ideal, then $\mathcal{I}^* := \{\mathbb{N} \setminus A\colon A \in \mathcal{I}\}$ forms a filter.

Definition 2. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We say that a sequence (x_n) of points of a metric space (X, ρ) is* $\mathcal{I}$-convergent *to $x \in X$ if*

$$\forall \varepsilon > 0\ \exists A \in \mathcal{I}\ \forall n \in \mathbb{N} \setminus A\ \rho(x_n, x) < \varepsilon.$$

The notion of $\mathcal{I}$-convergence generalizes the usual convergence of sequences. Note that Fin-convergence means simply the usual convergence. In the case of $\mathcal{I}_d$-convergence we say about *statistical convergence* which was investigated by several authors starting from Fast [6], Schoenberg [28], Šalát [27] and Fridy [11].

A nice survey on ideal convergence is contained in [8]. Here we focus only on selected topics. Our task is to emphasize a contribution of mathematicians from Łódz to these studies in the recent 16 years.

The definition of $\mathcal{I}$-convergence appeared in the article [18] by P. Kostyrko, T. Šalát and W. Wilczyński. An equivalent notion for filters was considered (independently) by F. Nurrey and W. F. Ruckle [25] but in fact had been studied much earlier by M. Katětov [15].

The article [18] had an important influence on further investigations in this direction. Now it has 107 citations registered by the MathSciNet. In particular, it was an inspiration for several further studies by mathematicians from Łódź and Gdańsk.

Definition 3. *An ideal $\mathcal{I}$ on $\mathbb{N}$ is called a* P-ideal *if for every sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathrm{Fin}$ for all $n \in \mathbb{N}$.*

Among the ideals described in the above examples, those given in (a)–(c) are P-ideals while the one defined in (d) is not.

Let us recall a useful property of P-ideals.

Theorem 1. [18] *If $\mathcal{I}$ is a P-ideal on $\mathbb{N}$, then a sequence (x_n) of points in a metric space X is $\mathcal{I}$-convergent to $x \in X$ if and only if there exists $A \in \mathcal{I}$ such that $\lim_{n\in\mathbb{N}\setminus A} x_n = x$.*

Let X be an uncountable Polish space. By $\mathcal{I}$-$B_1(X)$ we denote the set of limits of pointwise $\mathcal{I}$-convergent sequences of functions of the form $f_n\colon X \to \mathbb{R}$, $n \in \mathbb{N}$. In particular, Fin-$B_1(X)$ is the usual Baire 1 class $B_1(X)$. Let us mention some results concerning connections between $\mathcal{I}$-Baire classes and the usual Baire classes:

- It was shown in [18] that for some class of ideals $\mathcal{I}$ containing $\mathcal{I}_d$ we have $\mathcal{I}$-$B_1(X) = B_1(X)$.
- M. Laczkovich and I. Recław [20] gave a characterization of ideals $\mathcal{I}$ for which this equality holds.
- Similar characterizations were obtained by R. Filipów and P. Szuca [10] for Baire classes of higher levels, also for equal and discrete convergence. These investigations have been continued in [24] and in the PhD thesis of M. Staniszewski (its defence should be held in November '16).

Systematic studies on $\mathcal{I}$-convergence in Gdańsk were initiated by I. Recław (he passed away in 2012). In his research group, some elegant characterizations of ideals with the so-called Bolzano-Weierstrass property were proved; see [7].

The following definition was introduced in [9].

Definition 4. *We say that an ideal $\mathcal{I}$ has* property (R) *if for every series $\sum_{n\in\mathbb{N}} x_n$ which is conditionally convergent in $\mathbb{R}$ and every $r \in \overline{\mathbb{R}}$ there is a permutation p of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} x_{p(n)} = r$ and $\{n\colon p(n) \neq n\} \in \mathcal{I}$.*

- W. Wilczyński [30] proved that the ideal $\mathcal{I}_d$ has property (R) which improves the classic theorem of Riemann.
- R. Filipów and P. Szuca [9] proved that an ideal has property (R) if and only if it cannot be extended to a summable ideal. This solves the problem posed by W. Wilczyński [30].
- A multidimensional version of property (W) was studied by P. Klinga [16].

Note that P. Szuca (in 2012) and R. Filipów (in 2016) finished, in the University of Gdańsk, their habilitations based on a series of articles on ideal convergenge.

In 2005 K. Dems (Łódź University of Technology) defended her PhD thesis "On some kinds of convergence of sequences". The following Cauchy type condition is one of her results.

Theorem 2. [4] *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence (x_n) in a complete metric space is $\mathcal{I}$-convergent if and only if*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \{n \in \mathbb{N}\colon \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I}.$$

The PhD thesis of K. Dems contains some results of the joint paper by M. Balcerzak, K. Dems, A. Komisarski [1] (30 citations in the MathSciNet). The following properties were studied in this article:

- uniform $\mathcal{I}$-convergence of sequences of functions;
- a statistical version of the Egorov theorem;
- $\mathcal{I}$-convergence in measure of sequences of measurable functions.

The Egorov theorem for $\mathcal{I}$-convergence was investigated later by N. Mrożek [23] (he defended his PhD thesis in 2010 in Gdańsk) and by V. Kadets and A. Leonov [14] – in the language of filters.

In 2008 A. Komisarski published another article [17] on pointwise $\mathcal{I}$-convergence and $\mathcal{I}$-convergence in measure. An intersting paper [12] by G. Horbaczewska and A. Skalski deals with a version of the Banach principle for ideal convergence in the classic approach and in a noncommutative setting.

The power set $\mathcal{P}(\mathbb{N})$ can be identified, via characteristic functions, with the Cantor space $\{0,1\}^{\mathbb{N}}$, and thanks to this, an ideal on $\mathbb{N}$, treated as a subset of $\{0,1\}^{\mathbb{N}}$, may be Borel, analytic, coanalytic, etc. An elegant characterization of analytic P-ideals on $\mathbb{N}$ was given by S. Solecki [29]. Other set-theoretical investigations of ideals on $\mathbb{N}$ were conducted by K. Mazur [21], W. Just and A. Krawczyk [13], I. Farah [5], and D. Meza-Alcántara [22].

In 2015 M. Balcerzak, P. Das, M. Filipczak and J. Swaczyna published a paper [2] on a class of density like ideals.

Definition 5. [2] *Let G denote the family of functions $g\colon \mathbb{N} \to (0,\infty)$ such that $g(n) \to \infty$ and $g(n)/n \nrightarrow \infty$. For $g \in G$ let*

$$\mathcal{I}_{\langle g\rangle} := \left\{ A \subset \mathbb{N}\colon \limsup_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{g(n)} = 0 \right\}.$$

Then every family $\mathcal{I}_{\langle g\rangle}$ is an analytic P-ideal, and for $g = \mathrm{id}$ we obtain $\mathcal{I}_{\langle g\rangle} = \mathcal{I}_d$ (the classic density ideal). One of the results of [2] is the following.

Theorem 3. [2] *There exists a set $G_0 \subset G$ of cardinality $\mathfrak{c}$ such that there is no inclusion between $\mathcal{I}_d$ and $\mathcal{I}_{\langle g\rangle}$ for any $g \in G_0$ and there is no inclusion between $\mathcal{I}_{\langle f\rangle}$ and $\mathcal{I}_{\langle g\rangle}$ for any $f, g \in G_0$, $f \neq g$.*

Let us finish with some information about two recent papers. The former paper [3] is the effect of cooperation of researchers from Łódź – it is devoted to ideal invariant injections from $\mathbb{N}$ to $\mathbb{N}$. The latter paper [19] appeared as a very quick reaction to [3] in a research group from Gdańsk. In fact, all problems posed in [3] were solved in [19] and also, an interesting notion of a homogeneous ideal was investigated in [19].

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