Folia Mathematica
Vol. 19, No. 1, pp. 65–83
Acta Universitatis Lodziensis

GRADIENTS: THE ELLIPTICITY AND THE ELLIPTIC BOUNDARY CONDITIONS - A JIGSAW PUZZLE

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Abstract. Gradients form an important class of linear differential operators. They are, by the definition, irreducible summands of the covariant derivative. Many natural differential operators in geometry are gradients or their linear combinations or at last their compositions. They depend on the geometric structure of the manifold but their importance comes also from the fact that they can encode (e.g. in their spectra) geometric data. The article contains a review and a history of some results on the ellipticity of gradients or on their behavior at the boundary. We enlighten primarily the results with a contribution of the author and his colleagues.

1. Introduction

The article contains a review and a history of some results on a class of geometrically natural differential operators: the gradients. Its intention is enlightening few selected achievements of the author and his colleagues and students. And also some efforts that were undertaken to solve some problems of the theory. The XXth anniversary of the Faculty of Mathematics and Computer Science seems to be a right time for a scientific reflection and a good point for looking back. We believe that any such reflection may also be helpful to outline a better way for further continuation.

Gradients or generalized gradients in the sense of Stein and Weiss are first order differential operators that are irreducible summands of the covariant derivative ∇. More exactly, if one starts from any linear bundle E over M, a differential manifold, and terminates together with ∇ in the bundle $E \otimes T^*M$ and if, additionally, one has a Lie group $\mathfrak{G}$ acting both on E and $E \otimes T^*M$ (and such a group is always strictly associated to the geometric structure considered on M), then one can think on splitting both the origin bundle E and the target bundle $E \otimes T^*M$ onto direct sums of $\mathfrak{G}$-irreducible invariant

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subbundles. Then, the restriction of ∇ to any one of such subbundles of E composed with the projection onto any one of $E \otimes T^*M$ is just a $\mathfrak{G}$-*gradient*. We are mainly interested in $SO(n)$-gradients, i.e., in the case $\mathfrak{G} = SO(n)$ (sometimes also in the case $\mathfrak{G} = GL(n)$). The exact definition is given below.

Gradients are then the simplest bricks the covariant derivative is build of.

$SO(n)$-gradients were introduced first in 1968 by Stein and Weiss in their famous paper *Generalization of the Cauchy-Riemann equations and representations of the rotation group* [21]. Their theory developed next into branches of global analysis, geometry, differential operators or the representation theory. Many natural first order linear differential operators in Riemannian geometry are either gradients or their linear combinations and compositions. For example, the exterior and interior derivatives d and δ, respectively, the Cauchy-Riemann operator $\overline{\partial}$ are gradients while the classical Dirac operator on exterior forms, namely, $d + \delta$ is their sum. The Laplace operator on skew-symmetric forms is the composition $\Delta = (d+\delta) \circ (d+\delta)$. Gradients depend on the geometry of M (the group $\mathfrak{G}$) and this is obvious, but, on the other hand, they can themselves, e.g. by their spectral properties, determine to some extent the geometry (volume, area of the boundary, scalar curvature), cf. [17], [15], [4], [16]. The nice algebraic properties of gradients make that their theory is still successfully developing.

All the object and morphism in the paper are assumed to be smooth, i.e. of class C^∞.

2. Gradients: the notion and examples

Let M be a finite dimensional n and let E be a vector bundle over M.

Assume that ∇ is a covariant derivative in E, i.e., Assume $\mathfrak{G}$ is a Lie group acting both on T^*M and E (such a group is always strictly associated to the geometric structure considered on M).

The covariant derivative ∇ is, by definition, a first order linear differential operator

$$\nabla : E \to E \otimes T^*M$$

satisfying

$$\nabla(fs) = f\nabla s \otimes df$$

for any function f on M and any section s of E.

Our notation convention here is that a bundle itself and the spaces of sections of the bundle is denoted by the same letter. For example, the symbols E or $E \otimes T^*M$ denote the bundles themselves and the spaces of their sections: $C^\infty(E)$ or $C^\infty(E \otimes T^*M)$, respectively. Of course ∇ above is the operator between the spaces of sections. We hope that the proper understanding of the symbol will easily come from the context in each case.

Split both the origin bundle E:

$$E = V_1 \oplus \cdots \oplus V_\mu \oplus \cdots \oplus V_r \tag{1}$$

and the target bundle $F = E \otimes T^*M$:

$$F = W_1 \oplus \cdots \oplus W_\nu \oplus \cdots \oplus W_s \tag{2}$$

into a direct sum of $\mathfrak{G}$-irreducible invariant subbundles of E and F, respectively. Notice that (1) and (2) define uniquely analogous splittings for the spaces of sections.

The restriction of ∇ to any one of such subbundles of E composed with the projection onto any one of subbundles of F is just a $\mathfrak{G}$-*gradient*, shortly, *gradient*.

The described situation can be illustrated by the following diagram:

$$V_1 \oplus \cdots \oplus V_\mu \oplus \cdots \oplus V_r$$

$$\searrow\downarrow\swarrow \tag{3}$$

$$E$$

$$\nabla \downarrow$$

$$F$$

$$\swarrow\downarrow\searrow \tag{4}$$

$$W_1 \oplus \cdots \oplus W_\nu \oplus \cdots \oplus W_s$$

The arrows in row (3) of the diagram represent the natural injections defined by the splitting (1) while the arrows in row (3) - the natural projections defined by the splitting (2). To get a gradient choose one of the injections (3) compose it with ∇ and next with one of the projections (4).

In this talk we will mainly be interested in $GL(n)$-and $SO(n)$-gradients, i.e., in the case $\mathfrak{G} = GL(n)$ or $\mathfrak{G} = SO(n)$. In the other case we assume that the manifold M is oriented and equipped with a Riemannian structure represented by a scalar product $g = < \cdot, \cdot >$.

3. Gradients in tensor bundles

Consider now the case: the origin bundle E is a tensor bundle over M.

For any $k = 0, 1, 2 \ldots$, ∇ can be treated as the operator

$$\nabla : \Gamma(\bigotimes^k T^*M) \to \Gamma(\bigotimes^{k+1} T^*M).$$

By limiting considerations to this case we can derive many explicit formulas for gradients for many irreducible subbundles of $\bigotimes^k T^*M$. In particular in the bundles of skewsymmetric- or trace free symmetric tensors.

The fibers of TM are Euclidean spaces, $SO(n)$ acts on them in a natural way. Obviously, the action can be extended naturally to $\bigotimes^k T^*M$.

Decompose the space $T^*M^k = \bigotimes^k T^*M$ into a direct sum of irreducible invariant subspaces:

$$T^*M^k = \bigoplus_\mu V_\mu.$$

For every μ, denote by $j_\mu : V_\mu \to T^*M^k$ the natural injection defined by the splitting.

Next, take any μ and split the bundle $V_\mu \otimes T^*M$ into a direct sum of invariant irreducible subbundles

$$V_\mu \otimes T^*M = \bigoplus_\nu W_\nu.$$

For every ν, denote by $\pi_\nu : V_\mu \otimes T^*M \to W_\nu$ the natural projection defined by the splitting.

If the multiplicities are one - and it is almost always the case in our considerations - this decomposition is unique. More detailed information on decomposition into irreducibles of any representation (action) of $SO(n)$ in a tensor bundle may be found in [23].

For any μ, ν the first order differential operator

$$\nabla^{\mu\nu} = P_{\mu\nu} = \pi_\nu \circ \nabla \circ j_\mu : V_\mu \longrightarrow W_\nu$$

is just a gradient.

Without loss of generality we can always confine considerations to the case when the origin bundle is irreducible:

The splitting receives then a simpler form, namely:

$$\nabla = G_1 + \cdots + G_\nu + \cdots + G_r \tag{5}$$

From now on we will always assume that the origin bundle is irreducible.

The simplest example is the case $k = 1$. The origin bundle $T^*M^1 = T^*M$ is irreducible ($SO(n)$ acts on the unit sphere in T^*M transitively) but the target bundle $T^*M^2 = T^*M \otimes T^*M$ splits into three $SO(n)$-irreducible invariant subbundles:

$$T^*M^2 = \bigwedge^2 \oplus \mathsf{S}_0^2 \oplus \mathsf{S}_{\mathrm{tr}}^2,$$

where
$\bigwedge^2$ is the subbundle of skew-symmetric two-tensors,
S_0^2 is the subbundle of symmetric and trace-free two-tensors,
$\mathsf{S}_{\mathrm{tr}}^2$ is the subbundle of pure traces, i.e. two-tensors of the form cg, $c \in \mathbb{R}$.
The three projections π_1,π_2, π_3 define the following three gradients

$$G_1 = \pi_1\nabla = \frac{1}{2}d, \qquad G_2 = \pi_2\nabla = S, \qquad G_3 = \pi_3\nabla = -\frac{1}{n}g\delta,$$

so

$$\nabla = \frac{1}{2}d + S - \frac{1}{n}g\delta, \tag{6}$$

where d and δ are usual operators of exterior derivative and coderivative, respectively. The operator S known as the *Cauchy-Ahlfors* operator given by

$$S = \nabla^s + \frac{1}{n}\delta \cdot g$$

where ∇^s is the symmetrized ∇, cf. [17].

In the case of a Riemannian manifold of dimension n the operator ∇^* formally adjoint to ∇ can symbolically be written as

$$\nabla^* = \frac{1}{2}\delta + S^* - \frac{1}{n}d\ \mathrm{tr}$$

where tr denotes the trace of a two-tensor with respect to the Riemannian metric. The equality is to be read that the restrictions of ∇^* to the invariant subspaces coincide sequentially with the operators from the right hand side of the equality.

As a result we have the following splitting for the second order operator $\nabla^*\nabla$:

$$\nabla^*\nabla = \frac{1}{2}\delta d + S^*S + \frac{1}{n}d\delta.$$

The last formula together with the well-known Weitzenböck formula

$$\nabla^*\nabla = \delta d + d\delta - \mathrm{ric},$$

where ric is the Ricci action on one-tensors ($(\mathrm{ric}\phi)_i = \mathrm{ric}_i^j\phi_j$ for any one-tensor ϕ on M), give that the second order strongly elliptic operator S^*S called the *Ahlfors Laplacian* is of form

$$S^*S = \frac{1}{2}\delta d + \frac{n-1}{n}d\delta - \mathrm{ric}.$$

The geometry of the Ahlfors Laplacian is described in [17] or [18]. Its boundary behavior is investigated in [4] and [16].

4. Conformal covariance

One of the interesting fact on $SO(n)$-gradients is that they can be characterized by their *conformal covariance.*

The following fundamental fact was proved by Fegan [6].

Theorem 1. *Each $SO(n)$-gradient G is conformally covariant, in the sense that there are constants c and c^* with*

$$G = \Omega^{-(c+1)} \underline{G} \Omega^{c}, \qquad G^* = \Omega^{-(c^*+1)} \underline{G}^* \Omega^{c^*},$$

whenever we have two conformally equivalent metrics, i.e. metrics g and $\underline{g}$ related by $g = \Omega^2 \underline{g}$ for some positive smooth function Ω on M.

Conversely, any conformally invariant operator from an $SO(n)$-irreducible bundle is a composition of a gradient and a bundle map.

5. The main problems of the theory

The Fegan's theorem established an new perspective in the theory of gradients. It gave a complex unified view on the whole family of gradients and brought a hope that other interesting features can be selected and characterized. Let as give an example: Gradients are differential operators. If we start with a covariant derivative (which, by the way, is an elliptic operator) from an irreducible bundle then the splitting (5) contains in general both the elliptic and the not elliptic summands (gradients). A natural question is *how many of them* and next *which of them* are elliptic or more general which linear combinations of the gradients are elliptic. These questions relate to the so called problem of ellipticity. For example, the operator S from splitting (6) is elliptic. The two gradients d and δ are not elliptic while their sum $d + \delta$, the Dirac type operator, is.

Let us mention the following three main problems of the theory:

- **the problem of ellipticity of particular gradients**

- **the problem of finding a possibly compete list of elliptic boundary conditions for elliptic gradients**

- **the problem of dependence of the gradients on the geometric and other structures (the direct and the inverse problem)**

Although all the three problems (domains) are, in our opinion, equally interesting and although the author has in each of them his small contribution, we decided to focus our attention on the first two only. The reason is simple: the results from the two chosen domains of problems can be summarized in an affordable way. They are more spectacular and more suitable for being presented to a general audience and also have a common future. They can be, roughly speaking, described in a form of a jigsaw puzzle.

6. Ellipticity

Elliptic operators has many nice analytic and geometric properties especially from the point of view of existence or uniqueness of solutions to differential equations involving such operators.

Now, similarly to [14], introduce the notions of symbol and of ellipticity.

Let E and F be two vector bundles over M and $P : E \to F$ be a linear differential operator of order m.

Let $p \in M$ and $\xi \in T^*_p M$. Let $\mathfrak{m}_p$ denote the ring of germs of smooth functions vanishing at p. Let $f \in \mathfrak{m}_p$ be a smooth function defining ξ, i.e. $\xi = df(p)$. Let $e \in E_p$ and and s be such a section of E that $s(p) = e$.

The *symbol* of P at p is the map $\sigma : E_p \times T^*_p M \to F_p$ defined by

$$\sigma(e, \xi) = P(f^m s)(p). \tag{7}$$

One can proof that the definition is correct, i.e. the right hand side of (7) is independent of the choice of f and s. We will also write $\sigma(p, \xi)$ instead of $\sigma(e, \xi)$ to stress the dependence on p and ξ.

A linear differential operator P is called *elliptic at* p if the map

$$E_p \ni e \mapsto \sigma(e, \xi) \in F_p$$

is injective for every $\xi \in T^*_p M$, $\xi \neq 0$.

We say that P is *elliptic* if it is elliptic at every p.

A flagship example is a covariant derivative ∇. It is an elliptic operator in the above sense. Indeed, one can calculate that its symbol $\sigma(e, \xi) = e \otimes \xi$ is just the tensor multiplication be ξ and then defines an injective linear operator for for $\xi \neq 0$.

Now the question arises which gradients in the splitting (5) are elliptic.

Notice that if G_ν is elliptic then the second order differential operator

$$G^*_\nu G_\nu,$$

where G^*_ν denotes the operator formally adjoint to G_ν, is strongly elliptic.

The problem was completely solved within the three following years: 1995, 1996 and 1997 in the three following papers: [9], [10] and [3].

The first answer to the question on the ellipticity of particular gradients was given by **J. Kalina, A. Pierzchalski, P. Walczak**. The paper was send for publication in 1995.

It was proved there that in the case of $\mathfrak{G} = GL(n)$ and ∇ starting from $\mathfrak{G}$-irreducible bundle there is exactly one elliptic gradient in the splitting (5). To detect the elliptic gradient we used the Young diagram method. Let us describe it shortly.

Let W be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) of dimension n. Fix $k \in \mathbb{N}$ and take a sequence of integers $\alpha = (\alpha_1, \dots, \alpha_r)$, $\alpha_1 \geq \cdots \geq \alpha_r \geq 1$, $\alpha_1 + \cdots + \alpha_r = k$. Such an α is called a *Young scheme of length k.* In some references a Young scheme is called a *decomposition.* It can be represented by a figure consisting of r rows of squares and such that the number of the squares in the j-th row is α_j.

A Young scheme can be filled with numbers $1, \dots, k$ distributed in any order. A scheme filled with numbers is called a *Young diagram.* Without loss of generality, we can assume that the numbers grow, both, in rows and in columns.

Take a Young diagram α and denote by H_α and V_α the subgroup of the symmetric groups S_k consisting of all permutations preserving rows and columns, respectively. The diagram α determines the linear operator (called the *Young symmetrizer* $P_\alpha : W^k \to W^k$, $W^k = \bigotimes^k W$, given by

$$P_\alpha = \sum_{\tau \in H_\alpha, \sigma \in V_\alpha} sgn\sigma \cdot \tau\sigma,$$

where the action of any permutation $\rho \in S_k$ on simple tensors is given by

$$\rho(w_1 \otimes \cdots \otimes w_k) = w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(k)}$$

for all $w_1, \dots, w_k \in W$. It is well known that

$$P_\alpha^2 = m_\alpha P_\alpha$$

for some $m_\alpha \in \mathbb{N}$ and that $W_\alpha = imP_\alpha$ is an invariant subspace of W^k for the standard representation of $GL(n)$ in W^k. This representation is irreducible on W_α. Moreover,

$$W^k = \bigoplus_\alpha W_\alpha.$$

Repeat the same construction for the space $W^{k+1} = W^k \otimes W$ to get an analogous splitting into irreducible invariant subspaces:

$$W^{k+1} = \bigoplus_\beta W_\beta.$$

The symbol of ∇ is just "tensoring by a covector". More exactly the action of the symbol abstracts in this case as follows.

Take arbitrary $v \in W$ and consider a linear mapping $\otimes_w : W^k \to W^{k+1}$ defined by

$$\otimes_w(w_1 \otimes \cdots \otimes w_k) = w_1 \otimes \cdots \otimes w_k \otimes w.$$

Theorem 2. *For any $v \neq 0$ the mapping*

$$P_\beta \circ \otimes_w|_{W_\alpha} : W_\alpha \to W_\beta$$

is injective if and only if β is the distinguished extension of α.

Recall that the Young diagram $\beta = (\beta_1, \ldots, \beta_s)$ of the length $k+1$ is the distinguished extension of $\alpha = (\alpha_1, \ldots, \alpha_r)$ of the length k if it is obtained from α by an extension by a single square. The extended diagram should have the number $k+1$ in the added square, while the ordering in the remaining part of the diagram is the same as in α, i.e. if

$$s = r, \quad \beta_1 = \alpha_1 + 1 \quad \beta_2 = \alpha_2, \ldots \beta_s = \alpha_s.$$

In other words, looking at the diagrams, the extension of α to β is distinguished when the β rise from α by adding a single square and the square is added to α at the end of its first row.

Notice that Theorem 2 applied to any irreducible tensor bundle give a direct rule for determining the elliptic gradient. Notice also the striking simplicity of the rule. It is like a jigsaw puzzle: *just add a square to the diagram at its upper right corner.*

Let us state that as the following

Corollary 1. *The operator (gradient) $G_{\alpha\beta} = P_\beta \circ \nabla|_{W_\alpha}$ is elliptic if and only if β is the distinguished extension of α.*

Analogous results were also obtained in [9] for some $SO(n)$-gradients. In particular, for such gradients in the bundle of skew-symmetric tensors and in the bundle of symmetric tensors.

A year later a more general fact was proved for the case of compact semisimple group $\mathfrak{G}$ by **J. Kalina, A. Pierzchalski. B. Ørsted, P. Walczak, G. Zang** [10].

Let $\mathfrak{G}$ be a compact semisimple Lie group and U and V two irreducible fnite-dimensional unitary representations with highest weights μ and ν respectively. The tensor product $U \otimes V$ contains the unique invariant subspace W on which $\mathfrak{G}$ acts by the irreducible representation of highest weight $\mu+\nu$. The following observation was proved there.

Theorem 3. *Let P be the orthogonal projection on an $\mathfrak{G}$-irreducible invariant subspace of the tensor product $U \otimes V$. Then the implication:*

$$u \otimes v \neq 0, \quad u \in U, v \in V \quad \Rightarrow \quad P(u \otimes v) \neq 0$$

holds if only if P is the projection to W.

This observation leads directly to an ellipticity criterion for $\mathfrak{G}$-gradients.

Next, again a year later, the solution to the problem of ellipticity was completed by **T.P.Branson** with the investigating also linear combinations of gradients.

In his beautiful paper [3] he proved among other that for $\mathfrak{G} = SO(n)$ and the corresponding splitting of type (5) the following fact.

Theorem 4. *There are sets $B_1, \dots, B_p \subset \{1, ..., r\}$, each of cardinality 1 or 2, such that*

$$\sum_{\nu \in A} G_\nu^* G_\nu$$

is elliptic if and only if $B_u \subset A$ for some u.

Furthermore, excluding some exceptional cases the sets B_u partition $\{1, \dots, r\}$, i.e. B_u are pairwise disjoint and $\{1, \dots, r\}$ is the sum of all B_u.

Notice that the last theorem has in its essence again a form of a jigsaw puzzle.

Now we are ready to pass to the second problem from our list.

7. Systems of elliptic boundary conditions for elliptic gradients

By a *boundary condition* for a given differential operator $P : E \to F$ we will mean here a condition that have to be satisfied at the boundary by smooth solutions u to a related differential equation (eg. of form $P\phi = \psi$). The examples are the conditions $\mathcal{D}$, $\mathcal{A}$, $\mathcal{R}$ or $\mathcal{N}$ considered in Section 8.

Let us review here some investigations related to the second of the three mentioned problems. In particular let us describe shortly a method of finding a general rule for constructing systems of "natural" and "basic" boundary conditions for the elliptic $SO(n)$-gradients.

The method was suggested in 2004 by **T.P. Branson and A. Pierzchalski** and described in the manuscript [5]. We have concerned there on operators of form

$$G^* G$$

where G is a gradient, but the method is rather general and fully applicable to other differential operators.

Moreover the obtained systems of boundary conditions seem to be complete.

Let us also mention here that the method was summarized later in the introduction to the paper [13].

To describe the method we need, for a given boundary condition, the notion of *ellipticity at the boundary* introduced in [7].

Let M be a smooth compact and oriented manifold with smooth boundary ∂M. Let G be an elliptic gradient and let $P = G^*G$. P is then a second order strongly elliptic operator.

Let $\mathbb{R}_{+/-}$ denote the non zero positive/negative real numbers. It is immediate that then

$$det\{\sigma(x,\xi) - \lambda I\} \neq 0 \tag{8}$$

for $(\xi, \lambda) \neq (0,0) \in T^*M \times \{\mathbb{C} \setminus \mathbb{R}_+ \setminus \mathbb{R}_-\}$.

Fix a fiber metric on E and a volume element on M to define the global inner product $(\cdot,\cdot)$ on $L^2(E)$. P is, by its definition, formally self-adjoint:

$$(P\phi, \psi) = (\phi, P\psi) \tag{9}$$

for ϕ and ψ smooth sections with supports disjoint from the boundary ∂M.

To define the notion of ellipticity at the boundary we will work with a so called half-geodesic coordinate system:

Near ∂M we let $x = (y, r)$ where $y = (y_1, \ldots, y_{n-1})$ is a system of local coordinates on ∂M and where r is the normal distance to the boundary. We assume $\partial M = \{x : r(x) = 0\}$ and that $\frac{\partial}{\partial_r}$ is the inward unit normal. We further normalize the choice of coordinate by requiring the curves $x(r) = (y_0, r)$ for $r \in [0, \delta)$ are unit speed geodesics for any $y_o \in \partial M$. The inward geodesic flow identifies a neighborhood of ∂M in M with the collar $\partial M \times [0, \delta)$ for some $\delta > 0$. The collaring gives a splitting of $TM = T\partial M \oplus T\mathbb{R}$ and a dual splitting $T^*M = T^*\partial M \oplus T^*\mathbb{R}$. To reflect this splitting we let for $\xi \in T^*M$ that $\xi = (\zeta, z)$ where $\zeta \in T^*\partial M$, $z \in T^*\mathbb{R}$.

For a linear differential operator L, we consider the ordinary differential equation:

$$\sigma(y, 0, \zeta, D_r) f(r) = \lambda f(r) \quad \text{with} \quad \lim_{r\to\infty} f(r) = 0 \tag{10}$$

where

$$(\zeta, \lambda) \neq (0,0) \in T^*\partial M \times \mathbb{C} \setminus \mathbb{R}_+ \setminus \mathbb{R}_-.$$

We say that a boundary condition is *self-adjoint and elliptic* with respect to $\mathbb{C} \setminus \mathbb{R}_+ \setminus \mathbb{R}_-$ if:

• the condition is "self-adjoint" for P, i.e. (9) holds for all smooth sections ϕ and ψ satisfying that boundary condition,

• the symbol of P is elliptic in the interior of M, i.e.
(8) holds for $x \in M \setminus \partial M$,

• on the boundary there always exists a unique solution to (10) satisfying the boundary condition.

The described "ellipticity at the boundary" does not imply automatically the existence or the uniqueness (of the solutions) to the related boundary value problem. The results depend generally also on the operator itself, the shape of the boundary condition and the shape of the boundary. The described ellipticity has yet a strong consequence. It ensures namely the existence of a basis in L^2 composed of smooth sections satisfying the boundary condition.

We have namely, (cf. [7]) the following.

Lemma 1. *Let $P : E \to E$ be an elliptic partial differential operator of order $d > 0$. Consider a boundary condition for P that is a self-adjoint and elliptic with respect to $\mathbb{C} \setminus \mathbb{R}_+ \setminus \mathbb{R}_-$. Then*
(a) there exist a complete orthonormal system $(\phi_n)_{n=1}^{\infty}$ for $L^2(E)$ with $P\phi_n = \lambda_n \phi_n$,
(b) ϕ_n are smooth (i.e. of class C^∞) and satisfy the boundary condition,
(c) $\lambda_n \in \mathbb{R}$ and $\lim_{n\to\infty} |\phi_n| = \infty$.

To describe our method we will need some integral formulas.

Recall that if ∇ starts from an irreducible bundle and if we split our target bundle onto irreducible orthogonal summands we get an orthogonal splitting:

$$\nabla\phi = G_1\phi + \cdots + G_s\phi.$$

It is interesting that then for each differential operator $G = G_i$, $i = 1, \ldots, s$ which will be in fact a (Stein - Weiss) gradient we have, for any smooth sections ϕ and ψ, a universal integral formula (cf. [16]):

$$(G^*G\phi, \psi) - (\phi, G^*G\psi) = - \int_{\partial M} (\langle \phi, i_\nu G\psi\rangle - \langle i_\nu G\phi, \psi\rangle)\phi_{\partial M} \tag{11}$$

where

$$(\cdot,\cdot) = \int_M \langle \cdot,\cdot\rangle \, \phi_M$$

is the global scalar product.

Now we are ready to introduce our systems of boundary conditions.

First of all, the systems should be so constructed, that one of the consequence should be the self-adjointness of G^*G. i.e the integral on the right hand side of (11) should vanish. The trick is to find conditions that are "not to weak" (uniqueness) and "not to strong" (existence).

At the boundary, the action of the special orthogonal group $SO(n)$ is replaced by the action of of its subgroup that keep the normal vector invariant. So, our up to now irreducible bundle splits now onto, say s, orthogonal subbundles. Denote by $p_1, \dots, p_s$ the suitable projections.

Now, split both ϕ and $i_\nu G\phi$ by taking their compositions with the projections onto the just obtained (orthogonal and irreducible) subbundles.

We get then that at the boundary:

$$\phi = p_1\phi + \cdots + p_s\phi$$

and, similarly

$$i_\nu G\phi = p_1 i_\nu G\phi + \cdots + p_s i_\nu G\phi.$$

Analogous splittings are obtained for ϕ replaced by ψ.

As a result, we get the following decomposition for the scalar products that appear as the integrand in (11):

$$\begin{aligned}\langle \phi, i_\nu G\psi\rangle = & \\ & + \langle p_1\phi, p_1 i_\nu G\psi\rangle \\ & + \langle p_2\phi, p_2 i_\nu G\psi\rangle \\ & \cdots \\ & + \langle p_s\phi, p_s i_\nu G\psi\rangle \,.\end{aligned}$$

The right hand side of the last equality can be written (symbolically) in a form of a two column matrix

$$\begin{bmatrix} p_1\phi & p_1 i_\nu G\psi \\ p_2\phi & p_2 i_\nu G\psi \\ \vdots & \vdots \\ p_s\phi & p_s i_\nu G\psi \end{bmatrix}.$$

.

A natural boundary condition will be obtained by the demand that exactly one term of each row of the matrix is equal to zero.

We get that way 2^s natural boundary conditions, namely, by exhausting all the possibilities of prescribing the zero value to exactly one term in each row.

Consider an example:

$$\begin{bmatrix} 0 & * \\ 0 & * \\ \vdots & \vdots \\ 0 & * \end{bmatrix}, \dots, \begin{bmatrix} 0 & * \\ * & 0 \\ \vdots & \vdots \\ 0 & * \end{bmatrix}, \dots, \begin{bmatrix} * & 0 \\ * & 0 \\ \vdots & \vdots \\ * & 0 \end{bmatrix}$$

The first matrix defines the condition: $p_1\phi = 0,\ p_2\phi = 0,\ \dots, p_s\phi = 0$ on ∂M or, equivalently, $\phi = 0$ on ∂M. It simply coincides with the Dirichlet type condition.

Similarly, the last matrix defines the condition: $p_1 i_\nu G\phi = 0,\ p_2 i_\nu G\phi = 0,\ \dots, p_s i_\nu G\phi = 0$ on ∂M or, equivalently, $i_\nu G\phi = 0$ na ∂M, It simply coincides with the Neumann type condition.

These two conditions together with the remained $2^s - 2$ ones create a set of natural boundary conditions under which the operator G^*G is self-adjoint with respect to the global scalar product.

Example 1. *Consider the simplest case of the usual gradient operator acting on functions on M treating as sections of the trivial bundle $M \times \mathbb{R}$. Its sections are simply (smooth) functions on M. We have then*

$$\text{grad} : M \times \mathbb{R} \to TM \tag{12}$$

where TM is the tangent bundle. Since the bundles $M \times \mathbb{R}$ and TM are both irreducible the operator (12) is a gradient in our sense. Its formal adjoint is the negative divergence operator

$$\text{grad}^* = -\text{div}$$

So, the composition -div grad *is the negative classical Laplacian Δ on functions. The known Stokes formula says in this case that*

$$\int_M \Delta f\ g - \int_M f\ \Delta g = -\int_{\partial M} (f\ \nabla_\nu g - \nabla_\nu f\ g).$$

Since the bundle $M \times R$ is irreducible, we have a one row matrix

$$\begin{bmatrix} * & * \end{bmatrix}.$$

The natural boundary conditions take then the form:

$$\begin{bmatrix} 0 & * \end{bmatrix} \text{or} \begin{bmatrix} * & 0 \end{bmatrix},$$

or, explicitly,

$$f = 0 \text{ on } \partial M \text{ or } \nabla_\nu f = 0 \text{ on } \partial M.$$

That way we get the well known: Dirichlet or Neumann boundary conditions, respectively.

Let us come back to the general situation.

The systems of boundary conditions constructed above seem to be complete. Indeed, it is enough to scroll the way of their construction. Moreover, many up to known geometrically natural classical conditions appeared to be (directly or after a suitable modification) just particular cases or linear combinations of the particular cases.

More interesting and important is yet the problem of their ellipticity at the boundary. In each up to now investigated case the answer was affirmative. Let as give a short review in the next section.

8. Applications

We are going to set out two important cases:

- the case of gradients in the bundle of k-skew-symmetric tensors.
- the case of gradients in the bundle of k-symmetric tensors

In the first case the only elliptic gradient of all the three possible leads by a composition with its adjoint to a second order elliptic operator which up to a zero order term generalizes to the so called *weighted Lapacian*:

$$\Delta_{ab} = a\delta d + bd\delta \tag{13}$$

where a and b are positive constants called its *weights*.

A boundary behavior of its particular cases were investigated for $k = 1$ and domains in $\mathbb{R}^3$ by **H. Weyl** at the beginning of the XX-th Century (cf. eg. [22]). Next for domains in $\mathbb{R}^n$ by **L.V. Ahlfors** in a series of his papers from the seventies of the XX-th Century (cf. eg. [1]).

For $k = 1$ and an arbitrary Riemann manifold a boundary behavior of the Ahlfors Laplacian was investigated by **B. Orsted and A. Pierzchalski** in [16].

In each case variants of three different geometrically motivated boundary conditions were used.

Our method from [5] enabled revealing the forth condition for elliptic operators in the bundle of skew symmetric forms of any degree. It completed the list and made that it took on a new and interesting symmetry.

The obtained system of boundary conditions have been successfully tested by **W. Kozowski** and **A. Pierzchalski** [13]. For a and b positive constants the weighted Laplacian:

$$\Delta_{ab} = a\ \delta d + b\ d\delta$$

acting on differential skew-symmetric tensors of any degree in the Euclidean ball in $\mathbb{R}^n$ was investigated there in detail. In particular, the problem of existence and uniqueness of the solutions for each of the four boundary conditions was solved there.

Let us come back yet to our general situation.

For arbitrary k the bundle of skew-symmetric k-forms splits (at the boundary) into two summands. So $s = 2$ and the four ($= 2^2$) boundary conditions appear naturally.

For the wighted Laplacian the conditions are:

Dirichlet boundary condition ($\mathcal{D}$):

$$\phi^T = 0 \quad and \quad \phi^N = 0 \quad \text{on} \quad \partial M.$$

Absolute boundary condition ($\mathcal{A}$):

$$\phi^N = 0 \quad and \quad (d\phi)^N = 0 \quad \text{on} \quad \partial M.$$

Relative boundary condition ($\mathcal{R}$):

$$(\delta\phi)^T = 0 \quad and \quad \phi^T = 0 \quad \text{on} \quad \partial M.$$

Neumann boundary condition ($\mathcal{N}$):

$$(\delta\phi)^T = 0 \quad and \quad (d\phi)^N = 0 \quad \text{on} \quad \partial M.$$

Here ϕ^T and ϕ^N denote the tangent and the normal parts of ϕ at the boundary, respectively.

The first three conditions are known to geometers. In particular they appear in the Weyl's paper mentioned above. The fourth one seems to be unknown. But, being natural, it should find its meaning in geometry or physics.

Observe also a surprising symmetry with respect to the Hodge star operator $*$. Namely, by the following known relations:

$$** = \pm 1, \quad (*\phi)^T = \pm * (\phi^N), \quad (*\phi)^N = \pm * (\phi^T)$$

and

$$\delta\phi = \pm \star d \star \phi, \quad d\phi = \pm \star \delta \star \phi,$$

it follows easily that the set of all the four boundary conditions

$$\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{N}\}$$

is star - invariant. More precisely, each of the conditions $\mathcal{D}, \mathcal{B}$ is star-invariant, while the conditions $\mathcal{A}$ and $\mathcal{R}$ are star-symmetric each to the other.

The ellipticity of conditions $\{\mathcal{D}, \mathcal{A}, \mathcal{R},\}$ was proved by **B. Ørsted, A. Pierzchalski** in [16] for $k = 1$ and the case of a general compact Riemannian manifold with a smooth boundary. The proof constructed there extends also on the condition $\{\mathcal{N}\}$. Moreover, it extends also to the proof of the ellipticity of each of the four conditions and the operators acting on forms of an arbitrary degree k. So we get the following nice

Conclusion 1. *All the four investigated boundary conditions are elliptic.*

A. Klekot in her PhD thesis from 2014 (cf. [12]) considered another elliptic operator

$$\mathrm{divgrad}$$

in the bundle of skew-symmetric tensors (forms) on a Riemannian manifold M with a boundary. The notions of the gradient and the divergence from her paper are understood in the **H. Rummler** sense from [20]. She also obtained four boundary conditions. The operator investigated by her is yet different than that one investigated in [13]. The conditions she completed and investigated are also different. Yet, since the bundle is the same as in [13], the number of conditions on the list is also the same. Also in this case the results of study can be summarized as follows:

Conclusion 2. *All the four investigated boundary conditions are elliptic.*

Let us pass to the case of operators acting on the bundle of symmetric forms of arbitrary degree k. This case is more difficult than the previous ones and so, by nature, much less investigated. But recently it has been changing. The symmetric forms are studied more and more intensively and more and more papers has been appearing on this subject. Let us only mention the recent paper [8] on Killing and conformal Killing tensors which are the symmetric tensors from the kernel of some $SO(n)$-gradient.

The first difference is that the graded algebra of all symmetric tenors is infinitely dimensional. The next and, in our case, a more essential difference is that the bundle of symmetric tensors of degree k splits onto $k+1$ summands at the boundary, so $s = k+1$, and – in contrast to the skew-symmetric case – the number of summands in the splitting depends on the degree of forms. In consequence there are 2^{k+1} natural boundary conditions for the bundle of symmetric tensors of degree k. For big k this gives a huge number of conditions.

Recently **A.Kimaczyska** in her PhD thesis [11] found the explicit shape for each of them.

She investigated obtained conditions in detail and proved the ellipticity. Her interesting original construction of the so called auxiliary bundle enabled proving the ellipticity for all the particular boundary conditions simultaneously. As a result we can state here the following result.

Conclusion 3. *All the 2^{k+1} investigated boundary conditions are elliptic!*

To conclude the paper let us state only:

Remark 1. *The problem of ellipticity of the described systems of boundary conditions for an arbitrary elliptic gradient remains still open.*

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