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ON A GENERALIZED STURM-LIOUVILLE PROBLEM

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Abstract. Basic results of our paper are devoted to a generalized Sturm-Liouville problem for an equation of the form $-(p(t)y'(t))'+q(t)y(t) = F(t, y(\cdot))$ with conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

where $\alpha_1^2 + \alpha_2^2, \beta_1^2 + \beta_2^2 > 0$, $p(t) \neq 0$ for $t \in [a, b]$, $q \in C([a, b])$ and F is a continuous transformation from $[a, b] \times C([a, b])$ to $C([a, b])$.

It is required that the Green's function associated with this problem be nonnegative.

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1. Introduction.

Let F be a continuous transformation from $[a, b] \times C([a, b])$ to $C([a, b])$ with the supremum norm. The main problem considered in this paper is the existence of a solution of the generalized differential equation of the form

$$-(py')' + qy = F(\cdot, y) \quad \text{for } y \in C^2([a, b]) \tag{1}$$

with boundary conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases} \tag{2}$$

where $\alpha_1^2 + \alpha_2^2, \beta_1^2 + \beta_2^2 > 0$, $p \in C^1([a, b])$, $p(t) \neq 0$ for $t \in [a, b], q \in C([a, b])$.

The modification of the Sturm–Liouville problem we consider is motivoted by results of Fijałkowski–Przeradzki [2] and Fijałkowski, Przeradzki and Stańczy [3] on nonlocal elliptic equations. In our considerations we apply the following classical result.

Theorem 1. *([1] p.41) Let P be a cone in a Banach space X, i.e. P is a closed convex set such that:*

(i) *$\lambda P \subset P$ for $\lambda \geq 0$,*
(ii) *$P \cap (-P) = \{0\}$,*

and this cone is normal, i.e. there exists a positive constant C such that $||v|| \leq C||w||$, for $v, w \in P$, $v \leq w$. Let, for $v, w \in X$, the relation $v \leq w$ denote that $w - v \in P$. Suppose that a mapping $T\colon P \to X$ is completely continuous and nondecreasing, i.e.

$$T(v) \leq T(w), \text{ for } v \leq w.$$

If there exist points $v_1, v_2 \in P, v_1 \leq v_2$, for which $v_1 \leq T(v_1)$ and $T(v_2) \leq v_2$, then the mapping T has a fixed point $v_0 \in P$ such that $v_1 \leq v_0 \leq v_2$.

2. Main results

Before presentation of the main results concerning (1) we need the following lemma:

Lemma 1 (On a sublinear transformation)**.** *Let $T\colon C([a,b]) \to C([a,b])$ be a completely continuous and monotonic transformation such that*

(1) $T(0) \geq 0$,
(2) $T(y) \leq \alpha + \beta\upsilon(y)$ *for some* $\alpha, \beta \in C([a,b]), \alpha, \beta \geq 0$, *where υ is a seminorm defined on $C([a,b])$ satisfying the condition*
(3) $\upsilon(\beta) < 1$.

Then there exists a function $y_0 \geq 0$ which is a fixed point of the transformation T.

Proof. We can notice that for $w_1 = 0$ we have $T(w_1) \geq w_1$. We are looking for a $w_2 = c(\beta + \varepsilon), c, \varepsilon \in \mathbb{R}^+$, such that $T(w_2) \leq w_2$. By the assumptions

$$T(w_2) \leq \alpha + \beta\upsilon(c(\beta + \varepsilon)),$$

hence $T(w_2) \leq \alpha + c\beta(\upsilon(\beta) + \varepsilon\upsilon(1))$.

On account of (1) it is not difficult to choose $\varepsilon_0 > 0$ such that $\upsilon(\beta) < 1 - \varepsilon_0\upsilon(1)$. Then for that ε_0 the inequality $T(w_2) \leq \alpha + c\beta$ holds. Then we can take a constant c_0 big enough to satisfy the inequality $\alpha \leq c_0\varepsilon_0$. Thus for $w_2 = c_0(\beta + \varepsilon_0)$ we have $T(w_2) \leq w_2$.

We denote by $C^+([a,b])$ the set of all non-negative functions in $C([a,b])$. Clearly T is completely continuous and monotonic on $C^+([a,b])$. Moreover $T(w_1) \geq w_1$ and $T(w_2) \leq w_2$ for $w_1 = 0$ and $w_2 = c_0(\beta + \varepsilon_0)$, so the assumptions of Theorem 1 are fulfiled. Hence there exists a function $y_0 \in C^+([a,b])$ such that $w_1 \leq y_0 \leq w_2$ and $T(y_0) = y_0$. □

Theorem 2. *Let a continuous function $F\colon [a,b] \times C([a,b]) \to C([a,b])$ satisfy the conditions:*

(i) $0 \leq F(\cdot, y_1) \leq F(\cdot, y_2)$ *for* $0 \leq y_1 \leq y_2, y_1, y_2 \in C([a,b])$,
(ii) $F(\cdot, y) \leq f + h \cdot \upsilon(y)$ *for some nonnegative functions $f, h \in L^1([a,b])$, where υ is a seminorm on $C([a,b])$.*

If the Green's function G associated with problem (1-2) *is nonnegative and* $\upsilon(\int_a^b G(\cdot,t)h(t)dt) < 1$, *then there exists a* C^2*-solution to* (1-2). *The solution is nonnegative.*

Proof. Let us consider an operator $T\colon C([a,b]) \to C([a,b])$ of the form:

$$T(y)(s) = \int_a^b G(s,t)\cdot F(t,y)dt \quad \text{for } y \in C([a,b]), s \in [a,b].$$

We observe that any fixed point of T is a solution of (1-2). By properties of the function G and the assumptions about the function F, the transformation T is completely continuous. Furthermore

$$T(y) \leq \int_a^b G(\cdot,t)f(t)dt + \upsilon(y)\int_a^b G(\cdot,t)h(t)dt.$$

Using the notation $\alpha = \int_a^b G(\cdot,t)f(t)dt$ and $\beta = \int_a^b G(\cdot,t)h(t)dt$, we can see that T satisfies the assumptions of the lemma on a sublinear transformation. Consequently, there exists a fixed point of T, which gives the existence of a solution to problem (1-2). □

Theorem 3. *If the transformation F discribed in the assumptions of Theorem 2 is of the form* $F(t,y) = f(t)+h(t)\upsilon(y)$, *and υ is an additive seminorm on* $C^+([a,b])$ *such that*

$$\upsilon\left(\int_a^b G(\cdot,t)f(t)dt\right) \neq 0,$$

then the condition $(*)$ *is also necessary for existence of the solution* y_0 *to problem* (1-2). *For any such solution* $\upsilon(y_0) \neq 0$.

Proof. Under the above assumptions any solution y_0 to problem (1-2) satisfies the equation

$$y_0(x) = \int_a^b G(x,t)\cdot F(t,y_0)dt \text{ for } x \in [a,b].$$

In view of the form of F and properties of the seminorm υ, we have

$$\upsilon(y_0) = \upsilon\left(\int_a^b G(\cdot,t)f(t)dt\right) + \upsilon\left(\int_a^b G(\cdot,t)h(t)dt\right)\upsilon(y_0).$$

Hence

$$\upsilon(y_0)\left(1-\upsilon\left(\int\limits_a^b G(\cdot,t)h(t)dt\right)\right)=\upsilon\left(\int\limits_a^b G(\cdot,t)f(t)dt\right).$$

Since $\upsilon(\int_a^b G(\cdot,t)f(t)dt) > 0$ and $\upsilon(y_0) \geq 0$, it follows from the above equality that $\upsilon(\int_a^b G(\cdot,t)h(t)dt) < 1$ and $\upsilon(y_0) > 0$. □

Corollary 1. *Let $f, g, h \in L^1([a,b])$ be nonnegative functions. For any differential- integral problem*

$$-(py')' + qy = f + h||g \cdot y||_{L^1} \text{ for } y \in C([a,b]) \tag{3}$$

with boundary conditions (2) and nonnegative Green's function G let us denote $\alpha = \int_a^b G(\cdot,t)f(t)dt, \beta = \int_a^b G(\cdot,t)h(t)dt$. The problem has a solution if and only if one of the following conditions:

(i) *$g \cdot \alpha = 0$ a.e., or*
(ii) *$||g \cdot \beta||_{L^1} < 1$*

holds.

Proof. Observe that the function α is the solution to the differential-only part of the problem. □

Theorem 4. *Let the function F in (1) satisfy the following conditions:*

(i) *$0 \leq F(\cdot,y_1) \leq F(\cdot,y_2)$ if only $0 \leq y_1 \leq y_2$ for $y_1, y_2 \in C([a,b])$, and*
(ii) *$F(\cdot,y) \leq f + \int_a^b A(\cdot,s)y(s)ds$, for some functions $A \in C([a,b] \times [a,b]), f \in C^+([a,b])$ and $y \in C([a,b])$.*

Let $\Gamma(A)(u,s) = \int_a^b G(u,t)A(t,s)dt$ for $u,s \in [a,b]$. If either

(a) *there exist $p > 1$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$\int_a^b ||\Gamma(A)(u,\cdot)du||_{L^p}^q < 1,$$

or

(b) $\max\limits_{u\in[a,b]} ||\Gamma(A)(u,\cdot)||_{L^1} < 1,$

then problem (1-2) has a nonnegative solution in $C^2([a,b])$.

Proof. Let an operator $T\colon C([a,b]) \to C([a,b])$ be defined in the following way:

$$T(y)(u) = \int_a^b G(u,t)F(t,y)dt.$$

Then

$$T(y)(u) \leq \int_a^b G(s,t)f(t)dt + \int_a^b \left(\int_a^b G(u,t)A(t,s)dt \right) y(s)ds \text{ for } u \in [a,b],$$

and thus for $p \geq 1$ and suitable q

$$T(y)(u) \leq \int_a^b G(s,t)f(t)dt + ||\Gamma(A)(u,\cdot)||_{L^p}||y||_{L^q}.$$

The transformation T satisfies the assumptions of Lemma on sublinear transformation. Therefore it has a fixed point in $C^+([a,b])$, and so problem (1-2) has a solution in $C^2([a,b])$. □

References

[1] D.Guo and V.Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Acad. Press, Orlando, 1988.

[2] P.Fijałkowski and B.Przeradzki, *On a radial positive solution to a nonlocal elliptic equation*, Topological Methods in Nonlinear Analysis Journal of the Juliusz Schauder Center **21** (2003), pp. 293-300.

[3] P.Fijałkowski, B.Przeradzki and R.Stańczy, *A nonlocal elliptic equation in a bounded domain*, Banach Center Publications. **66** (2004), pp. 127-133.

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