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# A NEW TYPE OF DISTANCE AS A TOOL IN THE METRIC FIXED POINT THEORY

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**Abstract.** The basic instrument in metric fixed point theory is, as the name suggests, a metric. A metric is a fairly strict tool that requires, inter alia symmetry. Relatively recently, in fixed point theory, there appeared the concept of replacing in contraction conditions a traditional metric by a new type of mapping, which are generalizations of the traditional metric. One can mention results I. Vályi [1] (in uniform spaces) and D. Tataru [2], O. Kada, T. Suzuki and W. Takahashi [3], T. Suzuki [4] and L.-J. Lin i W.-S. Du [5] (in metric space). Using these new distances, the authors of numerous works, provided the new theorems, which are important extensions of the well known facts.

## 1. Introduction

Let $X$ be nonempty set. A distance on $X$ is called a map $p : X \times X \to [0, \infty)$. The set $X$ with defined distances is called a distance space (M.M. Deza and E. Deza [6], and W.A. Kirk and N. Shahzad [7]).

The following distance spaces are important for several reasons.

**Definition 1.** *Let $X$ be a (nonempty) set, and let $p : X^2 \to [0; \infty)$.*

(**A**) $(X, p)$ *is called metric if:*

(i) $\forall_{u,w \in X} \{p(u, w) = 0$ *iff* $u = w\}$;

(ii) $\forall_{u,w \in X} \{p(u, w) = p(w, u)\}$; *and*

(iii) $\forall_{u,v,w \in X} \{p(u, w) \leq p(u, v) + p(v, w)\}$.

(**B**) (*Roovij* [11]) $(X, p)$ *is called* ultra metric *if:*

(i) $\forall_{u,w \in X} \{p(u, w) = 0$ *iff* $u = w\}$;

(ii) $\forall_{u,w \in X} \{p(u, w) = p(w, u)\}$; *and*

(iii) $\forall_{u,v,w \in X} \{p(u, w) \leq \max\{p(u, v), p(v, w)\}$.

(**C**) (*Bakhtin* [12], *Czerwik* [13]) $(X, p)$ *is called* $b$-metric *with parameter* $c \in [1; \infty)$ *if:*

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(i) $\forall_{u,w \in X}\{p(u,w) = 0$ *iff* $u = w\}$*;*
(ii) $\forall_{u,w \in X}\{p(u,w) = p(w,u)\}$*; and*
(iii) $\forall_{u,v,w \in X}\{p(u,w) \leq c[p(u,v) + p(v,w)]\}$.

(**D**) (*Matthews* [14]) $(X,p)$ *is called* partial metric *if:*

(i) $\forall_{u,w \in X}\{u = w$ *iff* $p(u,u) = p(u,w) = p(w,w)\}$*;*
(ii) $\forall_{u,w \in X}\{p(u,u) \leq p(u,w)\}$*;*
(iii) $\forall_{u,w \in X}\{p(u,w) = p(w,u)\}$*; and*
(iv) $\forall_{u,v,w \in X}\{p(u,w) \leq p(u,v) + p(v,w) - p(v,v)\}$.

(**E**) (*Shukla* [15]) $(X,p)$ *is called* partial $b$-metric *with parameter* $C \in [1;\infty)$ *if:*

(i) $\forall_{u,w \in X}\{u = w$ *iff* $p(u,u) = p(u,w) = p(w,w)\}$*;*
(ii) $\forall_{u,w \in X}\{p(u,u) \leq p(u,w)\}$*;*
(iii) $\forall_{u,w \in X}\{p(u,w) = p(w,u)\}$*; and*
(iv) $\forall_{u,v,w \in X}\{p(u,w) \leq C[p(u,v) + p(v,w)] - p(v,v)\}$.

(**F**) (*Wilson* [16]) $(X,p)$ *is called* quasi-metric *if:*

(i) $\forall_{u,w \in X}\{p(u,w) = 0$ *iff* $u = w\}$*; and*
(ii) $\forall_{u,v,w \in X}\{p(u,w) \leq p(u,v) + p(v,w)\}$.

(**G**) $(X,p)$ *is called* ultra quasi-metric *if:*

(i) $\forall_{u,w \in X}\{p(u,w) = 0$ *iff* $u = w\}$*; and*
(ii) $\forall_{u,v,w \in X}\{p(u,w) \leq \max\{p(u,v), p(v,w)\}$.

(**H**) *The distance* $p$ *is called* pseudometric (*or, the* gauge) *on* $X$ *if:*

(i) $\forall_{u \in X}\{p(u,u) = 0\}$*;*
(ii) $\forall_{u,w \in X}\{p(u,w) = p(w,u)\}$*; and*
(iii) $\forall_{u,v,w \in X}\{p(u,w) \leq p(u,v) + p(v,w)\}$.

(**I**) *The distance* $p$ *is called* quasi-pseudometric (*or, the* quasi-gauge) *on* $X$ *if:*

(i) $\forall_{u \in X}\{p(u,u) = 0\}$*; and*
(ii) $\forall_{u,v,w \in X}\{p(u,w) \leq p(u,v) + p(v,w)\}$.

(**J**) (*Künzi and Otafudu* [17]) *The distance* $p$ *is called* ultra quasi-pseudometric (*or, the* ultra quasi-gauge) *on* $X$ *if:*

(i) $\forall_{u \in X}\{p(u,u) = 0\}$*; and*
(ii) $\forall_{u,v,w \in X}\{p(u,w) \leq \max\{p(u,v), p(v,w)\}\}$.

**Definition 2.** (*Dugundji* [18]) *Let* $X$ *be a* (*nonempty*) *set, and let* $\mathcal{A}$ *be an index set.*

(**A**) *Each family* $\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}$ *of pseudometrics* $d_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathcal{A}$*, is called* gauge *on* $X$*. The gauge* $\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}$ *on* $X$ *is called separating if* $\forall_{u,w \in X}\{u \neq w \Rightarrow \exists_{\alpha \in \mathcal{A}}\{d_\alpha(u,w) > 0\}\}$.

(**B**) *Let the family* $\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}$ *be separating gauge on* $X$*. The topology* $\mathcal{T}(\mathcal{D})$ *having as a subbase the family* $\mathcal{B}(\mathcal{D}) = \{B(u, d_\alpha, \varepsilon_\alpha) : u \in X,$ $\varepsilon_\alpha > 0,\ \alpha \in \mathcal{A}\}$ *of all balls* $B(u, d_\alpha, \varepsilon_\alpha) = \{v \in X : d_\alpha(u, v) < \varepsilon_\alpha\}$ *with* $u \in X$, $\varepsilon_\alpha > 0$, *and* $\alpha \in \mathcal{A}$ *is called topology* induced *by* $\mathcal{D}$ *on* $X$*; the topology* $\mathcal{T}(\mathcal{D})$ *is Hausdorff.*

(**C**) *A topological space* $(X, \mathcal{T})$ *such that there is a separating gauge* $\mathcal{D}$ *on* $X$ *with* $\mathcal{T} = \mathcal{T}(\mathcal{D})$ *is called a* gauge space *and is denoted by* $(X, \mathcal{D})$.

**Definition 3.** (*Reilly* [19]) *Let* $X$ *be a (nonempty) set, and let* $\mathcal{A}$ *be an index set.*

(**A**) *Each family* $\mathcal{P} = \{p_\alpha, \alpha \in \mathcal{A}\}$ *of quasi-pseudometrics* $p_\alpha : X^2 \to [0, \infty)$, $\alpha \in \mathcal{A}$, *is called* quasi-gauge *on* $X$.

(**B**) *Let the family* $\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}$ *be quasi-gauge on* $X$*. The topology* $\mathcal{T}(\mathcal{P})$ *having as a subbase of the family* $\mathcal{B}(\mathcal{P}) = \{B(u, p_\alpha, \varepsilon_\alpha) : u \in X,$ $\varepsilon_\alpha > 0,\ \alpha \in \mathcal{A}\}$ *of all balls* $B(u, p_\alpha, \varepsilon_\alpha) = \{v \in X : p_\alpha(u, v) < \varepsilon_\alpha\}$ *with* $u \in X$, $\varepsilon_\alpha > 0$ *and* $\alpha \in \mathcal{A}$ *is called topology* induced *by* $\mathcal{P}$ *on* $X$.

(**C**) *A topological space* $(X, \mathcal{T})$ *such that there is a quasi-gauge* $\mathcal{P}$ *on* $X$ *with* $\mathcal{T} = \mathcal{T}(\mathcal{P})$ *is called* quasi-gauge space *and is denoted by* $(X, \mathcal{P})$.

**Remark 1.** (*Reilly* [19, Theorems 4.2 and 2.6]) *Each quasi-uniform space and each topological space is the quasi-gauge space.*

The following theorem is major, central, simple and extremely inspiring result in fixed point theory.

**Theorem 1** (S. Banach [20], R. Caccioppoli [21])**.** *Let (X,d) be a complete metric space and let* $(X, T)$ *be a single-valued dynamic system that meets:*

$$\exists_{0 \leqslant \lambda < 1} \forall_{x,y \in X} \ \{d(T(x), T(y)) \leqslant \lambda d(x, y)\}. \tag{1.1}$$

*Then:*

(**i**) $(X, T)$ *has a unique fixed point* $w \in X$, *i.e.* $Fix(T) = \{w\}$;

(**ii**) $\forall_{w^0 \in X} \{\lim_{m \to \infty} d(T^{[m]}(w^0), w) = 0\}$.

The maps satisfying the condition (1.1) are called in the literature a Banach's contractions.

In recent decades, we observed the dynamic development of fixed point theory. Thanks to the introduction of new of ideas, methods and research tools, we have obtained many interesting results on the fixed point, end-points and periodic points for set-valued dynamical systems, as well as the best proximity points for cyclic and acyclic mappings (single-valued and set-valued).

## 2. New type of distances

New ideas and research tools that allowed to prove theorems which gave some answers to these questions were presented by I. Vályi [1] (in uniform spaces) and D. Tataru [2], O. Kada, T. Suzuki and W. Takahashi [3], T. Suzuki [4] and L.-J. Lin and W.-S. Du [5] (in metric space).

In 1985, I. Vályi [1] introduced in unfiorm spaces the concept of $V$-distance, which in metric space, can be defined in the following way.

**Definition 4** ([1]). *Let $(X, d)$ be a metric space. A map $p : X \times X \to [0, \infty)$ is called $V$-distance if the following conditions hold:*

(V1) $\forall_{x,y,z \in X}\{p(x, z) \leqslant p(x, y) + p(y, z)\}$*;*
(V2) *$p$ is lower semicontinuous in its secound variable;*
(V3) $\forall_{x,y \in X}\{p(x, y) \geqslant 0 \wedge [p(x, y) = 0 \Leftrightarrow x = y]\}$*; and*
(V4) $\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y \in X}\{p(x, y) < \delta \Rightarrow d(x, y) < \varepsilon\}$.

In 1992, D. Tataru [2], using a strongly continuous semigroups of nonexpansive maps on $X$, introduced the $T$-distance.

**Definition 5** ([2]). *Let $(X, ||\cdot||)$ be a subset of Banach space and let $\{T(t) : t \in [0, \infty)\}$ be a strongly continuous semigroup of nonexpansive maps on $X$, i.e.*

(sg1) *for each $t \in [0, \infty)$, $T$ is a nonexpansive map on $X$;*
(sg2) *$T(0)x = x$, for each $x \in X$;*
(sg3) *$T(s + t) = T(s) \circ T(t)$, for each $s, t \in \mathbb{R}_+$;*
(sg4) *for each $x \in X$, the map $T(\cdot)x : [0, \infty) \to X$ is continuous.*

*A map $p : X \times X \to [0, \infty)$ defined by the formula*

$$p(x, y) = \inf\{t + ||T(t)x - y|| : t \in \mathbb{R}_+\},\ x, y \in X,$$

*we called a $T$-distance.*

In 1996, O. Kada, T. Suzuki and W. Takahashi [3] introduced the following concept of $w$-distance.

**Definition 6** ([3]). *Let $(X, d)$ be a metric space. Then a map $p : X \times X \to [0, \infty)$ is called $w$-distance on $X$ if it satisfies the following conditions:*

(p1) $\forall_{x,y,z \in X}\{p(x, y) \leqslant p(x, z) + p(z, y)\}$*;*
(p2) *$p$ is lower semicontinuous in its secound variable;*
(p3) $\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y,z \in X}\{[p(z, x) \leqslant \delta \wedge p(z, y) \leqslant \delta] \Rightarrow d(x, y) \leqslant \varepsilon\}$.

Of course, a metric $d \in W = \{p : p \text{ is } w\text{-distance on } (X, d)\}$.

In 2001, T. Suzuki [4] defined a $\tau$-distance.

**Definition 7** ([4]). *Let $(X, d)$ be a metric space. Then a map $p : X \times X \to [0, \infty)$ is called a $\tau$-distance on $X$ if there exists a map $\eta : X \times [0, \infty) \to [0, \infty)$ and the following conditions hold:*

(S1) $\forall_{x,y,z\in X}\{p(x,z) \leqslant p(x,y) + p(y,z)\}$;
(S2) $\forall_{x\in X}\forall_{t>0}\{\eta(x,0) = 0 \wedge \eta(x,t) \geqslant t\}$ *and* $\eta$ *is concave and continuous in its secound variable;*
(S3) $\{\lim_{n\to\infty} x_n = x \wedge \lim_{n\to\infty}\sup_{m\geqslant n}\eta(z_n, p(z_n, x_m)) = 0\} \Rightarrow \{\forall_{w\in X}\{p(w,x) \leqslant \liminf_{n\to\infty} p(w, x_n)\}\}$;
(S4) $\{\lim_{n\to\infty}\sup_{m\geqslant n} p(x_n, y_m)) = 0 \wedge \lim_{n\to\infty}\eta(x_n, t_n) = 0\} \Rightarrow \{\lim_{n\to\infty}\eta(y_n, t_n) = 0\}$;
(S5) $\{\lim_{n\to\infty}\eta(z_n, p(z_n, x_n)) = 0 \wedge \lim_{n\to\infty}\eta(z_n, p(z_n, y_n)) = 0\} \Rightarrow \{\lim_{n\to\infty} d(x_n, y_n) = 0\}$.

In 2006, L.-J. Lin and W.-S. Du [5] introduced a $\tau$-function.

**Definition 8** ([5])**.** *Let* $(X, d)$ *be a metric space. Then a map* $p : X \times X \to [0,\infty)$ *is called* $\tau$*-function if the following conditions hold:*
(L1) $\forall_{x,y,z\in X}\{p(x,y) \leqslant p(x,z) + p(z,y)\}$;
(L2) *if* $x \in X$ *and* $(y_n : n \in \mathbb{N})$ *be a sequence in* $X$ *with* $\lim_{n\to\infty} y_n = y$ *and* $p(x, y_n) \leqslant M$ *for some* $M = M(x) > 0$*, then* $p(x,y) \leqslant M$;
(L3) *for any sequence* $(x_n : n \in \mathbb{N})$ *in* $X$ *with* $\lim_{n\to\infty}\sup\{p(x_n, x_m) : m > n\} = 0$*, if there exists a sequence* $(y_n : n \in \mathbb{N})$ *in* $X$ *such that* $\lim_{n\to\infty} p(x_n, y_n) = 0$*, then* $\lim_{n\to\infty} d(x_n, y_n) = 0$;
(L4) $\forall_{x,y,z\in X}\{[p(x,y) = 0 \wedge p(x,z) = 0] \Rightarrow y = z\}$.

The following remark includes some comparisons of these distances.

**Remark 2.** (a) *In metric spaces, a metric* $d$ *is* $w$*-distance;*
(b) *In metric spaces, any* $w$*-distance is* $\tau$*-distance (compare:* [4, Proposition 1]*);*
(c) *In Banach spaces, any* $T$*-distance is* $\tau$*-distance (compare:* [4, Proposition 2]*);*
(d) *In a matic space, any* $w$*-distance is* $\tau$*-function (compare:* [5, Remark 2.1]*).*

Replacing in contraction condition (1) the function of $d$ by the distances given in the Definitions 3-7, and using innovative methods, more general extensions of Banach's result were received. It is worth noting that in these results, the assumption of completeness of space was still important.

In 2010, K. Włodarczyk and R. Plebaniak introduced new concepts of distance, which turned out to be simple and convenient research tools. Due to them it was possible to obtain more general conclusions with the resignation of restrictive assumptions. These new distances were first defined in the uniform spaces (equipped family pseudometrics).

Let $(X, \mathcal{D})$ be a Hausdorff uniform space equipment with a family $\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}$ uniform continuous on $X \times X$ pseudometrics $d_\alpha$, $\alpha \in \mathcal{A}$, $\mathcal{A}$-index set.

**Definition 9.** [8, Definition 4.1] *Let $(X,\mathcal{D})$ be a Hausdorff uniform space. A family $\mathcal{J}=\{J_\alpha : \alpha\in\mathcal{A}\}$ of maps*

$$J_\alpha : X\times X\to[0,\infty), \alpha\in\mathcal{A},$$

*is called a $\mathcal{J}$-family of generalized pseudodistances on $X$ ($\mathcal{J}$-family, for short) if the following two conditions hold:*

($\mathcal{J}$**1**) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\{J_\alpha(x,z)\leqslant J_\alpha(x,y)+J_\alpha(y,z)\}$*; and*

($\mathcal{J}$**2**) *for any sequences $(x_m : m\in\mathbb{N})$ and $(y_m : m\in\mathbb{N})$ in $X$ sasisfasies*

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{n\to\infty}\sup_{m>n}J_\alpha(x_n,x_m)=0\}$$

*and*

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}J_\alpha(x_m,y_m)=0\},$$

*the following hods*

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}d_\alpha(x_m,y_m)=0\}.$$

In a metric spaces, a generalized pseudodistances is define in the following way:

**Definition 10.** *Let $(X,d)$ be a matric spaces. A map $J : X\times X\to[0,\infty)$ is called a generalized pseduodistance on $X$ if the following two conditions hold:*

(**J1**) $\forall_{x,y,z\in X}\{J(x,z)\leqslant J(x,y)+J(y,z)\}$*; and*

(**J2**) *for any sequences $(x_m : m\in\mathbb{N})$ and $(y_m : m\in\mathbb{N})$ in $X$ such that*

$$\lim_{n\to\infty}\sup_{m>n}J(x_n,x_m)=0$$

*and*

$$\lim_{m\to\infty}J(x_m,y_m)=0,$$

*we have*

$$\lim_{m\to\infty}d(x_m,y_m)=0.$$

Now, we give some important properties of $\mathcal{J}$-family.

**Remark 3.** *Let $(X,\mathcal{D})$ be a Hausdorff uniform spaces and let $\mathcal{J}=\{J_\alpha : X\times X\to[0,\infty),\alpha\in\mathcal{A}\}$ be a $\mathcal{J}$-family on $X$.*

(**a**) *From the properts ($\mathcal{J}$1) and ($\mathcal{J}$2) we obtain*

$$\forall_{x,y\in X}\{[x\neq y]\Rightarrow\exists_{\alpha\in\mathcal{A}}\{J_\alpha(x,y)>0\vee J_\alpha(y,x)>0\}\}.$$

(**b**) *If $\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\{J_\alpha(x,x)=0\}$ then, for each $\alpha\in\mathcal{A}$, a map $J_\alpha$ is a quasi-pseudometrcic. An example of $\mathcal{J}$-family, such that a maps $J_\alpha$, $\alpha\in\mathcal{A}$, are not a quasi-pseudometrics, we will present in the rest of the paper (for details, see Example 1).*

(**c**) *A family $\mathcal{D}$ is $\mathcal{J}$-family on $X$.*

The paper [9], presents the following comparisons.

**Remark 4** ([9, Theorem 6.11]). *Let $(X, d)$ be a metric spaces.*

(a) *If $p : X \times X \to [0, \infty)$ is $\tau$-distance, then $p$ is a generalized pseudodistance;*
(b) *If $p : X \times X \to [0, \infty)$ is $\tau$-function, then $p$ is a generalized pseudodistance;*
(c) *If $p : X \times X \to [0, \infty)$ is $V$-distance, then $p$ is a generalized pseudodistance;*
(d) *There exists a generalized pseudodistance, which is not $\tau$-distance;*
(e) *There exists a generalized pseudodistance, which is not $\tau$-function;*
(f) *There exists a generalized pseudodistance, which is not a $V$-distance.*

## 3. Example of usinig of new of distances

In 1975, P. V. Subrahmanyam [22] intorduced, in complete metric space $(X, d)$, the new type of single-valued contractions $T : X \to X$, such that

$$\exists_{\lambda, 0 \leqslant \lambda < 1} \forall_{x \in X} \{d(T(x), T^{[2]}(x)) \leqslant \lambda d(x, T(x))\}.$$

By assumption of continuity of this contractions, he provided the therem about existing a unigue fixed point $w \in X$ and convergence of any sequence of Picard's iterations to this fixed point.

In this section, we present a example of using of Suzuki's distance (i.e. $\tau$-distance) in some of generalized of Subrahmanyam's theorem.

**Theorem 2.** [10, Theorem 1.1.] *Let $(X, d)$ be a complete metric space and let $(X, T)$ be a single-valued dynamic system (i.e. $T : X \to X$). Suppose that there exist a $\tau$-distance $p$ on $X$ and $\lambda \in [0, 1)$ such that*

$$\forall_{x \in X} \{\tau(T(x), T^{[2]}(x)) \leqslant \lambda \tau(x, T(x))\}.$$

*Assume that one of the following two conditions (A1), (A2) or (A3) holds:*

*A1 if $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$, $\lim_{n\to\infty} p(x_n, T(x_n)) = 0$ and $\lim_{n\to\infty} p(x_n, y) = 0$, then $y = T(y)$;*
*A2 if $(x_n : n \in \mathbb{N})$ and $(T(x_n) : n \in \mathbb{N})$ converge to $y$, then $T(y) = y$;*
*A3 $T$ is continuous.*

*Then there exists $x_0 \in X$ such that $x = T(x)$, and $p(x_0, x_0) = 0$.*

## 4. Example of using a generalized pseduodistances

In the paper [10] K. Włodarczyk and R. Plebaniak generalize Suzuki's result.

**Theorem 3.** *Let* $(X,\mathcal{D})$, $\mathcal{D}=\{d_\alpha : X\times X\to[0,\infty),\ \alpha\in\mathcal{A}\}$, *be a Hausdorff uniform space and let the family* $\mathcal{J}=\{J_\alpha:\alpha\in\mathcal{A}\}$ *of maps* $J_\alpha : X\times\ X\to[0,\infty)$, $\alpha\in\mathcal{A}$, *be a* $\mathcal{J}$*-family on* $X$. *Let* $(X,T)$ *be a single-valued dynamic system and let:*

$$\textit{(S)}\quad \forall_{\alpha\in\mathcal{A}}\exists_{\lambda_\alpha,0\leqslant\lambda_\alpha<1}\forall_{x\in X}\{J_\alpha(T(x),T^{[2]}(x))\leqslant\lambda_\alpha J_\alpha(x,T(x))\}.$$

*Additionally, assume that one of the following two conditions (A1) or (A2) holds:*

*(A1) (A1*$_1$*)* $\forall_{w^0\in X}\exists_{w\in X}\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}J_\alpha(T^{[m]}(w^0),w)=0\}$ *and (A1*$_2$*)* $\forall_{w^0,w\in X}\ \{\forall_{\alpha\in\mathcal{A}}\{\lim_{n\to\infty}\sup_{m>n}J_\alpha(T^{[n]}(w^0),T^{[m]}(w^0))=\lim_{n\to\infty}J_\alpha(T^{[m]}(w^0),w)=0\}\Rightarrow T(w)=w\}$;

*(A2)* $\forall_{w^0\in X}\exists_{w\in X}\forall_{\alpha\in\mathcal{A}}\{lim_{m\to\infty}J_\alpha(w,T^{[m]}(w^0))=\lim_{m\to\infty}J_\alpha(T^{[m]}(w^0),T(w))=0\}$.

*Then:*

*(i)* $Fix(T)\neq\emptyset$;

*(ii)* $\forall_{w^0\in X}\exists_{w\in Fix(T)}\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}d_\alpha(T^{[m]}(w^0),w)=0\}$;

*(iii)* $\forall_{\alpha\in\mathcal{A}}\forall_{w\in Fix(T)}\{J_\alpha(w,w)=0\}$.

It is worth noticing the fact that in the above theorem, despite the expressing of contraction condition (S) by means of a family of generalized pseudodistances, received in the convergence of any iteration is a convergence with respect to family pseudometrics $\mathcal{D}=\{d_\alpha : X\times X\to[0,\infty)$. The benefits from the introduction of a generalized pseudodistances are easy to see on a simple example illustrating the situation of the above theorem. For clarity we limit our considerations to the metric space. We begin with an example of a generalized pseudodistance.

**Example 1.** ([10, Example 3.1]) Let $(X,d)$ be a metric space. Let the set $E\subset X$, containing at least two different points, be arbitrary and fixed and let $c>0$ satisfy $\delta(E)<c$ where $\delta(E)=\sup\{d(x,y):x,y\in E\}$. Let $J:X\times X\to[0,\infty)$ be defined by the formula

$$J(x,y)=\begin{cases} d(x,y) & \text{if } E\cap\{x,y\}=\{x,y\}\\ c & \text{if } E\cap\{x,y\}\neq\{x,y\}\end{cases},\ x,y\in X.$$

The family $\mathcal{J}=\{J\}$ is $\mathcal{J}$-family on $X$, and the map $J$ is a generalized pseudodistance on $X$ (for details see [10, Example 6.1]).

**Example 2.** [10, Example 3.2] Let $(X,d)$ be a noncomplete metric space where $X=[0,3)\cup(3,4)$ and $d(x,y)=|x-y|$, $x,y\in X$. Let $E=[0,1)\cup[2,3)$ and let

$$J(x,y)=\begin{cases} d(x,y) & \text{if } \{x,y\}\cap E=\{x,y\}\\ 4 & \text{if } \{x,y\}\cap E\neq\{x,y\}.\end{cases}\qquad(2)$$

By Example 1, the family $\mathcal{J} = \{J\}$ is $\mathcal{J}$-family on $X$, i.e. the map $J$ is a generalized pseudodistance on $X$. Let $T : X \to X$ be a map given by the formula

$$T(x) = \begin{cases} 0 & \text{if } x = 1 \\ 2 & \text{if } x \in X \setminus \{1\}. \end{cases}$$

Clearly (for details see [10, Example 3.2]), the assumptions $(A1)$ and $(A2)$ of Theorem 3 hold, the set $Fix(T) = \{2\}$, $\forall_{w^0 \in X} \exists_{w=2 \in Fix(T)} \{\lim_{m\to\infty} d(T^{[m]}(w^0), w) = 0\}$ and $J(2, 2) = 0$.

Let us observe that dynamic system $(X, T)$ does not satisfy the condition (S) for $\mathcal{J} = \{d\}$. In fact, if

$$\exists_{\lambda\in[0,1)} \forall_{x\in X} \{d(T(x), T^{[2]}(x)) \leqslant \lambda d(x, T(x))\},$$

then, in particular, for $x_0 = 1 \in X$, we get

$$2 = d(0, 2) = d(T(x_0), T^{[2]}(x_0)) \leqslant \lambda d(x_0, T(x_0)) = \lambda d(1, 0) = \lambda < 1,$$

which is absurd.

Above considerations we conclude by the remark.

**Remark 5.** *We see that in Example 2, the map $T : X \to X$ is not continuous; $T$ satisfies (S) for$\mathcal{J}$-family defined by (2); $T$ does not satisfy (C2) for $\mathcal{J} = \{d\}$; $X$ is not complete; all assumptions of Theorem 3 are satisfied, but the assumptions of S. Banach* [20]*, P.V. Subrahmanyam* [22] *i T. Suzuki's* [4, Theorem 1] *theorems do not hold.*

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