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SPATIAL AND AGE-DEPENDENT POPULATION DYNAMICS MODEL WITH AN ADDITIONAL STRUCTURE: CAN THERE BE A UNIQUE SOLUTION?

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Abstract. A simple age-dependent population dynamics model with an additional structure or physiological variable is presented in its variational formulation. Although the model is well-posed, the closed form solution with space variable is difficult to obtain explicitly, we prove the uniqueness of its solutions using the fundamental Green's formula. The space variable is taken into account in the extended model with the assumption that the coefficient of diffusivity is unity.

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1. Introduction

Mathematical models for epidemics and population dynamics have been evolving in complexity in the last century. This paper introduces and investigates an age-physiology structured population model, where the physiological variable (or factor, also referred to as additional structure, g, say, which could represent size, mass, caloric content or any other attribute that affects the dynamics of individuals in the population (Tchuenche, 2007_b)) is introduced. We assume that the evolution of the population depends on some external constraint, $u \leq \psi$. Without this condition, the problem considered herein would be a linear one with little or no mathematical interest. Thus, the reason for this constraint is that, it transforms the classical problem of proving the existence and uniqueness of a solution via the contraction mapping principle into a variational inequality (Tchuenche, 2007_b). By defining an appropriate operator, we prove the uniqueness of solution of the model using both the Green's formula (also known as Ostrograskii's method) and the contraction mapping principle.

Hernandez (1986) considered the existence of solutions of a population dynamics model with age dependence and diffusion. He introduced a birth modulus which models the case in which individuals are more fertile at younger ages, and derived a scheme to solve the approximating problem. To this scheme, he applied the Schauder Fixed Point Theorem. Motivated by

the fact that variational formulations have been among the least favoured method of solutions to population dynamics problems, we decided to carry out this analysis. Variational formulation of system (1) is rare or non existent, while the proof of uniqueness of solutions via Ostrograskii's method of similar problem is also rare. This paper deals with a population dynamics problem with age-dependence, a, an additional structuring variable, g, and a space variable x (for the extended case). The goal is to prove the uniqueness of solutions to the resulting mathematical problem using the topological fixed point theory. The results in this paper greatly extend a previous result by Tchuenche (2005_a).
The model framework, basic notations and hypotheses are respectively provided in Sections 1.1 and 1.2. The classical problem is transformed into its variational equivalent and analyzed in Section 2. Conclusion follows in Section 3.

1.1. **Model Framework.** Let $u(t, a, g)$ represents the population density of individuals at time t, aged a, with physiological variable g. The dynamics of the population is described by a function $u(t, a, g)$ such that for every interval $[a_1, a_2]$, and any open set $\Omega \subset \mathbb{R}^+$, the integral

$$\int_{a_1}^{a_2} \int_{\Omega} u(t, a, g)dgda,$$

gives the number of individuals of age between ages a_1 and a_2 at time t with physiological factor $g \in \Omega \subset \mathbb{R}^+$. Such a population is ruled by the following first order quasi-linear partial differential equation (Sinko and Streiffer, 1967; Tchuenche, 2005_b), where the variable g can also be referred to as the additional structure

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + G(a)\frac{\partial u}{\partial g} + \mu u = h(t, a, g), \tag{1}$$

with the initial and boundary conditions given respectively by:

$$u(0, a, g) = u_0(a, g), \quad u(t, 0, g) = B(t, g),$$

where the parameter g appearing in the renewal term probably describes how such a factor might be distributed among newborns (Tchuenche, 2007_b). Note that the above model includes the demography (birth and death). $G(a)$ represents the rate of change of g, here assumed independent of g itself for simplicity. Since it is well-known that pregnancy tends to heighten mortality amongst women (Sowunmi, 1993), $h(t, a, g) \leq 0$ may represent the increase in death among females resulting from pregnancy. It is therefore not out of place to assume subsequently and wherever necessary (mainly for mathematical convenience) that this forcing term is a constant. $t \in [0, T], a \in [0, A], g \in \Omega = [\underline{g}, \bar{g}]$. For the preliminaries, we use $u(t, a, g)$ as a basis for elaboration, which enables us to extend the analysis to the case where $u = u(t, a, x, g)$ and the model equation now incorporates diffusion to allow for spatial mobility of individuals, with $x \in (0, L)$. Without the additional structuring variable, the result of Garroni and Langlais (1982) is obtained.

1.2. **Some Hypotheses and Notations.** Unless otherwise stated, the assumptions listed in this section hold throughout the paper:

- $\mu \geq 0$ is the rate of mortality, characteristic of the species.
- $h(t, a, g)$ is a monotone function, possibly zero, that takes into account possible external decrease of population (for a complete derivation of h see Tchuenche, 2005_b). Herein, we assume h is a negative constant. This enables us to carry out the estimate in Equation (12) below.
- birth is described by the renewal equation

$$B(t, g) = u(t, 0, g) = \int_0^A \beta u(t, a, g) da, \tag{2}$$

  where $\beta(> 0)$ represents the rate of fertility.
- the initial density of the population $u_0(a, g) > 0$ is known.

We introduce the following notations:

(H$_1$) $w(t, a, g) \in C_0^\infty(Q)$ is a test function, which is continuously differentiable with compact support, with $Q := [0, T] \times [0, A] \times \Omega$. A natural assumption for w is that it is a smooth positive function, which is uniformly bounded along with its first partial derivatives (Meade and Milner, 1992).

(H$_2$) Denote $[0, A] \times [0, T]$ by Φ, and $[0, A] \times \Omega := \Phi_0$, $\Gamma = [0, A] \times [0, T] \times [0, L]$.

(H$_3$) $Q \times [0, L] := Q^*$, $\quad u_0 \in L^1(Q)$.

(H$_4$) $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-1}(Q)$ and its dual $H_0^1(Q)$, while $\| \cdot \|_Q$ is the usual norm in Q.

(H$_5$) Since $\beta > 0$, then either

$$\int_Q \int_0^\infty \beta u_0(a, x, g) dadx = u_0(0, g) \text{ in } H^{-1}(Q),$$

or,

$$\int_\Omega \int_0^\infty \beta u_0(a, \cdot, g) dadg = u_0(0, x) \text{ in } H^{-1}(Q).$$

This problem can be solved in terms of variational inequalities as follows:
Find a function u such that:

$$
\begin{aligned}
&u \leq \psi, \\
&\left.\begin{aligned}
&u_t + u_a + G(a)u_g + \mu u - h \leq 0 \\
&(u_t + u_a + G(a)u_g + \mu u - h)(u - \psi) = 0
\end{aligned}\right\} ; t > 0, 0 < a < A, g \in \Omega, \\
&u(0, a, g) = u_0(a, g), \\
&B(t, g) = u(t, 0, g) = \int_0^A \beta u(t, a, g) da,
\end{aligned}
\tag{3}
$$

where ψ is a regular function often referred to as the obstacle. Equation (3) can be interpreted to mean that the population develops with a constraint depending for instance on the environment (Friedman, 1988), i.e., $u \leq \psi$. A proof of existence of solutions of (1) can be found in (Tchuenche, 2007_b). Consequently, we assume that its variational counterpart (3) has a solution. For a proof of (3) with $h = 0$, see Tchuenche (2005_a).
Variational methods consist essentially of finding a functional whose variation yields the equation of interest, substituting the trial solution into the functional, and by taking variation with respect to some adjustable parameters, determining these parameters and thereby the *best* approximate solution of the equation (with the given trial function). A weak solution of (1) is a function u, such that: u is continuous on Q, and for the extended case, u^2 is differentiable with respect to the space variable x in the sense of distribution (Rektorys, 1980), a valid reason to account for the space variable in the proof of the main Theorem.

2. Variational Formulation

We define the weak or mild generalized solution of equation (1) by multiplying it by a suitable test function w,

$$\langle u_t + u_a + G(a)u_g, w\rangle = -\langle \mu u + h, w\rangle; w := w(t, a, g). \tag{4}$$

For all $w \in H_0^1(Q)$, u is a distribution in $L^2(Q)$ (Rektorys, 1980), that is,

$$\int_0^\infty \int_0^\infty (u_t \cdot w + u_a \cdot w + Gu_g \cdot w) dadt = \int_0^\infty \int_0^\infty [-\mu u + h] \cdot wdadt. \tag{5}$$

The upper limits of the integrals are ∞ instead of T, or A simply for convenience (for more details on what follows, see Tchuenche, 2007_b). Then,

$$I_1 := \int_0^T \int_0^A u_t \cdot wdadt = \int_0^A \left(\int_0^T u_t \cdot wdt \right) da,$$

integration by parts yields:

$$I_1 = \int_0^A (w_T u_T - w_0 u_0)da - \int_0^T \int_0^A u \cdot w_t dtda, \tag{6}$$

$$I_2 := \int_0^T \left(\int_0^A w \cdot u_a da \right) dt = -\int_0^T u(t,0,g)w(t,0,g)dt$$
$$-\int_0^\infty \int_0^\infty u \cdot w_a dadt, \tag{7}$$

and

$$I_3 := \int_0^\infty \int_0^\infty Gw \cdot u_g dadt. \tag{8}$$

Hence,

$$\begin{aligned} I &= \sum_{n=1}^3 I_n = \int_0^\infty \int_0^\infty [Gwu_g - u(w_a + w_t)dadt \\ &+ \int_0^\infty (w_T u_T - w_0 u_0)da - \int_0^T u(t,0,g)w(t,0,g)dt, \end{aligned} \tag{9}$$

where w_a and w_t are taken in the sense of distributions, and the change in the order of integration can always be justified by the continuity of the integrals (which are understood as functionals), or by Fubini's Theorem (Pitt, 1963). These integrals are functionals on the space

$$\Theta := W^{2,2}([0,A] \times \Omega) \cap W^{1,2}([0,A] \times \Omega) \cap C([0,A] \times \Omega).$$

By Sobolev's imbedding Theorem, $W^{2,2}(Q)$ is compactly imbedded in $L^\infty(Q)$ (Adams, 1975). Let $L(u,w)$ be a bilinear form on $L^\infty(\Theta)$, then

$$\begin{aligned} L(u,w) &:= \int_0^\infty \int_0^\infty [Gw \cdot u_g - u(w_t + w_a)]dadt \\ &+ \int_0^\infty (w_\infty u_\infty - w_0 u_0)da - w(0,g) \\ &+ \int_0^\infty \int_0^\infty (\mu u - h) \cdot wdadt, \end{aligned} \tag{10}$$

and we have the following existence Lemma. The following assumption would be of interest later.

(H_6) $u, w \in H_0^1(Q)$ and $Lu \in H^{-1}(Q)$.

Lemma 1. *A weakly differentiable function $u \in L^1(Q)$ is a mild solution of equation (4) in Q, if there exists a $w \in \Theta$ satisfying equation (5), such that: $\forall\, u \in L^1(Q)$ and any fixed $w \in W^{1,2}(Q)$, $w \mapsto \langle Lu, w\rangle := L(u,w)$ is a bounded linear functional on Θ.*

Proof. This problem more often than not is related to the *a priori* estimates and without loss of generality, we assume that

$$\int_0^\infty u(t,0,g)dt = 1. \tag{11}$$

For mathematical tractability, assume also that the population has a constant migration rate or additional death modulus from pregnancy h, we then have

$$\begin{aligned}
|L(u,w)|_{L^\infty(\Theta)} &\leq \|Gw \cdot u_g - u(w_t + w_a)\|_{W^{2,2}(Q)} \\
&\quad + \|(w_\infty u_\infty - w_0 u_0)\|_{W^{1,2}(Q)} + \sup_{0\leq a\leq A} ||w(t,a,g)|| \\
&\quad + |\mu + h| \|u\|_{W^{2,2}(Q)} \|w\|_{W^{2,2}(Q)}, \\
&\leq \|Gw \cdot u_g - u(w_t + w_a)\|_{W^{2,2}(Q)} + \sup_{0\leq a\leq A} ||w(t,a,g)|| + C_3 \\
&\quad + |\mu + h| \|u\|_{W^{2,2}(Q)} \|w\|_{W^{2,2}(Q)}. \qquad (12)
\end{aligned}$$

By approximating arbitrary functions u, w and their derivatives in $W^{2,2}(Q)$, $W^{1,2}(Q)$ and $C^2(Q)$ by functions in Θ, where u_a, u_t and u_g can be made arbitrarily small for t, a, g in their respective domains (see Daners, 1996 for the argument leading to this hypothesis), with the fact that;
$|G|\|u_g\| \leq C_1$; $\|w_t + w_a\| \leq C_2$; $\|w_\infty u_\infty - w_0 u_0\| = C_3$;
$\sup_{0\leq a\leq A} |w(t,a,g)| \leq C_4$; $|\mu + h| \leq C_5$, the right-hand side of (12) becomes:

$$\begin{aligned}
|L(u,w)|_{L^\infty(\Theta)} &\leq C_1\|w\|_{W^{2,2}(Q)} + C_2\|u\|_{W^{2,2}(Q)} + C_3, \\
&\quad + C_4 + C_5\|u\|_{W^{2,2}(Q)}\|w\|_{W^{2,2}(Q)}, \\
&\leq C_6(\|u\| + \|w\|^2) + C_5\|u\|\|w\|, \\
&\leq C_7\|u\|\|w\| + C_5\|u\|\|w\|, \\
&\leq C_8\|u\|_{W^{2,2}(Q)}\|w\|_{W^{2,2}(Q)}.
\end{aligned}$$

$L(u,w)$ is thus a bilinear form on $L^\infty(\Theta)$, which is bounded (Kinderleherer and Stampacchia, 1980). □

The constants C_1-C_8 are positive, and depend on the parameters of the equation. Without any ambiguity, and without any need of discussing the technicality of incorporating the space variable x into the population density (which leads to a model with diffusion basically to avoid overcrowding), we are now well equipped to move on to the extended case. With the space variable, system (1) takes the form (13) given below.

3. Existence and Uniqueness of Solution in $H^{-1}(Q)$

Problem Setting. Let $g = (g_1, g_2, \cdots, g_n)$ and replace Ω by $\Omega_n := [\underline{g}_1, \bar{g}_1] \times \cdots \times [\underline{g}_n, \bar{g}_n]$. Consider the following population dynamics problem P_1 with spatial variable x and diffusion. For this case, take $h = 0$ and the constant of diffusivity to be unity ($\kappa = 1$) for convenience. Compatibility of the initial and boundary data is assumed, since for $\kappa > 1$, the boundary conditions imply that $u_x = 0$ on the boundary ∂D (homogeneous Neumann condition). Now, we are well-equipped to define the extended model (by including the space variable).
Find $u \in Q^*$, such that:

$$
\begin{aligned}
& u_t + u_a + \sum_{l=1}^{n} G_l(a) u_{g_l} + \kappa u_{xx} + \mu u = 0, \\
& u(0, a, x, g) = u_0(a, x, g), \\
& u(t, 0, x, g) = \int_\Omega \int_0^\infty \beta u(t, a, x, g) da dg \quad \text{on} \quad [0, T] \times \Omega_n,
\end{aligned}
\tag{13}
$$

with Dirichlet conditions or non negative distributions $u(t, a, 0, g) = 0 = u(t, a, L, g)$, representing a close environment. The diffusion term here allows for movements of individuals in an out of the closed region. Note that we shall use the L^2-norm because $\frac{\partial}{\partial t}\left(\int_0^L \frac{1}{2} u^2 dx\right) = -\int_0^L (u_x)^2 dx \leq 0$, which implies that the L^2-norm of u is decreasing and therefore bounded by its initial value (see Morton and Meyers, 1996). By a solution of (13), we mean a function $u(t, a, x, g)$ in Q which is non-negative and satisfy system (13). Existence and uniqueness is guaranteed if u_x is continuous in Q. But it is important to note that (13) may not be globally solvable in general (Meade and Milner, 1992), except when u_x is uniformly Lipschitz continuous in x and uniformly bounded. When the initial data is not strictly positive, solutions exist only in a weak sense, and in this case, we have compactly supported population (Meade and Milner, 1992). Denote $[0, L]$ by D and let C_c be the space of continuous and weakly sequentially compact maps, then:

Theorem 1. *Let $u_n(t, a, x, g)$ be a Cauchy sequence, then, equation (13) has a unique solution if,*

$$\exists\, u_n(t, a, x, g) \to u(t, a, x, g) \;\; \textit{in} \;\; C_c(D; L^2((0, T); H^{-1}(Q^*))).$$

Consequently, (4) has a unique solution.

In order to prove this Theorem, and to apply the penalization technique thereafter, we need an extension of Ostrogradski formula (Langlais, 1980) also known as Green' or Divergence Formula. The proof shall come after the following.

Lemma 2. *Let* $u(t,a,g) \in L^2(D; H_0^1(Q^*))$ *such that* $u_t + u_a + Gu_g$ *belongs to* $L^2(D; H^{-1}(Q^*))$*, then*

(i) For all t_0 *in* $(0,T)$ *and* a_0 *in* $(0,A)$*,* u *has a trace at* $t = t_0$ *belonging to* $L^2(\Phi_0)$*, and at* $a = a_0$ *belonging to* $L^2((0,T) \times \Omega_n)$ *and at* $g = \tilde{g}$ *belonging to* $L^2(\Phi)$*.*

(ii) The following Ostrogradskii formula holds

$$\begin{aligned}
&\int_{\Phi\times\Omega} \langle u_t + u_a + \sum_1^n G_l u_{g_l}, u\rangle dtdadg \\
&= \frac{1}{2}\left\{\int_{(0,A)\times\Omega} u^2(T,a,g)dadg + \int_{(0,A)\times\Omega} u^2(t,A,g)dtdg\right. \\
&\left.+ \sum_1^n \int_{[0,A]\times\Omega_n} G_l(a) u_l^2(t,a,\bar{g})dtda\right\} \\
&- \frac{1}{2}\left\{\int_{(0,A)\times\Omega_n} u^2(0,a,g)dadg + \int_{(0,A)\times\Omega_n} u^2(t,0,g)dtdg\right. \\
&\left.+ \sum_1^n \int_{[0,T]\times\Omega_n} (G_l(a) u^2(t,a,\underline{g})dtda\right\},
\end{aligned} \tag{14}$$

where $(\underline{g}, \bar{g}) \subset \Omega_n$*,* $\underline{g} := \inf g_l$*;* $\bar{g} = \sup g_l, 1 \le l \le n$*.*
Similar proof to what follows with $u = u(t,a,x)$ *has been given by Garroni and Langlais (1982) and ours in essence follows the same pattern, but does not overlap.*

Proof. By applying Ostrogradskii's formula with $G_l(a)$ as a constant, that is, $G_l(a) = G_l \neq 0$, the left-hand side of equation (14) becomes

$$\begin{aligned}
&\int_{\Phi\times\Omega_n} \langle u_t + u_a + \sum_{l=1}^n G_l u_{g_l}, u\rangle dtdadg = \\
&= \int_{\Phi\times\Omega_n} uu_t dtdadg + \int_{\Phi\times\Omega_n} uu_a dadtdg + \sum_1^n \int_{\Phi\times\Omega_n} G_l uu_g dgdtda, \\
&= \frac{1}{2}\int_{(0,A)\times\Omega_n} u^2(t,a,g)\Big|_0^T dadg + \frac{1}{2}\int_{(0,T)\times\Omega_n} u^2(t,a,g)\Big|_0^A dtdg
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{1}^{n} G_l \int_{\Omega_n} u^2(t,a,g)\Big|_{\underline{g}}^{\bar{g}} dtda,\\
&=\frac{1}{2}\left\{\int_{(0,A)\times\Omega_n} u^2(T,a,g)dadg + \int_{(0,T)\times\Omega_n} u^2(t,A,g)dtda\right.\\
&\left.+\sum\int_{\Omega_n} G_l u^2(t,a,\bar{g})dtda\right\}\\
&-\frac{1}{2}\left\{\int_{(0,A)\times\Omega_n} u^2(0,a,g)dadg + \int_{(0,T)\times\Omega_n} u^2(t,0,g)dtda\right.\\
&\left.+\sum\int_{\Omega_n} G_l u^2(t,a,\underline{g})dtda\right\}. \qquad (15)
\end{aligned}
$$

□

We are now well equipped to prove our main result.

Proof. (Theorem 1) Without loss of reality, let $l = 1$. In this case, the underlying set for the variable g is Ω. In order to have a well-defined representation of $L(u, w)$, we have to choose a suitable linear operator $\mathcal{A}$, together with the associated function spaces. When the thermal diffusivity in the classical diffusion equation is unity, we have $u_t = u_{xx}$, which without loss of reality implies that equation (13) can be written as

$$2u_t + u_a + G\sum u_{g_l} + \mu h = h. \qquad (16)$$

where for mathematical convenience, we made the crude assumption that $G_l(a) := G_l$, a constant. Here, we suggest that $\mathcal{A}$ is of the form

$$\mathcal{A}u := -u_t - u_a - Gu_g - \mu u, \qquad (17)$$

$$D(\mathcal{A}) = \{u \in Q : \exists\ 2u_t + u_A + Gu_g \in Q, u \text{ satisfies equation (1)}\}.$$

The operator $\mathcal{A}$ is accretive, a condition closely related to the contraction mapping principle, which enables us to prove uniqueness of solutions of (13) when $h \neq 0$. From hypothesis H_5, we have:

$$\int_{\Omega}\int_0^A \beta u da dg = u(t,0,x,\cdot) \in L^2((0,T);H^{-1}(Q^*)),$$

or

$$\int_{Q^*}\int_0^A \beta u da dx = u(t,0,\cdot,g) \in L^2((0,T);H^{-1}(Q^*)),$$

and

$$u(0,a,\cdot,g) = u_0(a,\cdot,g) \text{ in } L^2(D).$$

Also, from Lemma 2 above, let $u_1,\ u_2 \in X$ be any two solutions of (13), then

$$\begin{aligned}
&\langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\rangle = \\
&= -\langle (2u_1 - u_2)_t + (u_1 - u_2)_a + G(u_1 - u_2)_g \\
&+\mu(u_1 - u_2) - (h_1 - h_2), u_1 - u_2\rangle \\
&\leq -\int_0^A \|u_1(T, a, x, g) - u_2(T, a, x, g)\|^2_{H^{-1}(\Gamma)} dadg \\
&-\frac{1}{2}\int_0^T \|u_1(t, A, x, g) - u_2(t, A, x, g)\|^2_{H^{-1}(\Gamma)} dtdg \\
&-\frac{G}{2}\int_\Omega \|u_1(t, a, x, \bar{g}) - u_2(t, a, x, \bar{g})\|^2_{H^{-1}(\Gamma)} dadt \\
&+\frac{\mu}{2}\int_{(0,T)\times\Gamma} \|[u_1(t, a, x, g) - u_2(t, a, x, g)]\|^2_{H^{-1}(\Gamma)} dadg \\
&+(h_1 - h_2)\int_\Gamma |[u_1(t, a, x, g) - u_2(t, a, x, g)]\|_{H^{-1}(\Gamma)} \\
&\leq -(h_2 - h_1 + \frac{G + 3 + \mu}{2})\int_\Gamma \|[u_1(t, a, \cdot, g) - u_2(t, a, x, g)]\|^2_{H^{-1}(\Gamma)} \\
(18)\quad &= -c\|u_1 - u_2\|^2_Q \leq 0;
\end{aligned}$$

$g \in \Omega_n$ and c is a constant depending only on the parameters of the equation t, a, x and g.

Furthermore, using Hypothesis (H_6), if u_n is a sequence of weak solutions such that $u_n \to u \in H_0^1(Q^*)$, then $\mathcal{A}u_n \to \mathcal{A}u$ weakly in $H^{-1}(Q^*)$, and thus $\mathcal{A}$ is continuous on a finite dimensional subspace of $H_0^1(Q^*)$, with $u_0 \in D(\mathcal{A})$ in (17). From (18), we note that the spatial structure and the parameter h probably add to the dynamics of the model. We now apply the contraction mapping principle to prove the uniqueness of a weak solution of equation (1) via the penalization method (Adams, 1975).

Consider

$$\lambda u - \mathcal{A}u = h, \quad \lambda > 0, \quad h > 0, \tag{19}$$

then,

$$2u_t + u_a + G(a)u_g + (\lambda + \mu)u = h. \tag{20}$$

For a given $w \in L^2(\Phi; H_0^1(Q^*))$, let S be an application of $L^2(Q^*; H_0^1(Q^*))$ into itself with Sw denoting the solution of the regularized equation below, with $\epsilon > 0$ as the regularization parameter. We now show that limit as $\epsilon \to 0$ of the solution of (19) exists. This technique is referred to as the *penalization method.*

$$\int_{Q^*} \langle 2u_t + u_a + \sum G u_g, v\rangle dtdadg + \int_{Q^*} \left[(\lambda+\mu)uv + \frac{1}{\epsilon}(u-\psi)v\right] dtdadg$$

$$= \int_{Q^*} hv\; dtdadg \tag{21}$$

$$u(0,a,\cdot,g) = u_0(a,\cdot,g), \qquad u(t,0,\cdot,g) = B(t,\cdot,g).$$

Since h is a constant, with $v := v(t,a,x,g)$, the right hand side of (21) equals $hv(x)$, a function of x only. Let w_1 and $w_2 \in L^2(Q^*; H_0^1(Q^*))$, be any two solutions of (20). Then, by freshman computations, we have

$$\int_{Q^*} \langle [2\partial_t + \partial_a + \sum G\partial_g](Sw_1 - Sw_2), Sw_1 - Sw_2\rangle dtdadg$$
$$+ \int_{Q^*} (\lambda+\mu)(Sw_1 - Sw_2)^2 dtdadg$$
$$+ \frac{1}{\epsilon}\int_{Q^*} ([Sw_1 - \psi) - (Sw_2 - \psi)] \cdot [Sw_1 - Sw_2] dtdadg$$
$$= h(w_1(x) - w_2(x)). \tag{22}$$

The last term on the left hand side is zero, since

$$[Sw_1 - Sw_2](0,a,\cdot,g) = 0,\; [Sw_1 - Sw_2](t,0,\cdot,g) = \int_\Omega \int_0^{\mathcal{A}} \mu(w_1 - w_2) dadg.$$

Using (16), it is now easy to complete the proof of the Theorem.

$$\int_{\Phi\times\Omega} \langle (2\partial_t + \partial_a + \sum G\partial_g)(Sw_1 - Sw_2)), Sw_1 - Sw_2\rangle dtdadg =$$
$$= \int_{[0,A]\times\Omega} [Sw_1 - Sw_2]^2 (T,a,\cdot,g) dadg +$$
$$+ \frac{1}{2}\int_{[0,A]\times\Omega} [Sw_1 - Sw_2]^2 (t,A,\cdot,g) dtdg$$
$$+ \frac{1}{2}\sum G \int_\Omega [Sw_1 - Sw_2]^2 (t,a,\cdot,\bar{g}) dtda$$
$$- \frac{1}{2}\int_{[0,A]\times\Omega} \left[\int_\Omega \int_0^A \mu(w_1 - w_2) dadg\right]^2 dtdg. \tag{23}$$

Substituting (22) into (21), we obtain

$$\int_{Q^*} (\lambda + \mu)(Sw_1 - Sw_2)^2 dtdadg - h[w_1(x) - w_2(x)] \leq$$

$$\leq \frac{\tilde{c}}{2} \int_{Q^*} (w_1 - w_2)^2 dtdadg \geq 0, \tag{24}$$

where $\tilde{c} = \lambda + \mu$. Since $\mu < 1$, the analysis above suggests that for the result to hold, we should choose λ such that $\lambda + \mu \ll 1$. From (18) and (24), equality holds iff $u_1 = u_2$, and $w_1 = w_2$, respectively. Hence, the result follows. □

4. Conclusion

Motivated by the fact that variational formulations have been among the least favoured method of solutions to population dynamics problems, a deterministic age-physiological strucutrured population dynamics problem is transformed into its variational form and analyzed. It is assumed that the evolution of the population depends on some external constraint. By defining an appropriate operator, we prove the uniqueness of solutions of the model using both the Green's formula (also known as Ostrograskii's method) and the contraction mapping principle.

There are some litiations to this study. One of them being the fact that the space variable is one-dimensional, which may limit its applicability since in reality, the notion of space is multi-dimensional. A potential but not easy extension of this study is the case where the rate of change $G(a)$ of the physiological varable g, is not a constant. Also, the non-linear model where this rate depends on age and g itself $(G(a, g))$ may be mathematically enticing.

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