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A PERIODICITY THEOREM FOR THE EULER-LAGRANGE EQUATION WITH THE LAGRANGE FUNCTION: $L(x,\dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$

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Abstract. This paper proves a new periodicity theorem for the Euler-Lagrange equation with the Lagrange function: $L(x,\dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$ where x is one dimensional, $x \in R$, $\dot{x} \in R$ and $g \in C^2(R)$. L fulfills some assumptions. It is proven that all Euler-Lagrange solutions are periodic.

1. Introduction.

We shall study the solutions of the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{dL_B}{d\dot{x}}\left(x(t),\dot{x}(t)\right)\right) = \frac{dL_B}{dx}\left(x(t),\dot{x}(t)\right)$$

with the Lagrange function $L_B = g(\dot{x}) - \frac{1}{2}x^2$. We will prove that all the solutions $(x(t),\dot{x}(t))$ of the Euler-Lagrange equation for x in 1 dimension, under some assumptions on L_B, are periodic. The Euler-Lagrange equation with the Lagrange function L_B is important for the theory of calculus of variations. This is because the solution of the Euler-Lagrange equation, $x \in C^2([a,b],R)$, is the critical point of the integral $\int_a^b L_B(t,x(t),\dot{x}(t))dt$, where $L_B \in C^2([a,b]\times R\times R)$.

The assumptions on L_B are:

(1) L is autonomous w.r.t t.
(2) L is separable w.r.t x and $\dot{x}$. This tells us that $L(x,\dot{x}) = g(\dot{x}) + h(x)$ where we have that $x \in R$, $\dot{x} \in R$, $g \in C^2(R)$ and $h(x) = -\frac{1}{2}x^2$.
(3) $0 < \frac{d^2g}{d\dot{x}d\dot{x}} < \infty$.
(4) $\frac{g(\dot{x})}{\dot{x}} \longrightarrow \infty \; for \; \dot{x} \longrightarrow \pm\infty$.

According to [1], for the problem $(\Phi(u'))' = f(t,u,u')$ with $u(0)-u(T) = 0 = u'(0) - u'(T)$, it has been proven that, under some conditions, there exist a periodic solution. The Euler-Lagrange equation where L fulfils our

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assumptions can be expressed as $(\Phi(u'))' = f(u)$. This is a special case of $(\Phi(u'))' = f(t, u, u')$. $u(0) - u(T) = 0 = u'(0) - u'(T)$ is the periodicity condition. This periodicity condition is weaker than the periodicity condition used in our paper. In our paper, a new theorem is proven with the periodicity condition $x(t+T) = x(t)$ for all $t \in R$ and $\dot{x}(t) = \dot{x}(t+T)$ for all $t \in R$ and some $T > 0$. For the Euler-Lagrange equation, where L fulfils our assumptions and some further conditions, we have according to [1] that there exist a periodic solution.

There exist other methods to analyse periodicity issues of the Euler-Lagrange Equation, see [2]. Some of these methods are based on functional analysis.

We first introduce $L_A(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$. It is trivial to show that $L_A(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$ has only periodic solutions and that L_A also fulfil the assumptions we have put on L_B above. We perform a reconstruction of the Euler-Lagrange curves (solutions), we have for L_A to some new Euler-Lagrange curves for $L_B(x, \dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$. This tells us L_B has only periodic Euler-Lagrange curves.

In our case, there is a one-to-one relation between any point of the Euler-Lagrange curves for some t in the $(x, \dot{x})$ space and in the $(\dot{x}, \dot{\tilde{p}})$ space. $\dot{\tilde{p}}$ is simply $\dot{\tilde{p}}(\dot{x}(t)) = \frac{d}{dt}\frac{dg}{d\dot{x}}(\dot{x})$, it will be described later in more detail in this paper. The reconstruction of the Euler-Lagrange curves takes place in the $(\dot{x}, \dot{\tilde{p}})$ space.

Without any loss of generality, with regard to the periodicity of the Euler-Lagrange equation, we can assume $g_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$ and $h_{\max}(x) = 0$ for $x = 0$. This makes our proof formally simpler.

We introduce the situation A and B. In the situation A, we are following the Euler-Lagrange curves for L_A and in the situation B, we are following the Euler-Lagrange curves for L_B. The reconstruction of the Euler-Lagrange curves for L_A to L_B start at two arbitrary points, on the $\dot{\tilde{p}}$ axis, that are equally distanced from the origo in both the situations A and B; see figure (1) below. Reconstructing the periodic curves for L_A to L_B, in principle, step by step, we follow the curves in both L_A and L_B having equal $\dot{\tilde{p}}$ from the starting points. We then show analytically that doing so will have the curves meet in L_B.

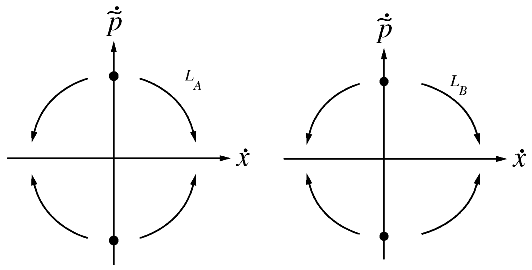

FIGURE 1. Reconstruction of the Euler-Lagrange curves for L_A to L_B

From this we obtain the new theorem:

Theorem 1. *Let L_B be the Lagrange function: $L_B(x, \dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$ for $L_B(x, \dot{x}) \in R^2$ fulfilling our assumptions. Then are all the Euler-Lagrange solutions periodic.*

2. LANGUAGES AND EXPRESSIONS.

2.1. **Some definitions and some basic expressions.** The set V_{L_B} is defined as $\{(x, \dot{x})\ |$there exist an Euler-Lagrange curve $x(t)$ where the point $(x, \dot{x})$ is equal to $(x(t), \dot{x}(t))$ for some $t \in R$, $x \in R$ and $\dot{x} \in R\}$.

The derivatives of L_B are: $\frac{dL_B}{dx} = \frac{dh}{dx} = -x$ and $\frac{dL_B}{d\dot{x}} = \frac{dg}{d\dot{x}}$. We have that $\frac{dh}{dx} \in C^{\infty}(R)$, $\frac{dg}{d\dot{x}} \in C^1(R)$ and that $\frac{dh}{dx}$ is a function of x and $\frac{dg}{d\dot{x}}$ is a function of $\dot{x}$.

We use $\frac{dL_B}{dx} = \frac{dh}{dx}$ and $\frac{dL_B}{d\dot{x}} = \frac{dg}{d\dot{x}}$ to get:

$$\begin{aligned} \frac{d}{dt}\left(\frac{dL_B}{d\dot{x}}(x(t), \dot{x}(t))\right) &= \frac{dL_B}{dx}(x(t), \dot{x}(t)) \Longleftrightarrow \\ \Leftrightarrow \frac{d}{dt}\left(\frac{dg}{d\dot{x}}(\dot{x}(t))\right) &= \frac{dh}{dx}(x(t)). \end{aligned} \tag{1}$$

We define $\widetilde{p} \in C^1(R)$ as a function of the value $\dot{x}$. It is the function:

$$\widetilde{p}(\dot{x}) = \frac{dg}{d\dot{x}}(\dot{x}).$$

We define $\dot{\widetilde{p}} \in C^0(R)$ as a function of the value $\dot{x}$ along an Euler-Lagrange curve with parameter t. It is the function:

$$\dot{\widetilde{p}}(\dot{x}(t)) = \frac{d}{dt}\widetilde{p}(\dot{x}(t)).$$

Substituting $\widetilde{p}$ into the Euler-Lagrange equation (1) along an Euler-Lagrange curve we get another expression of the Euler-Lagrange equation.

We obtain for $t \in R$ that:

$$\frac{d}{dt}\left(\frac{dg}{d\dot{x}}(\dot{x}(t))\right) = \frac{dh}{dx}(x(t)) \Leftrightarrow \frac{d}{dt}\widetilde{p}(\dot{x}(t)) = \frac{dh}{dx}(x(t)) = -x(t). \tag{2}$$

We define $\dot{\widehat{p}} \in C^1(R)$ as a function of the value of x. It is the function:

$$\dot{\widehat{p}}(x) = \frac{dh}{dx}(x) = -x. \tag{3}$$

Note that $\dot{\widehat{p}}$ is not defined as $\frac{d}{dt}\widehat{p}$. Everywhere else in this paper is "." equal to "$\frac{d}{dt}$".

We define $\ddot{\widehat{p}} \in C^0(R)$ as a function of the value $\dot{x}$ along an Euler-Lagrange curve with parameter t. It is the function:

$$\ddot{\widehat{p}}(x(t)) = \frac{d}{dt}\dot{\widehat{p}}(x(t)) = -\dot{x}(t). \tag{4}$$

Along an Euler-Lagrange curve the Euler-Lagrange equation (2) gives us for $t \in R$:

$$\begin{aligned} \frac{d}{dt}\widetilde{p}(\dot{x}(t)) = \frac{dh}{dx}(x(t)) \Leftrightarrow \frac{d}{dt}(\widetilde{p}(\dot{x}(t)) = \dot{\widehat{p}}(x(t)) \Leftrightarrow \\ \Leftrightarrow \dot{\widetilde{p}}(\dot{x}(t)) = \dot{\widehat{p}}(x(t)) = -x(t). \end{aligned} \tag{5}$$

The set U_L is defined as $\{(\dot{x}, \dot{\widetilde{p}}) \mid \dot{\widetilde{p}} = \dot{\widetilde{p}}(\dot{x}) = \frac{dh}{dx}(x) = -x(t), (x, \dot{x}) \in V_L\}$. We can define an orbit in the $(x, \dot{x})$ space as $\{(x(t), \dot{x}(t)) \mid t \in [a, \infty[, x(a) = x_0, \dot{x}(a) = \dot{x}_0\}$. Similarly, we can define an orbit in the $(\dot{x}, \dot{\widetilde{p}})$ space as $\{(\dot{x}(t), \dot{\widetilde{p}}(\dot{x}(t))) \mid t \in [a, \infty[, \dot{x}(a) = \dot{x}_0, \dot{\widetilde{p}}(\dot{x}(a)) = \dot{\widetilde{p}}_0\}$. We can define a backward orbit in the $(x, \dot{x})$ space as $\{(x(t), \dot{x}(t)) \mid t \in]-\infty, b], x(b) = x_0, \dot{x}(b) = \dot{x}_0\}$. Similarly, we can define a backward orbit in the $(\dot{x}, \dot{\widetilde{p}})$ space as $\{(\dot{x}(t), \dot{\widetilde{p}}(\dot{x}(t))) \mid t \in]-\infty, b], \dot{x}(b) = \dot{x}_0, \dot{\widetilde{p}}(\dot{x}(b)) = \dot{\widetilde{p}}_0\}$. A backward orbit in a $(x, \dot{x})$ or $(\dot{x}, \dot{\widetilde{p}})$ space is the graph of a curve with parameter t, that we follow backward in t from some point in that space. We can define a periodic orbit in the $(x, \dot{x})$ space for some $T \in]0, \infty[$ as $\{(x(t), \dot{x}(t)) \mid t \in]-\infty, \infty[, x(t) = x(t+T), \dot{x}(t) = \dot{x}(t+T)\}$. Similarly, we can define a periodic orbit in the $(\dot{x}, \dot{\widetilde{p}})$ space for some $T \in]0, \infty[$ as $\{(\dot{x}(t), \dot{\widetilde{p}}(\dot{x}(t))) \mid t \in]-\infty, \infty[, \dot{x}(t) = \dot{x}(t+T)\}$. For $t \in]-\infty, \infty[$ and for some $T \in]0, \infty[$ is $\dot{x}(t) = \dot{x}(t+T)$. This gives us for $t \in]-\infty, \infty[$ and for some $T \in]0, \infty[$ that $\frac{d}{dt}\widetilde{p}(\dot{x}(t)) = \frac{d}{dt}\widetilde{p}(\dot{x}(t+T))$.

2.2. **Periodicity.** The Euler-Lagrange curve is periodic, with some period T if $x(t+T) = x(t)$ for all $t \in R$ and $\dot{x}(t) = \dot{x}(t+T)$ for all $t \in R$. This is a strong periodicity condition.

Along an Euler-Lagrange curve is

$$(\dot{x}(t), \dot{\widetilde{p}}(\dot{x}(t))) = (\dot{x}(t), \dot{\widehat{p}}\,(x(t))) = (\dot{x}(t), -x(t)).$$

From this we obtain that if the Euler-Lagrange curve $(\dot{x}(t), \dot{\widetilde{p}}(\dot{x}(t)))$ has a periodic orbit in the $(\dot{x}, \dot{\widetilde{p}})$ space with L_B satisfying our assumptions, then the curve $(x(t), \dot{x}(t))$ has a periodic orbit in the $(x, \dot{x})$ space.

2.3. **Some properties of $\widetilde{p}$ and $\dot{\widehat{p}}$.** Because $\widetilde{p}\,(\dot{x}) = \frac{dg}{d\dot{x}}\,(\dot{x})\,,\ \frac{dg}{d\dot{x}} \in C^1(R)$ and assumption (3): $\frac{d^2g}{d\dot{x}d\dot{x}}\,(\dot{x}) = \frac{d\widetilde{p}(\dot{x})}{d\dot{x}} > 0$, we obtain that $\widetilde{p}\,(\dot{x})$ is an increasing differentiable function of $\dot{x}$. $\widetilde{p} = \widetilde{p}\,(\dot{x})$ therefore has an inverse function $\dot{x}^* = \dot{x}^*(\widetilde{p})$ and the inverse function is differentiable and has the derivative $\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p})$. We show in the section (3.2) that $L_B \in C^2(V_L)$ gives us $x \in C^2(R)$. Along a parameter curve $x(t) \in C^2(R)$ we obtain:

$$\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) = \frac{1}{\dfrac{d\widetilde{p}}{d\dot{x}}(\dot{x}(t))}. \tag{6}$$

Because $\dot{\widehat{p}}\,(x) = \frac{dh}{dx}\,(x) = -x(t)$ with $\frac{dh}{dx} \in C^\infty(R)$, we obtain that $\dot{\widehat{p}}\,(x)$ is a decreasing differentiable function of x. $\dot{\widehat{p}} = \dot{\widehat{p}}\,(x)$ therefore has an inverse function $x^* = x^*(\dot{\widehat{p}})$ and the inverse function is differentiable and has the derivative $\frac{dx^*}{d\dot{\widehat{p}}}(\dot{\widehat{p}})$. Along a parameter curve $x(t) \in C^2(R)$ we obtain:

$$\frac{dx^*}{d\dot{\widehat{p}}}(\dot{\widehat{p}}(x(t))) = \frac{1}{\dfrac{d\dot{\widehat{p}}}{dx}(x(t))}.$$

We follow any curve $x(t) \in C^2(R)$ obtaining $\frac{d}{dt}\frac{dg}{d\dot{x}}\,(\dot{x}(t)) = \frac{d\frac{dg}{d\dot{x}}(\dot{x}(t))}{d\dot{x}}\frac{d\dot{x}}{dt}$, $\frac{d\widetilde{p}(\dot{x}(t))}{dt} = \frac{d\widetilde{p}(\dot{x}(t))}{d\dot{x}}\frac{d\dot{x}}{dt}$, $\dot{\widetilde{p}}(\dot{x}(t)) = \frac{d\widetilde{p}(\dot{x}(t))}{d\dot{x}}\ddot{x}(t)$.

From this and equation (5) along an Euler-Lagrange curve $x(t) \in C^2(R)$, $(x, \dot{x}) \in V_L$ we obtain:

$$\begin{aligned} &\dot{\widetilde{p}}\,(\dot{x}(t)) = \dot{\widehat{p}}\,(x(t)) \Leftrightarrow \dot{\widehat{p}}\,(x(t)) = \frac{d\widetilde{p}\,(\dot{x}(t))}{d\dot{x}}\ddot{x}(t) \ \Leftrightarrow \\ &\Leftrightarrow \ddot{x}(t) = \dot{\widehat{p}}\,(x(t))\,\frac{1}{\dfrac{d\widetilde{p}\,(\dot{x}(t))}{d\dot{x}}} \Leftrightarrow \\ &\Leftrightarrow \ddot{x}(t) = \dot{\widehat{p}}\,(x(t))\,\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) = -x(t)\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))). \end{aligned} \tag{7}$$

We follow any curve $x(t) \in C^2(R)$ obtaining $\frac{d}{dt}\frac{dh}{dx}(x(t)) = \frac{d\frac{dh}{dx}(x(t))}{dx}\frac{dx(t)}{dt}$, $\frac{d\dot{\widehat{p}}(x(t))}{dt} = \frac{d\dot{\widehat{p}}(x(t))}{dx}\dot{x}(t)$ and $\ddot{\widehat{p}}(x(t)) = \dot{x}(t)\frac{d\dot{\widehat{p}}(x(t))}{dx} = -\dot{x}(t)$.

We express it in vector form:

$$\begin{pmatrix} \ddot{\widehat{p}}(x(t)) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t)\frac{d\dot{\widehat{p}}(x(t))}{dx} \\ \dot{\widehat{p}}(x(t))\,\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) \end{pmatrix}. \tag{8}$$

We have rewritten the Euler-Lagrange equation (1) into the Euler-Lagrange equation (8) in the set U_{L_B} along an Euler-Lagrange curve.

3. Preparing the Proof.

3.1. **The Euler-Lagrange equations in the situation A and B.** L_A is the Lagrange function: $L_A(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$ for $(x,\dot{x}) \in V_{L_A}$. Our assumptions are satisfied for L_A. We obtain $V_{L_A} = \{(x,\dot{x}) \mid (x,\dot{x}) \in R^2\}$ and $U_{L_A} = \{(\dot{x},\dot{\widehat{p}}) \mid (\dot{x},\dot{\widehat{p}}) \in R^2\}$.

We obtain $\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}) = 1$ and $\frac{d\dot{\widehat{p}}}{dx}(x) = -1$ everywhere. Using equation (8) we have along an Euler-Lagrange curve with parameter $t \in R$ in the case L_A that:

$$\begin{pmatrix} \ddot{\widehat{p}}(x(t)) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} -\dot{x}(t) \\ \dot{\widehat{p}}(x(t)) \end{pmatrix}. \tag{9}$$

L_B is the Lagrange function: $L_B(x,\dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$ for $(x,\dot{x}) \in V_{L_B}$. Our assumptions on L_B are satisfied. We obtain $V_{L_B} = \{(x,\dot{x}) \mid (x,\dot{x}) \in R^2\}$ and $U_{L_A} = \{(\dot{x},\dot{\widehat{p}}) \mid (\dot{x},\dot{\widehat{p}}) \in R^2\}$.

We obtain $\frac{d\dot{\widehat{p}}}{dx} = -1$. Using equation (8) we have along an Euler-Lagrange curve with parameter $t \in R$ in the case L_B that:

$$\begin{pmatrix} \ddot{\widehat{p}}(x(t)) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} -\dot{x}(t) \\ \dot{\widehat{p}}(x(t))\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) \end{pmatrix}. \tag{10}$$

It is trivial to show that for $L_A(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$ with $(x,\dot{x}) \in R^2$ that all the Euler-Lagrange curves, $(x(t),\dot{x}(t))$, $t \in R$, are periodic with $T = 2\pi$ in the space $(x,\dot{x}) \in R^2$ and all the Euler-Lagrange curves, $(\dot{x}(t),\dot{\widetilde{p}}(\dot{x}(t)))$, $t \in R$, are periodic with $T = 2\pi$ in the $(\dot{x},\dot{\widehat{p}}) \in R^2$ space.

3.2. **Differentiability.** Along an Euler-Lagrange curve, we have the equation (10): $\ddot{x}(t) = \dot{\widehat{p}}(x(t))\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t)))$. Having $\widetilde{p}(\dot{x}) \in C^1(R) \Longrightarrow \frac{d\widetilde{p}}{d\dot{x}^*}(\dot{x}) \in C^0(R) \Longrightarrow \frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x})) \in C^0(R) \Longrightarrow \frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) \in C^0(R)$ and equation (10) gives us $x(t) \in C^2(R)$. This tells us that $\ddot{x}_B(t_B) \in C^0(R)$. Having $\dot{\widehat{p}}_B(x_B) = \frac{dh_B}{dx}(x_B) = -x_B$ gives us $\dot{\widehat{p}}_B(x_B) \in C^\infty(R)$. From our assumption (2) we have $g \in C^2(R)$. Having $\widetilde{p}_B\,(\dot{x}_B) = \frac{dg}{d\dot{x}}\,(\dot{x}_B)$ gives us $\widetilde{p}_B\,(\dot{x}_B) \in C^1(R)$. Along an Euler-Lagrange curve we have the equation (10): $\ddot{\widehat{p}}_B(x_B(t_B)) = -\dot{x}_B(t_B)$, that gives us $\ddot{\widehat{p}}_B\,(x_B(t)) \in C^1(R)$ because we have $x_B(t) \in C^2(R)$. Along an Euler-Lagrange curve we have the equation (5): $\dot{\widetilde{p}}\,(\dot{x}(t)) = \dot{\widehat{p}}\,(x(t))$. This tells us that along an Euler-Lagrange curve is $\dot{\widehat{p}}_B\,(x_B(t_B)) \in C^2(R)$ and $\dot{\widetilde{p}}_B\,(\dot{x}_B(t_B)) \in C^2(R)$.

3.3. **Change of variables of the Lagrange functions L_B.** Without any loss of generality, we can assume that for L_B is $g_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$. The change of variables of L_B below makes the proof formally simpler. We establish some propositions exhibiting this simplicity.

$g(\dot{x})$ is any convex function with a $g_{\min}(\dot{x}) = b$ for $\dot{x} = d$, satisfying our growth condition that is our assumption (4). By the equation (10), we can, without changing any of the solutions, add a constant to $g(\dot{x})$. We can choose this constant to be $-g_{\min}(d)$. This equals assuming that $g(\dot{x})$ has a $g_{\min}(\dot{x}) = 0$ for $\dot{x} = d$.

Proposition 1. *For L_B we can assume that $g_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$ without any loss of generality, with regard to the periodicity of the Euler-Lagrange equation.*

Proof. $g(\dot{x})$ is a convex function with a $g_{\min}(\dot{x}) = 0$ for $\dot{x} = d$. The equation (10) contains the terms $\dot{x}(t)$ and $\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t)))$. The term $\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t)))$ in the equation (10): $\ddot{x}(t) = \dot{\widehat{p}}(x(t))\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t)))$ does not change the form of our solutions when we translate $g(\dot{x})$ along the $\dot{x}$ axis, but $\dot{x}(t)$ would change giving us a different solution by $\ddot{\widehat{p}}(x(t)) = -\dot{x}(t)$. Translating $g(\dot{x})$ along the $\dot{x}$ axis gives us different solutions. However,

$$\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) = \frac{1}{\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}(t))} = \frac{1}{\frac{dg}{d\dot{x}d\dot{x}}(\dot{x}(t))}$$

is just some function > 0, another $g^*(\dot{x}) = g(\dot{x}) - \frac{dg}{d\dot{x}}(0)\dot{x} - g(0)$ would have the same $\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t)))$. $g^*(\dot{x})$ is satisfying our growth condition that is the assumption (4). We have obtained that there exists an $\dot{x} \in R$ such that $\frac{dg^*(\dot{x})}{d\dot{x}} = 0$. $g^*(\dot{x})$ has $g^*_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$ because $g^*(0) = g(0) - \frac{d\widetilde{p}}{d\dot{x}}(0)0 - g(0) = 0$, $\frac{dg^*}{d\dot{x}}(0) = \frac{dg}{d\dot{x}}(0) - \frac{dg}{d\dot{x}}(0) - 0 = 0$, there exists an $\dot{x} \in R$ such that $\frac{dg^*(\dot{x})}{d\dot{x}} = 0$ and assumption (3): $0 < \frac{d^2g}{d\dot{x}d\dot{x}} < \infty$. $g^*(\dot{x})$ and $g(\dot{x})$ give us the same equations (10). We obtain the same solutions. If $g^*(\dot{x})$ is periodic then so is $g(\dot{x})$. We can therefore, just assume that $g_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$ without changing the generality of the proof. □

We already have that $h_{\max}(x) = \max(-\frac{1}{2}x^2) = 0$ for $x = 0$.

3.3.1. *An additional requirement to L_B.* The change of variables gives us an additional requirement we can put on L_B without changing the generality of the proof. We obtain:

L_B is the Lagrange function: $L_B\,(x,\dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$ for $L_B\,(x,\dot{x}) \in V_{L_B}$. Our assumptions are satisfied for L_B where g is an arbitrary function fulfilling assumptions (2-4) and $g_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$.

3.4. **The direction of the Euler-Lagrange trajectory.** We define the following set that is needed in our reconstruction proof: $Q_1 = \{(\dot{x}, \dot{\widetilde{p}}) \mid \dot{x} \geqq 0 \wedge \dot{\widetilde{p}} \geqq 0 \wedge (\dot{x} \neq 0 \vee \dot{\widetilde{p}} \neq 0)\}$. We define analogously the sets Q_2, Q_3 and Q_4 for the other quadrants. Variables describing curves belonging to Q_1 is denoted with the index 1. Analogously we denote variables describing curves belonging to Q_2, Q_3 and Q_4. The index 0 denotes our starting point. A point at some t belonging to several sets is denoted with all the indices of these sets. If the point on an Euler-Lagrange curve at some t is to enter, either forward or backward, a particular set, then the index of this set is denoted last. Both variables, points and t may be denoted with a situation A or B.

For L_A and L_B along an Euler-Lagrange curve for every $t \in R$, we obtain $:\frac{d\dot{\widehat{p}}}{dx}(x(t)) = -1 < 0$ and $\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) > 0$. For L_A and L_B along an Euler-Lagrange curve, we obtain: $\dot{x}(t) < 0 \Leftrightarrow \ddot{\widehat{p}}(x(t)) = -\dot{x}(t) > 0$ and $\dot{x}(t) > 0 \Leftrightarrow \ddot{\widehat{p}}(x(t)) = -\dot{x}(t) < 0$. For L_A and L_B along an Euler-Lagrange curve, we obtain: $\ddot{x}(t) > 0 \Leftrightarrow \dot{\widehat{p}}(x(t)) = \frac{1}{2}(-x(t))^2 > 0$ and $\ddot{x}(t) < 0 \Leftrightarrow \dot{\widehat{p}}(x(t)) = \frac{1}{2}(-x(t))^2 < 0$. We obtain that $(\dot{x}, \dot{\widetilde{p}}) \in Q_1 \Leftrightarrow \dot{\widehat{p}}\,(x(t))$ and $\dot{\widetilde{p}}\,(\dot{x}(t))$ decrease and $\dot{x}(t)$ increase in t. $(\dot{x}, \dot{\widetilde{p}}) \in Q_2 \Leftrightarrow \dot{\widehat{p}}\,(x(t))$ and $\dot{\widetilde{p}}\,(\dot{x}(t))$ decrease and $\dot{x}(t)$ decrease in t. $(\dot{x}, \dot{\widetilde{p}}) \in Q_3 \Leftrightarrow \dot{\widehat{p}}\,(x(t))$ and $\dot{\widetilde{p}}\,(\dot{x}(t))$ increase and $\dot{x}(t)$ decrease in t. $(\dot{x}, \dot{\widetilde{p}}) \in Q_4 \Leftrightarrow \dot{\widehat{p}}\,(x(t))$ and $\dot{\widetilde{p}}\,(\dot{x}(t))$ increase and $\dot{x}(t)$ increase in t.

Proposition 2. *The Euler-Lagrange orbit for L_B with a starting point in Q_1 belonging to U_{L_B} when $t = a$ will reach some point $(\dot{x}_{12} > 0, \dot{\widetilde{p}}_{12} = 0)$ when $t = t_{12}$.*

Proof. In Q_1 is the coordinate of the orbit $\dot{\widetilde{p}}_1$ decreasing in t and the coordinate of the orbit $\dot{x}_1$ increasing in t. We may have the following cases:

1) The orbit will reach some point $(\dot{x}_{12} > 0, \dot{\widetilde{p}}_{12} = 0) \in U_L$ when $t = t_{B_{12}}$.

2a) $\dot{\widetilde{p}}(\dot{x}(t)) \to w_0 \geqq 0$ for $t \to \infty$ and $\dot{x}(t) \to \infty$ for $t \to \infty$. This gives us using equation (5): $\dot{\widetilde{p}}\,(\dot{x}(t)) = \dot{\widehat{p}}\,(x(t))$ that $\dot{\widehat{p}}(x(t)) \to w_0 \geqq 0$ for $t \to \infty$. We obtain that $x(t) \to -w_0 \leqq 0$ for $t \to \infty$. $\dot{x}(t) \to \infty$ for $t \to \infty$ give us $x(t) \to \infty$ for $t \to \infty$. This is in contradiction with $x(t) \to -w_0 \leqq 0$ for $t \to \infty$.

2b) $\dot{\widetilde{p}}(\dot{x}(t)) \to w_0 \geqq 0$ for $t \to t_k$ and $\dot{x}(t) \to \infty$ for $t \to t_k$. This gives us using equation (5) that $\dot{\hat{p}}(x(t)) \to w_0 \geqq 0$ for $t \to t_k$. This gives us that $x(t) \to -w_0 \leqq 0$ for $t \to t_k$. Having the equation (10): $\ddot{x}(t) = \dot{\hat{p}}(x(t))\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t)))$ with $\dot{\hat{p}}(x(t)) \to w_0 \geqq 0$ for $t \to t_k$ and not having $\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}(\dot{x}(t))) \to \infty$ for $t \to t_k$ because we don't have that $\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}(t)) \to 0$ for $t \to t_k$ due to the assumption (4) tells us that we don't have that $\ddot{x}(t) \to \infty$ for $t \to t_k$. This is in contradiction with $\dot{x}(t) \to \infty$ for $t \to t_k$ that implies $\ddot{x}(t) \to \infty$ for $t \to t_k$.

3) $\dot{\widetilde{p}}(\dot{x}(t)) \to w_0 \geqq 0$ for $t \to \infty$ and $\dot{x}(t) \to j_0 > 0$ for $t \to \infty$. Using the equation (5) we have $\dot{\hat{p}}(x(t)) \to w_0 \geqq 0$ for $t \to \infty$. This gives us that $x(t) \to -w_0 \leqq 0$ for $t \to \infty$. This is in contradiction with $\dot{x}(t) \to j_0 > 0$ for $t \to \infty$ that implies $x(t) \to \infty$ for $t \to \infty$.

Only 1) is possible. With a starting point Q_1 belonging to U_{L_B} the orbit must reach some point $(\dot{x}_{12} > 0, \dot{\widetilde{p}}_{12} = 0) \in U_{L_B}$ when $t = t_{B_{12}}$. □

Using similar argumentation we get: The Euler-Lagrange orbit for L_B with a starting point in Q_2 belonging to U_{L_B} when $t = a$ will reach some point $(\dot{x}_{23} = 0, \dot{\widetilde{p}}_{23} < 0)$ when $t = t_{23}$. We obtain analogous results for Q_3 and Q_4. It follows that:

Proposition 3. *All the Euler-Lagrange orbits with starting points* $\in U_{L_B}$ *different from* $(\dot{x}_0 = 0, \dot{\widetilde{p}}_0 = 0)$, *reach all quadrants and passes both the negative and positive part of the* $\dot{x}$ *and* $\dot{\widetilde{p}}$ *axis.*

Analogously we obtain results following the Euler-Lagrange backward orbits.

4. Proof that the Euler-Lagrange curve is periodic.

4.1. **The situation A is tied to the situation B for each step.** We follow an Euler-Lagrange curve for the situation B. First, we follow one Euler-Lagrange curve $(\dot{x}_{B_1}(t), \dot{\widetilde{p}}_{B_1}(\dot{x}_{B_1}(t)))$ forwards in t at $(\dot{x} = 0, \dot{\widetilde{p}} = \dot{\widetilde{p}}_0 > 0)$ when $t = t_{B_{41}}$ and another Euler-Lagrange curve $(\dot{x}_{B_2}(t), \dot{\widetilde{p}}_{B_2}(\dot{x}_{B_2}(t)))$ backwards in t at $(\dot{x} = 0, \dot{\widetilde{p}} = -\dot{\widetilde{p}}_0 < 0)$ when $t = t_{B_{32}}$ in the $\dot{x}$ direction.

In the situation A, we also follow the Euler-Lagrange orbit forwards in t starting at $(\dot{x} = 0, \dot{\widetilde{p}} = \dot{\widetilde{p}}_0 > 0)$ when $t = t_{A_{41}} = 0$ and we follow the backward orbit in t starting at $(\dot{x} = 0, \dot{\widetilde{p}} = -\dot{\widetilde{p}}_0 < 0)$ when $t = t_{A_{32}} = \pi$ in the $\dot{x}$ direction.

Later we repeat the arguments following a backward orbit in t at $(\dot{x} = 0, \dot{\widetilde{p}} = \dot{\widetilde{p}}_0 > 0)$ and following a forwards orbit in t at $(\dot{x} = 0, \dot{\widetilde{p}} = -\dot{\widetilde{p}}_0 < 0)$ in the $-\dot{x}$ direction.

The figure below illustrates how we follow the Euler-Lagrange curve in t using backward orbits and forward orbits in our $A - B$ step by step orbit construction. The red arrow illustrates us following the backward orbits and the black arrow illustrates us following the forward orbits.

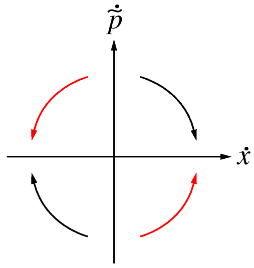

FIGURE 2. Backward and forward orbits.

4.2. **About the situation** A**.** An equation for the Lagrange function in the situation A is expressed as $L_A(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$ with $(x, \dot{x}) \in R^2$. We follow the orbit $(\dot{x}(t), \dot{\tilde{p}}(\dot{x}(t)))$ forwards in $t_{A_1} \in R$, starting at $t_{A_1} = 0$ at $(\dot{x}_{41} = \dot{x}_0 = 0, \dot{\tilde{p}}_{41} = \dot{\tilde{p}}_0 > 0)$ in the first quadrant. It is trivial to show that we obtain:

$$(\dot{x}(t), \dot{\tilde{p}}(\dot{x}(t))) = (\dot{\tilde{p}}_0 \sin(t), \dot{\tilde{p}}_0 \cos(t)). \tag{11}$$

As before we follow the orbit $(\dot{x}(t), \dot{\tilde{p}}(\dot{x}(t)))$ forwards in $t_{A_1} \in R$, starting at $t_{A_1} = 0$ at $(\dot{x}_{41} = \dot{x}_0 = 0, \dot{\tilde{p}}_{41} = \dot{\tilde{p}}_0 > 0)$ in the first quadrant. We also follow the backward orbit $(\dot{x}(t), \dot{\tilde{p}}(\dot{x}(t)))$ in $t_{A_2} \in R$, starting at $t_{A_2} = \pi$ at $(\dot{x}_{A_{32}} = 0, \dot{\tilde{p}}_{A_{32}} = -\dot{\tilde{p}}_0 < 0)$ in the second quadrant. It is trivial to show that we obtain:

$$t_{A_1} = \pi - t_{A_2} \Leftrightarrow \dot{x}_{A_1}(t_{A_1}) = \dot{x}_{A_2}(t_{A_2}) \wedge \dot{\tilde{p}}_{A_1}(t_{A_1}) = -\dot{\tilde{p}}_{A_2}(t_{A_2}). \tag{12}$$

$$\Delta\dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1}) = \Delta\dot{x}_{A_2}(t_{A_2}, \Delta t_{A_2}). \tag{13}$$

$$\Delta\dot{\tilde{p}}_{A_1}(t_{A_1}, \Delta t_{A_1}) = -\Delta\dot{\tilde{p}}_{A_2}(t_{A_2}, \Delta t_{A_2}). \tag{14}$$

4.3. **Assertions in the situation** A**.** We assert $t_{A_1} = \pi - t_{A_2}$ because we would like to have $\dot{x}_{A_1}(t_{A_1}) = \dot{x}_{A_2}(t_{A_2})$ and $\dot{\tilde{p}}_{A_1}(t_{A_1}) = -\dot{\tilde{p}}_{A_2}(t_{A_2})$. We also for each step assert $\Delta t_{A_1} = -\Delta t_{A_2}$ because we would like for each step to have $\Delta\dot{x}(t_{A_1}, \Delta t_{A_1}) = \Delta\dot{x}(t_{A_2}, \Delta t_{A_2})$ and $\Delta\dot{\tilde{p}}_{A_1}(t_{A_1}, \Delta t_{A_1}) = -\Delta\dot{\tilde{p}}_{A_2}(t_{A_2}, \Delta t_{A_2})$.

4.4. **About the situation** B**.** It is trivial to prove that:

Proposition 4. *The assertions in the situation A are fulfilled. For all $t_{A_1} \in [0, \frac{\pi}{2}]$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in [t_{B_{41}}, t_{B_{12}}]$ and $\Delta t_{B_2} \in [t_{B_{21}} - t_{B_2}, 0[$ such that:*

$$\dot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1})) = \dot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1})). \tag{15}$$

$$\Delta\dot{\widehat{p}}_{A_1}(x(t_{A_1}), \Delta t_{A_1}) = \Delta\dot{\widehat{p}}_{B_1}(x(t_{B_1}), \Delta t_{B_1})). \tag{16}$$

For all $t_{A_2} \in [\frac{\pi}{2}, \pi]$ and $\Delta t_{A_2} \in [\frac{\pi}{2} - t_{A_2}, 0[$ there exists one and only one $t_{B_2} \in [t_{B_{21}}, t_{B_{32}}]$ and $\Delta t_{B_2} \in [t_{B_{21}} - t_{B_2}, 0[$ such that:

$$\dot{\widehat{p}}_{A_2}(x_{A_2}(t_{A_2})) = \dot{\widehat{p}}_{B_2}(x_{B_2}(t_{B_2})).$$

$$\Delta\dot{\widehat{p}}_{A_2}(x(t_{A_2}), \Delta t_{A_2}) = \Delta\dot{\widehat{p}}_{B_2}(x(t_{B_2}), \Delta t_{B_2}).$$

Analogous arguments and results can be found following the backward orbit $(\dot{x}(t), \dot{\widetilde{p}}(\dot{x}(t)))$ in t_{B_4} at $(\dot{x}_{B_4} = 0, \dot{\widetilde{p}}_{B_4} = \dot{\widetilde{p}}_0)$ when $t_{B_4} = t_{B_{14}}$ and forwards in t_{B_3} at $(\dot{x}_{B_3} = 0, \dot{\widetilde{p}}_{B_3} = -\dot{\widetilde{p}}_0)$ when $t_{B_3} = t_{B_{23}}$ in the $-\dot{x}$ direction in the $(\dot{x}, \dot{\widetilde{p}})$ space. Figure 3 illustrate, a step in the situation A and B.

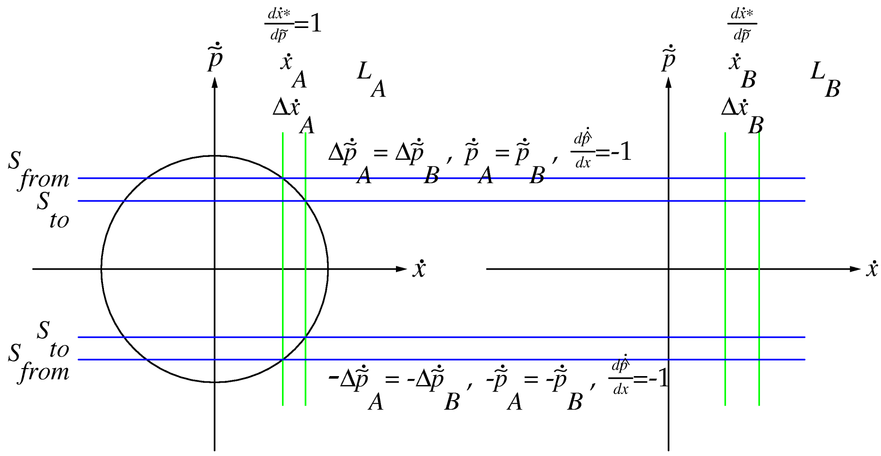

FIGURE 3. A step in the situation A and B.

4.5. **Assertions in the situation A to B.** We select suitable $t_{A_1}, t_{B_1}, t_{A_2}, t_{B_2}, \Delta t_{A_1}, \Delta t_{B_1}, \Delta t_{A_2}$ and Δt_{B_2} with $t_{A_1} = \pi - t_{A_2}$ to make some assertions. We assert:

$\dot{\widetilde{p}}_{A_1}(\dot{x}_{A_1}(t_{A_1})) = \dot{\widetilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))$.

$\dot{\widetilde{p}}_{A_2}(\dot{x}_{A_2}(t_{A_2})) = \dot{\widetilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))$.

For each step, we assert:

$\Delta\dot{\tilde{p}}_{A_1}(\dot{x}(t_{A_1}), \Delta t_{A_1}) = \Delta\dot{\tilde{p}}_{B_1}(\dot{x}(t_{B_1}), \Delta t_{B_1})$.
$\Delta\dot{\tilde{p}}_{A_2}(\dot{x}(t_{A_2}), \Delta t_{A_2}) = \Delta\dot{\tilde{p}}_{B_2}(\dot{x}(t_{B_2}), \Delta t_{B_2})$.
The assertions in section (4.3) for the situation A applies.

4.6. **Orbit construction.** Using the equation (9) and the equation (10) it is trivial to prove that:

Proposition 5. *The assertions in the situation A are fulfilled. We obtain*

$$-\dot{x}_{A_{12}}(t_{A_{12}}) \leqq \ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1})) < 0, \ \textit{for } t_{A_1} \in \left]0, \frac{\pi}{2}\right].$$

$$0 < \ddot{x}_{A_1}(t_{A_1}) \leqq \dot{\tilde{p}}_0, \ \textit{for } t_{A_1} \in \left[0, \frac{\pi}{2}\right[.$$

$$-\dot{x}_{B_{12}}(t_{B_{12}}) \leqq \ddot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1})) < 0, \ \textit{for } t_{B_1} \in \left]t_{B_{41}}, t_{B_{12}}\right].$$

$$0 < \ddot{x}_{B_1}(t_{B_1}) < \infty, \ \textit{for } t_{B_1} \in \left[t_{B_{41}}, t_{B_{12}}\right[.$$

We prove that our deltas are well behaved.

Proposition 6. *The assertions in the situation A to B are fulfilled. For* $t_{A_1} \in]0, \frac{\pi}{2}[$, $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}[$, $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ *and* $\Delta t_{B_1} \in]t_{B_{41}}, t_{B_{12}} - t_{B_1}[$, *we obtain:*

$$-\infty < \Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1})) < 0.$$

$$0 < \Delta\dot{x}_{A_1}(t_{A_1}) < \dot{\tilde{p}}_0 \Delta t_{A_1}.$$

$$-\infty < \Delta\dot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1})) < 0.$$

$$0 < \Delta\dot{x}_{B_1}(t_{B_1}) < \dot{\tilde{p}}_0 \Delta t_{B_1}.$$

Proof. We use proposition (5) $-\dot{x}_{A_{12}}(t_{A_{12}}) \leqq \ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1})) < 0$ for $t_{A_1} \in]0, \frac{\pi}{2}]$ where $-\dot{x}_{A_{12}}(t_{A_{12}}) = -\dot{x}_{A_{12}}(\frac{\pi}{2}) = -\dot{\tilde{p}}_0 \sin(\frac{\pi}{2}) = -\dot{\tilde{p}}_0$, see equation (11). We obtain $-\dot{x}_{A_{12}}(t_{A_{12}}) \leqq \ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1} + t_{\gamma A_1})) < 0$ for all $t_{A_1} + t_{\gamma A_1} \in]0, \frac{\pi}{2}]$. We will have that $t_{A_1} + t_{\gamma A_1} \in]0, \frac{\pi}{2}]$ when $t_{\gamma A_1} \in [0, \Delta t_{A_1}]$. We have that $\Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}), \Delta t_{A_1}) = \dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1} + \Delta t_{A_1})) - \dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}))$. With $-\dot{x}_{A_{12}}(t_{A_{12}}) < \ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1} + t_{\gamma A_1})) < 0$ for $t_{\gamma A_1} \in [0, \Delta t_{A_1}]$ will $-\infty < \dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1} + \Delta t_{A_1})) < \dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}))$. This tells us that $-\infty < \Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1})) < 0$ for $t_{A_1} \in]0, \frac{\pi}{2}]$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}[$.

We use proposition (5) $0 < \ddot{x}_{A_1}(t_{A_1}) \leqq \dot{\tilde{p}}_0$ for $t_{A_1} \in [0, \frac{\pi}{2}[$ and obtain $0 < \ddot{x}_{A_1}(t_{A_1} + t_{\gamma A_1}) < \dot{\tilde{p}}_0$ for $t_{A_1} + t_{\gamma A_1} \in [0, \frac{\pi}{2}[$. We will have that $t_{A_1} + t_{\gamma A_1} \in [0, \frac{\pi}{2}[$ when $t_{\gamma A_1} \in [0, \Delta t_{A_1}]$. We have that $\Delta\dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1}) = \dot{x}_{A_1}(t_{A_1} + \Delta t_{A_1}) - \dot{x}_{A_1}(t_{A_1})$ for $t_{A_1} \in [0, \frac{\pi}{2}[$. With $0 < \ddot{x}_{A_1}(t_{A_1} + t_{\gamma A_1}) \leqq \dot{\tilde{p}}$ for $t_{\gamma A_1} \in [0, \Delta t_{A_1}]$ will $\dot{x}_{A_1}(t_{A_1}) < \dot{x}_{A_1}(t_{A_1} + \Delta t_{A_1}) < \infty$. This tells us that $0 < \Delta\dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1}) < \infty$ for $t_{A_1} \in [0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}[$. Analogous results are obtained for the situation B. □

Obviously, for all $t_{A_1} \in [0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in [t_{B_{41}}, t_{B_{12}}[$ and $\Delta t_{B_1} \in]0, t_{B_{12}} - t_{B_1}]$ such that:

$$(17) \quad \frac{\Delta\dot{\widehat{p}}_{A_1}(\dot{x}(t_{A_1}), \Delta t_{A_1})}{\Delta\dot{\widehat{p}}_{B_1}(\dot{x}(t_{B_1}), \Delta t_{B_1})} = \frac{\dot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1} + \Delta t_{A_1})) - \dot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1}))}{\dot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1} + \Delta t_{B_1})) - \dot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1}))} = 1.$$

Notation 1. $\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}) = x_{B_1}^{-1}(\dot{\widehat{p}}_{B_1}^{-1}(\dot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1}+\Delta t_{A_1})))) - t_{B_1}$.

It is trivial to prove that:

Proposition 7. *The assertions in the situation A to B are fulfilled. For all $t_{A_1} \in [0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in [t_{B_{41}}, t_{B_{12}}[$ and $\Delta t_{B_1} \in]0, t_{B_{12}} - t_{B_1}]$ such that:*

$$(18) \qquad \Delta t_{B_1} = \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}).$$

$\psi_{A_1B_1}$ is a continuous differentiable function of t_{B_1}, t_{A_1} and Δt_{A_1}.

It is trivial to prove that $\lim\limits_{\Delta t_{A_1} \to 0} \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}) = 0$.

It is also trivial to prove that:

Proposition 8. *The assertions in the situation A to B are fulfilled. We obtain:*

$$\ddot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1})) = \lim_{\Delta t_{A_1} \to 0} \frac{\Delta\dot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1}), \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))}{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}.$$

Proposition 9. *The assertions in the situation A to B are fulfilled. For all $t_{A_1} \in]0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ such that:*

$$(19) \qquad 0 < \frac{\ddot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1}))}{\ddot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1}))} = \lim_{\Delta t_{A_1} \to 0} \frac{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}} < \infty.$$

Proof. By proposition (5) we have $-\dot{x}_{A_{12}}(t_{A_{12}}) \leqq \ddot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1})) < 0$, for $t_{A_1} \in]0, \frac{\pi}{2}]$ and we have $-\dot{x}_{B_{12}}(t_{B_{12}}) \leqq \ddot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1})) < 0$, for $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}]$.

$$\frac{\ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}))}{\ddot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1}))} = \frac{\lim\limits_{\Delta t_{A_1}\to 0} \frac{\Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}),\Delta t_{A_1})}{\Delta t_{A_1}}}{\lim\limits_{\Delta t_{B_1}\to 0} \frac{\Delta\dot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1}),\Delta t_{B_1})}{\Delta t_{B_1}}} \overset{(18)}{=}$$

$$= \frac{\lim\limits_{\Delta t_{A_1}\to 0} \frac{\Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}),\Delta t_{A_1})}{\Delta t_{A_1}}}{\lim\limits_{\Delta t_{A_1}\to 0} \frac{\Delta\dot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1}),\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))}{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}} =$$

$$= \lim_{\Delta t_{A_1}\to 0} \frac{\frac{\Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}),\Delta t_{A_1})}{\Delta t_{A_1}}}{\frac{\Delta\dot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1})\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))}{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}} =$$

$$= \lim_{\Delta t_{A_1}\to 0} \frac{\Delta\dot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}),\Delta t_{A_1})}{\Delta\dot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1}),\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))} \frac{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}{\Delta t_{A_1}} \overset{(17)}{=}$$

$$= \lim_{\Delta t_{A_1}\to 0} \frac{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}{\Delta t_{A_1}}.$$

This gives us for all $t_{A_1} \in]0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ such that:

$$\frac{\ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}))}{\ddot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1}))} = \lim_{\Delta t_{A_1}\to 0} \frac{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}{\Delta t_{A_1}}.$$

□

Proposition 10. *The assertions in the situation A to B are fulfilled. For all* $t_{A_1} \in]0, \frac{\pi}{2}[$ *and* $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ *there exists one and only one* $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ *such that:*

$$0 < \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})} = \lim_{\Delta t_{A_1}\to 0} \frac{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}{\Delta t_{A_1}} < \infty. \tag{20}$$

Proof. Since $0 < \dot{x}_{B_1}(t_{B_1}) < \infty$ when $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ and $0 < \dot{x}_{A_1}(t_{A_1}) < \infty$ when $t_{B_1} \in]0, \frac{\pi}{2}[$ because we are in the first quadrant. Using the Euler-Lagrange equations (9) and (10) we obtain:

$$\frac{\ddot{\hat{p}}_{A_1}(x_{A_1}(t_{A_1}))}{\ddot{\hat{p}}_{B_1}(x_{B_1}(t_{B_1}))} = \frac{-\dot{x}_{A_1}(t_{A_1})}{-\dot{x}_{B_1}(t_{B_1})} = \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})}$$

Using equation (19) we obtain for all $t_{A_1} \in]0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ such that for $\dot{\hat{p}}_{A_1}(x(t_{A_1})) =$

$\dot{\tilde{p}}_{B_1}(x(t_{B_1}))$:

$$\frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})} = \lim_{\Delta t_{A_1}\to 0} \frac{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}}.$$

□

Proposition 11. *The assertions in the situation A to B are fulfilled. For all* $t_{A_1} \in]0, \frac{\pi}{2}[$ *and* $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ *there exists one and only one* $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ *such that:*

$$0 < \frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})} = \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{A_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))} \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})} < \infty.$$

Proof. By proposition (5) we have $0 < \ddot{x}_{A_1}(t_{A_1}) \leqq \dot{\tilde{p}}_0$ for $t_{A_1} \in [0, \frac{\pi}{2}[$ and we have $0 < \ddot{x}_{B_1}(t_{B_1}) < \infty$ for $t_{B_1} \in [t_{B_{41}}, t_{B_{12}}[$.
We have that

$$\ddot{x}_{A_1}(t_{A_1}) = \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}}$$

and

$$\ddot{x}_{B_1}(t_{B_1}) = \lim_{\Delta t_{B_1}\to 0} \frac{\Delta \dot{x}_{B_1}(t_{B_1}, \Delta t_{B_1})}{\Delta t_{B_1}}.$$

$$\frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})} = \frac{\lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}}}{\lim_{\Delta t_{B_1}\to 0} \frac{\Delta \dot{x}_{B_1}(t_{B_1}, \Delta t_{B_1})}{\Delta t_{B_1}}} \stackrel{(18)}{=} \lim_{\Delta t_{A_1}\to 0} \frac{\frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}}}{\frac{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))}{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}} =$$
$$= \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))} \frac{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}}$$

[We obtain that $\lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))} \frac{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}}$ exists because $\frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})}$ exists because $\ddot{x}_{B_1}(t_{B_1}) > 0$ when $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$. We also obtain

$$\frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})} > 0 \Leftrightarrow \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))} \frac{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}{\Delta t_{A_1}} > 0.$$

Using equation (20) we obtain that $\lim_{\Delta t_{A_1}\to 0} \frac{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}{\Delta t_{A_1}}$ exists because $\frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})}$ exist. This tells us that $\lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))}$ exist.]

$$\lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))} \lim_{\Delta t_{A_1}\to 0} \frac{\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1})}{\Delta t_{A_1}} \overset{(20)}{=}$$
$$= \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})} \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{A_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))} =$$
$$= \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{A_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))} \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})}.$$

We obtain for all $t_{A_1} \in]0,\frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0,\frac{\pi}{2}-t_{A_1}]$ there exists one and only one $t_{B_1} \in]t_{B_{41}},t_{B_{12}}[$ such that:

$$\frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})} = \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{A_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))} \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})}.$$

□

Proposition 12. *The assertions in the situation A to B are fulfilled. For all* $t_{A_1} \in [0,\frac{\pi}{2}[$ *there exists one and only one* $t_{B_1} \in [t_{B_{41}},t_{B_{12}}[$ *such that:*

$$0 < \frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})} = \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1})) < \infty. \tag{21}$$

Proof. The proposition (11) and the equations (10), (9) and (15) gives us using the assertion in the situation *A to B* that for all $t_{A_1} \in [0,\frac{\pi}{2}[$ there exists one and only one $t_{B_1} \in [t_{B_{41}},t_{B_{12}}[$ such that:

$$\frac{\ddot{x}_{A_1}(t_{A_1})}{\ddot{x}_{B_1}(t_{B_1})} = \frac{\dot{\widehat{p}}_{A_1}(x_{A_1}(t_{A_1}))}{\dot{\widehat{p}}_{B_1}(x_{B_1}(t_{B_1}))\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}_{B_1}(\dot{x}_{B_1}(t_{B_1})))} =$$
$$= \frac{1}{\frac{d\dot{x}^*}{d\widetilde{p}}(\widetilde{p}_{B_1}(\dot{x}_{B_1}(t_{B_1})))} = \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1})).$$

□

Proposition 13. *The assertions in the situation A to B are fulfilled. For all* $t_{A_1} \in]0,\frac{\pi}{2}[$ *and* $\Delta t_{A_1} \in]0,\frac{\pi}{2}-t_{A_1}]$ *there exists one and only one* $t_{B_1} \in]t_{B_{41}},t_{B_{12}}[$ *such that:*

$$\lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))} = 1. \tag{22}$$

Proof. We obtain from proposition (11) and equation (21):

$$\lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{A_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))} \frac{\dot{x}_{A_1}(t_{A_1})}{\dot{x}_{B_1}(t_{B_1})} = \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1})).$$

This equation gives us for all $t_{A_1} \in]0, \frac{\pi}{2}[$ and $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ there exists one and only one $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ such that:

$$\lim_{\Delta t_{A_1} \to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1 B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))} = \frac{\dot{x}_B(t_B)}{\dot{x}_A(t_A)} \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1})) \Leftrightarrow$$
$$\Leftrightarrow \lim_{\Delta t_{A_1} \to 0} \frac{\Delta \dot{x}_{A_1}(t_{A_1}, \Delta t_{A_1}) \dot{x}_{A_1}(t_{A_1})}{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1 B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})) \dot{x}_{B_1}(t_{B_1}) \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))} = 1.$$

□

Analogously to the notation of $\psi_{A_1 B_1}$ we have:

Notation 2. $\psi_{A_2 B_2}(t_{B_2}, t_{A_2}, \Delta t_{A_2}) = x_{B_2}^{-1}(\dot{\widehat{p}}_{B_2}^{-1}(\dot{\widehat{p}}_{A_2}(x_{A_2}(t_{A_2} + \Delta t_{A_2})))) - t_{B_2}$.

Analogously to proposition (7) we obtain:

Proposition 14. *The assertions in the situation A to B are fulfilled. For all* $t_{A_2} \in]\frac{\pi}{2}, \pi]$ *and* $\Delta t_{A_2} \in [\frac{\pi}{2} - t_{A_2}, 0[$ *there exists one and only one* $t_{B_2} \in]t_{B_{21}}, t_{B_{32}}]$ *and* $\Delta t_{B_2} \in [t_{B_{21}} - t_{B_2}, 0[$ *such that:*

$$\Delta t_{B_2} = \psi_{A_2 B_2}(t_{B_2}, t_{A_2}, \Delta t_{A_2}).$$

$\psi_{A_2 B_2}$ *is a continuous differentiable function of* t_{B_2}, t_{A_2} *and* Δt_{A_2}.

$$\Delta t_{B_2} = \psi_{A_2 B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1}).$$

Analogously to proposition (13) we obtain:

Proposition 15. *The assertions in the situation A to B are fulfilled. For all* $t_{A_2} \in]\frac{\pi}{2}, \pi[$ *and* $\Delta t_{A_2} \in [\frac{\pi}{2} - t_{A_2}, 0[$ *there exists one and only one* $t_{B_2} \in]t_{B_{21}}, t_{B_{32}}[$ *such that:*

$$\lim_{\Delta t_{A_1} \to 0} \frac{\Delta \dot{x}_{A_2}(t_{A_2}, -\Delta t_{A_1}) \dot{x}_{A_2}(t_{A_2})}{\Delta \dot{x}_{B_2}(t_{B_2}, \psi_{A_2 B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1})) \dot{x}_{B_2}(t_{B_2}) \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))} = 1. \tag{23}$$

Proposition 16. *The assertions in the situation A to B are fulfilled. For all* $t_{A_1} \in]0, \frac{\pi}{2}[$ *and* $\Delta t_{A_1} \in]0, \frac{\pi}{2} - t_{A_1}]$ *there exists one and only one* $t_{B_1} \in]t_{B_{41}}, t_{B_{12}}[$ *and* $t_{B_2} \in]t_{B_{21}}, t_{B_{32}}[$ *such that:*

$$\lim_{\Delta t_{A_1} \to 0} \frac{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1 B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})) \dot{x}_{B_1}(t_{B_1}) \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta \dot{x}_{B_2}(t_{B_2}, \psi_{A_2 B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1})) \dot{x}_{B_2}(t_{B_2}) \frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))} = 1.$$

Proof. The equations (22) and (23) are combined to:

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}=$$
$$=\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}\Leftrightarrow$$
$$\Leftrightarrow\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}-$$
$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}=0\Leftrightarrow$$
$$\Leftrightarrow\lim_{\Delta t_{A_1}\to 0}\left(\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}-\right.$$
$$\left.\frac{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}\right)=0\Leftrightarrow$$

[We have

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}=$$
$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}=1,$$

because we have the assertions $\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})=\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})$ and $\dot{x}_{A_1}(t_{A_1})=\dot{x}_{A_2}(t_{A_2})$.]

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}\cdot$$
$$\lim_{\Delta t_{A_1}\to 0}\left(\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}-\right.$$
$$\left.\frac{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}\right)=0\Leftrightarrow$$
$$\Leftrightarrow\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}\cdot$$
$$\left(\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}-\right.$$
$$\left.\frac{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\widetilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}\right)=0\Leftrightarrow$$

$$\Leftrightarrow \lim_{\Delta t_{A_1}\to 0}\left(\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}-\right.$$

$$\left.\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}\right)=0\Leftrightarrow$$

[We obtain $\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}=1$, because we have the assertions $\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})=\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})$ and $\dot{x}_{A_1}(t_{A_1})=\dot{x}_{A_2}(t_{A_2})$.]

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{A_1}(t_{A_1},\Delta t_{A_1})\dot{x}_{A_1}(t_{A_1})}{\Delta\dot{x}_{A_2}(t_{A_2},-\Delta t_{A_1})\dot{x}_{A_2}(t_{A_2})}=$$

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}\Leftrightarrow$$

For all $t_{A_1}\in]0,\frac{\pi}{2}[$ and $\Delta t_{A_1}\in]0,\frac{\pi}{2}-t_{A_1}]$ there exists one and only one $t_{B_1}\in]t_{B_{41}},t_{B_{12}}[$ and $t_{B_2}\in]t_{B_{21}},t_{B_{32}}[$ such that:

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}=1. \tag{24}$$

□

Notation 3.

$$\Psi_{B_2B_1}(t_{B_2})=x_{B_1}^{-1}(\dot{\tilde{p}}_{B_1}^{-1}(\dot{\tilde{p}}_{A_1}(x_{A_1}(\pi-x_{A_2}^{-1}(\dot{\tilde{p}}_{A_2}^{-1}(\dot{\tilde{p}}_{B_2}(x_{B_2}(t_{B_2}))))))))). \tag{25}$$

It is trivial to prove that $\Psi_{B_2B_1}(t_{B_2})$ is a continuous differential function of t_{B_2}.

Proposition 17. *The assertions in the situation A to B are fulfilled. We have an Euler-Lagrange orbit starting at the point ($\dot{x}_{B_{41}}=\dot{x}_0=0,\dot{\tilde{p}}_{B_{41}}=\dot{\tilde{p}}_0>0$) at time $t_{B_{41}}$ and moving through the first quadrant ending at the point ($\dot{x}_{B_{12}},\dot{\tilde{p}}_{B_{12}}=0$) at time $t_{B_{12}}$. $\dot{x}_{B_{12}}$ is the $\dot{x}$ coordinate of the attained endpoint. We have a backward Euler-Lagrange orbit starting at the point ($\dot{x}_{B_{32}}=0,\dot{\tilde{p}}_{B_{32}}=-\dot{\tilde{p}}_0<0$) at time $t_{B_{32}}$ and moving through the second quadrant ending at the point ($\dot{x}_{B_{21}},\dot{\tilde{p}}_{B_{21}}=0$) at time $t_{B_{21}}$. $\dot{x}_{B_{21}}$ is the $\dot{x}$ coordinate of the attained endpoint. We obtain:*

$$\dot{x}_{B_{21}}=\dot{x}_{B_{12}}.$$

Proof. We use equation (24): For all $t_{A_1}\in]0,\frac{\pi}{2}[$ and $\Delta t_{A_1}\in]0,\frac{\pi}{2}-t_{A_1}]$ there exists one and only one $t_{B_1}\in]t_{B_{41}},t_{B_{12}}[$ and $t_{B_2}\in]t_{B_{21}},t_{B_{32}}[$ such that:

$$\lim_{\Delta t_{A_1}\to 0}\frac{\Delta\dot{x}_{B_1}(t_{B_1},\psi_{A_1B_1}(t_{B_1},t_{A_1},\Delta t_{A_1}))\dot{x}_{B_1}(t_{B_1})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta\dot{x}_{B_2}(t_{B_2},\psi_{A_2B_2}(t_{B_2},t_{A_2},-\Delta t_{A_1}))\dot{x}_{B_2}(t_{B_2})\frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))}=1\Leftrightarrow$$

$$\Leftrightarrow \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))}{\Delta \dot{x}_{B_2}(t_{B_2}, \psi_{A_2B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1}))} \frac{\Delta t_{B_2}}{\Delta t_{B_1}} \frac{\Delta t_{B_1}}{\Delta t_{B_2}} \frac{\dot{x}_{B_1}(t_{B_1}) \frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\dot{x}_{B_2}(t_{B_2}) \frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))} = 1 \Leftrightarrow$$

$$\Leftrightarrow \lim_{\Delta t_{A_1}\to 0} \frac{\frac{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))}{\Delta t_{B_1}}}{\frac{\Delta \dot{x}_{B_2}(t_{B_2}, \psi_{A_2B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1}))}{\Delta t_{B_2}}} \frac{\Delta t_{B_1}}{\Delta t_{B_2}} \frac{\dot{x}_{B_1}(t_{B_1}) \frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\dot{x}_{B_2}(t_{B_2}) \frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))} = 1 \Leftrightarrow$$

[We use $\ddot{x}_{B_1}(t_{B_1}) = \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{B_1}(t_{B_1}, \psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1}))}{\psi_{A_1B_1}(t_{B_1}, t_{A_1}, \Delta t_{A_1})}$ and $\ddot{x}_{B_2}(t_{B_2}) = \lim_{\Delta t_{A_1}\to 0} \frac{\Delta \dot{x}_{B_2}(t_{B_2}, \psi_{A_2B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1}))}{\psi_{A_2B_2}(t_{B_2}, t_{A_2}, -\Delta t_{A_1})}$.]

$$\Leftrightarrow \lim_{\Delta t_{A_1}\to 0} \frac{\Delta t_{B_1} \ddot{x}_{B_1}(t_{B_1},) \dot{x}_{B_1}(t_{B_1}) \frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))}{\Delta t_{B_2} \ddot{x}_{B_2}(t_{B_2}) \dot{x}_{B_2}(t_{B_2}) \frac{d\tilde{p}}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))} = 1 \Leftrightarrow$$

$$\Leftrightarrow \lim_{\Delta t_{A_1}\to 0} \frac{\Delta t_{B_1} \dot{x}_{B_1}(t_{B_1}) \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))}{\Delta t_{B_2} \dot{x}_{B_2}(t_{B_2}) \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))} = 1 \Leftrightarrow$$

$$\Leftrightarrow \lim_{\Delta t_{A_1}\to 0} \frac{\Delta t_{B_1}}{\Delta t_{B_2}} = \frac{\dot{x}_{B_2}(t_{B_2}) \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))}{\dot{x}_{B_1}(t_{B_1}) \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))} \Leftrightarrow$$

$$\Leftrightarrow \frac{dt_{B_1}}{dt_{B_2}} = \frac{\dot{x}_{B_2}(t_{B_2}) \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))}{\dot{x}_{B_1}(t_{B_1}) \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))} \Leftrightarrow$$

$$\Leftrightarrow \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1})) \frac{dt_{B_1}}{dt_{B_2}} \dot{x}_{B_1}(t_{B_1}) = \dot{x}_{B_2}(t_{B_2}) \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2})) \Leftrightarrow$$

$$\Leftrightarrow \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2}))) \frac{dt_{B_1}}{dt_{B_2}} \dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2})) = \dot{x}_{B_2}(t_{B_2}) \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))$$

[Notice that the integration follows the backward orbit.]

$$\int_{t_{B_{32}}}^{t_{B_{21}}} \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2}))) \frac{dt_{B_1}}{dt_{B_2}} \dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2})) dt_{B_2} =$$

$$= \int_{t_{B_{32}}}^{t_{B_{21}}} \dot{x}_{B_2}(t_{B_2}) \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2})) dt_{B_2} \Leftrightarrow$$

$$\Leftrightarrow \int_{t_{B_{32}}}^{t_{B_{21}}} \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2}))) \dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2})) \frac{d\Psi_{B_2B_1}(t_{B_2})}{dt_{B_2}} dt_{B_2} =$$

$$= \int_{t_{B_{32}}}^{t_{B_{21}}} \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2})) \dot{x}_{B_2}(t_{B_2}) dt_{B_2} \Leftrightarrow$$

[We use integration by substitution on the left side of the equation given by:

$$\int_a^b f(k(t_{B_2})) \frac{dk(t_{B_2})}{t_{B_2}} dt_{B_2} = \int_{k(a)}^{k(b)} f(t_{B_1}) dt_{B_1},$$

in our case with:
$f(k(t_{B_2})) = \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2}))) \dot{x}_{B_1}(\Psi_{B_2B_1}(t_{B_2}))$, $\frac{dk(t_{B_2})}{dt_{B_2}} = \frac{d\Psi_{B_2B_1}(t_{B_2})}{dt_{B_2}}$, $k(t_{B_2}) = \Psi_{B_2B_1}(t_{B_2}), f(t_{B_1}) = \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1})) \dot{x}_{B_1}(t_{B_1}), a = t_{B_{32}}, b = t_{B_{21}}$, $g(a) = \Psi_{B_2B_1}(t_{B_{32}})$ and $g(b) = \Psi_{B_2B_1}(t_{B_{21}})$.

Integration by substitution requires $k(t_{B_2}) = \Psi_{B_2B_1}(t_{B_2})$ to be a continuously differentiable function and that $k(t_{B_2})$ has an inverse function. $k(t_{B_2})$ is continuously differentiable. $k(t_{B_2})$ has an inverse function because $\Psi_{B_2B_1}(t_{B_2})$ is strictly decreasing. This is seen by inspecting the equation (25) for $\Psi_{B_2B_1}(t_{B_2})$. Integration by substitution also requires $f(t_{B_1}) = \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))\dot{x}_{B_1}(t_{B_1})$ to be an integrable function. This is the case for $f(t_{B_1})$ because it is a continuous function, as is seen in section (3.2).]

$$\int_{\Psi_{B_2B_1}(t_{B_{32}})}^{\Psi_{B_2B_1}(t_{B_{21}})} \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))\dot{x}_{B_1}(t_{B_1})dt_{B_1} = \int_{t_{B_{32}}}^{t_{B_{21}}} \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))\dot{x}_{B_2}(t_{B_2})dt_{B_2} \Leftrightarrow$$

[We use equation (25) to obtain:

$$\Psi_{B_2B_1}(t_{B_{32}}) = x_{B_1}^{-1}(\dot{\hat{p}}_{B_1}^{-1}(\dot{\hat{p}}_{A_1}(x_{A_1}(\pi - x_{A_2}^{-1}(\dot{\hat{p}}_{A_2}^{-1}(\dot{\hat{p}}_{B_2}(x_{B_2}(t_{B_2})))))))) = t_{B_{41}}$$

and

$$\Psi_{B_2B_1}(t_{B_{21}}) = x_{B_1}^{-1}(\dot{\hat{p}}_{B_1}^{-1}(\dot{\hat{p}}_{A_1}(x_{A_1}(\pi - x_{A_2}^{-1}(\dot{\hat{p}}_{A_2}^{-1}(\dot{\hat{p}}_{B_2}(x_{B_2}(t_{B_2})))))))) = t_{B_{12}}].$$

$$\Leftrightarrow \int_{t_{B_{41}}}^{t_{B_{12}}} \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))\dot{x}_{B_1}(t_{B_1})dt_{B_1} = \int_{t_{B_{32}}}^{t_{B_{21}}} \dot{\tilde{p}}_{B_2}(\dot{x}_{B_2}(t_{B_2}))\dot{x}_{B_2}(t_{B_2})dt_{B_2} \Leftrightarrow$$

[We use partial integration on both sides of the equation, given by:

$$\int_a^b f(t_{B_1})k(t_{B_1})dt_{B_1} = [k(t_{B_1})F(t_{B_1})]_a^b - \int_a^b F(t_{B_1})\frac{dk}{dt_{B_1}}(t_{B_1})dt_{B_1},$$

in our case, for the left side of the equation, with: $f(t_{B_1}) = \dot{\tilde{p}}_{B_1}(\dot{x}_{B_1}(t_{B_1}))$, $k(t_{B_1}) = \dot{x}_{B_1}(t_{B_1})$, $F(t_{B_1}) = \tilde{p}_{B_1}(\dot{x}_{B_1}(t_{B_1}))$ and $\frac{dk}{dt_{B_1}}(t_{B_1}) = \ddot{x}_{B_1}(t_{B_1})$. Partial integration requires that $F(t_{B_1}) = \tilde{p}_{B_1}(\dot{x}_{B_1}(t_{B_1}))$ and $k(t_{B_1}) = \dot{x}_{B_1}(t_{B_1})$ are continuously differentiable functions, this is seen in section (3.2).]

$$\Leftrightarrow [\tilde{p}_{B_1}(\dot{x}_{B_1}(t_{B_1}))\dot{x}_{B_1}(t_{B_1})]_{t_{B_{41}}}^{t_{B_{12}}} - \int_{t_{B_{41}}}^{t_{B_{12}}} \tilde{p}_{B_1}(\dot{x}_{B_1}(t_{B_1}))\ddot{x}_{B_1}(t_{B_1})dt_{B_1} =$$

$$= [\tilde{p}_{B_2}(\dot{x}_{B_2}(t_{B_2}))\dot{x}_{B_2}(t_{B_2})]_{t_{B_{32}}}^{t_{B_{21}}} - \int_{t_{B_{32}}}^{t_{B_{21}}} \tilde{p}_{B_2}(\dot{x}_{B_2}(t_{B_2}))\ddot{x}_{B_2}(t_{B_2})dt_{B_2} \Leftrightarrow$$

$$\Leftrightarrow [\frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))\dot{x}_{B_1}(t_{B_1})]_{t_{B_{41}}}^{t_{B_{12}}} - \int_{t_{B_{41}}}^{t_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))\ddot{x}_{B_1}(t_{B_1})dt_{B_1} =$$

$$= [\frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))\dot{x}_{B_2}(t_{B_2})]_{t_{B_{32}}}^{t_{B_{21}}} - \int_{t_{B_{32}}}^{t_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))\ddot{x}_{B_2}(t_{B_2})dt_{B_2} \Leftrightarrow$$

$$\Leftrightarrow \int_{t_{B_{41}}}^{t_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1}))\ddot{x}_{B_1}(t_{B_1})dt_{B_1} - \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_{12}}))\dot{x}_{B_1}(t_{B_{12}}) =$$

$$= \int_{t_{B_{32}}}^{t_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_2}))\ddot{x}_{B_2}(t_{B_2})dt_{B_2} - \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_{21}}))\dot{x}_{B_2}(t_{B_{21}}) \Leftrightarrow$$

[We use integration by substitution, on both sides of the equation, in our case, for the left side of the equation: $f(k(t_{B_1})) = \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_1})), \frac{dk(t_{B_1})}{dt_{B_1}} = \ddot{x}_{B_1}(t_{B_1}), k(t_{B_1}) = \dot{x}_{B_1}(t_{B_1}), f(\dot{x}_{B_1}) = \frac{dk}{d\dot{x}}(\dot{x}_{B_1}), a = t_{B_{41}}, b = t_{B_{12}}, k(a) = \dot{x}_{B_1}(t_{B_{41}}), k(b) = \dot{x}_{B_1}(t_{B_{12}})$.
Integration by substitution requires $k(t_{B_1}) = \dot{x}_{B_1}(t_{B_1})$ to be a continuously differentiable function and that $k(t_{B_1})$ has an inverse function. $k(t_{B_1})$ is continuously differentiable. This is seen in section (3.2). $k(t_{B_1})$ has an inverse function because $\dot{x}_{B_1}(t_{B_1})$ is strictly increasing. Integration by substitution also requires $f(\dot{x}_{B_1}) = \frac{dg}{d\dot{x}}(\dot{x}_{B_1})$ to be an integrable function. This is the case for $f(\dot{x}_{B_1})$ because it is a continuous function, as is also seen in section (3.2). Analogous considerations apply for the right side of the equation.]

$$\int_{\dot{x}_{B_1}(t_{B_{41}})}^{\dot{x}_{B_1}(t_{B_{12}})} \frac{dg}{d\dot{x}}(\dot{x}_{B_1})d\dot{x}_{B_1} - \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_{12}}))\dot{x}_{B_1}(t_{B_{12}}) =$$

$$= \int_{\dot{x}_{B_2}(t_{B_{32}})}^{\dot{x}_{B_2}(t_{B_{21}})} \frac{dg}{d\dot{x}}(\dot{x}_{B_2})d\dot{x}_{B_2} - \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_{21}}))\dot{x}_{B_2}(t_{B_{21}}) \Leftrightarrow$$

$$\Leftrightarrow g(\dot{x}_{B_1}(t_{B_{12}})) - g(\dot{x}_{B_1}(t_{B_{41}})) - \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_{12}}))\dot{x}_{B_1}(t_{B_{12}}) =$$

$$= g(\dot{x}_{B_2}(t_{B_{21}})) - g(\dot{x}_{B_2}(t_{B_{32}})) - \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_{21}}))\dot{x}_{B_2}(t_{B_{21}}) \Leftrightarrow$$

$$\Leftrightarrow g(\dot{x}_{B_1}(t_{B_{12}})) - g(0) - \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_{12}}))\dot{x}_{B_1}(t_{B_{12}}) =$$

$$= g(\dot{x}_{B_2}(t_{B_{21}})) - g(0) - \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_{21}}))\dot{x}_{B_2}(t_{B_{21}}) \Leftrightarrow$$

$$\Leftrightarrow g(\dot{x}_{B_1}(t_{B_{12}})) - \frac{dg}{d\dot{x}}(\dot{x}_{B_1}(t_{B_{12}}))\dot{x}_{B_1}(t_{B_{12}}) =$$

$$= g(\dot{x}_{B_2}(t_{B_{21}})) - \frac{dg}{d\dot{x}}(\dot{x}_{B_2}(t_{B_{21}}))\dot{x}_{B_2}(t_{B_{21}}) \Leftrightarrow$$

$$\Leftrightarrow g(\dot{x}_{B_{12}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})\dot{x}_{B_{12}} = g(\dot{x}_{B_{21}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})\dot{x}_{B_{21}} \Leftrightarrow$$

[We use

$$\int_0^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x})d\dot{x} = [g(x)]_0^{\dot{x}_{B_{12}}} = g(\dot{x}_{B_{12}}) - g(0) = g(\dot{x}_{B_{12}})$$

and

$$\int_0^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x})d\dot{x} = [g(x)]_0^{\dot{x}_{B_{21}}} = g(\dot{x}_{B_{21}}) - g(0) = g(\dot{x}_{B_{21}}).]$$

$$\int_0^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x})d\dot{x} - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})\dot{x}_{B_{12}} = \int_0^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x})d\dot{x} - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})\dot{x}_{B_{21}} \Leftrightarrow$$

[We use

$$\int_0^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})d\dot{x} = \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) \int_0^{\dot{x}_{B_{12}}} d\dot{x} = \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})[x]_0^{\dot{x}_{B_{12}}} =$$
$$= \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})(\dot{x}_{B_{12}} - 0) = \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})\dot{x}_{B_{12}}$$

and

$$\int_0^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})d\dot{x} = \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) \int_0^{\dot{x}_{B_{21}}} d\dot{x} = \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})[x]_0^{\dot{x}_{B_{21}}} =$$
$$= \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})(\dot{x}_{B_{12}} - 0) = \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})\dot{x}_{B_{12}}].$$

$$\int_0^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x})d\dot{x} - \int_0^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})d\dot{x} = \int_0^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x})d\dot{x} - \int_0^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})d\dot{x} \Leftrightarrow$$
$$\Leftrightarrow \int_0^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} = \int_0^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} \Leftrightarrow$$

[According to the assumption (3) is $0 < \frac{d^2g}{d\dot{x}d\dot{x}}(\dot{x})$, this tells us that $\frac{dg}{d\dot{x}}(\dot{x})$ is a strictly increasing function. According to proposition (1) is $g_{\min}(\dot{x}) = 0$ for $\dot{x} = 0$ this tells us that $\frac{dg}{d\dot{x}}(\dot{x}) = 0$ at $\dot{x} = 0$. This tells us that $0 < \frac{dg}{d\dot{x}}(\dot{x}) < \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})$ for $\dot{x} \in]0, \dot{x}_{B_{12}}[$ and $0 < \frac{dg}{d\dot{x}}(\dot{x}) < \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})$ for $\dot{x} \in]0, \dot{x}_{B_{21}}[$. This tells us that $\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) < 0$ for $\dot{x} \in]0, \dot{x}_{B_{12}}[$ and $\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) < 0$ for $\dot{x} \in]0, \dot{x}_{B_{21}}[$.]
We first assume that $0 < \dot{x}_{B_{12}} < \dot{x}_{B_{21}}$.

$$\int_0^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} = \int_0^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} \Leftrightarrow$$
$$\Leftrightarrow \int_0^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} =$$
$$= \int_0^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} + \int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} \Leftrightarrow$$
$$\Leftrightarrow \int_0^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} - \int_0^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} =$$

$$= \int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow -\int_{0}^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})d\dot{x} + \int_{0}^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})d\dot{x} = \int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow \int_{0}^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})d\dot{x} - \int_{0}^{\dot{x}_{B_{12}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})d\dot{x} = \int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow \int_{0}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} = \int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x}$$

We know that $\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) < 0$ for $\dot{x} \in]0, \dot{x}_{B_{21}}[$. This tells us that $\int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} < 0$. $\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) < 0$ for $\dot{x} \in]0, \dot{x}_{B_{21}}[$ tells us that $\frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) - \frac{dg}{d\dot{x}}(\dot{x}) > 0$ for $\dot{x} \in]0, \dot{x}_{B_{21}}[$. This and $0 < \dot{x}_{B_{12}} < \dot{x}_{B_{21}}$ tells us that $\frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) > 0$, this tells us that $\int_{0}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} > 0$.
This contradicts with

$$\int_{0}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} = \int_{\dot{x}_{B_{12}}}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x}.$$

This tells us that it is impossible that $0 < \dot{x}_{B_{12}} < \dot{x}_{B_{21}}$.
We then assume that $0 < \dot{x}_{B_{21}} < \dot{x}_{B_{12}}$.

$$\int_{0}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} = \int_{0}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow \int_{0}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} =$$

$$= \int_{0}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} + \int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow \int_{0}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} - \int_{0}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} =$$

$$= \int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow -\int_{0}^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})d\dot{x} + \int_{0}^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})d\dot{x} = \int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow \int_{0}^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}})d\dot{x} - \int_{0}^{\dot{x}_{B_{21}}} \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}})d\dot{x} = \int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} \Leftrightarrow$$

$$\Leftrightarrow \int_{0}^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} = \int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x}$$

We know that $\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) < 0$ for $\dot{x} \in]0, \dot{x}_{B_{12}}[$. This tells us that $\int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x} < 0$.
$\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) < 0$ for $\dot{x} \in]0, \dot{x}_{B_{12}}[$ tells us that $\frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) - \frac{dg}{d\dot{x}}(\dot{x}) > 0$ for $\dot{x} \in]0, \dot{x}_{B_{12}}[$. This and $0 < \dot{x}_{B_{21}} < \dot{x}_{B_{12}}$ tells us that $\int_0^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} > 0$.
This contradicts with

$$\int_0^{\dot{x}_{B_{21}}} (\frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{21}}))d\dot{x} = \int_{\dot{x}_{B_{21}}}^{\dot{x}_{B_{12}}} (\frac{dg}{d\dot{x}}(\dot{x}) - \frac{dg}{d\dot{x}}(\dot{x}_{B_{12}}))d\dot{x}.$$

This tells us that it is impossible that $0 < \dot{x}_{B_{21}} < \dot{x}_{B_{12}}$. We need $\dot{x}_{B_{21}} = \dot{x}_{B_{12}}$.]

$$\Leftrightarrow \dot{x}_{B_{21}} = \dot{x}_{B_{12}}.$$

□

4.6.1. *What $\dot{x}_{B_{21}} = \dot{x}_{B_{12}}$ gives us.* Analogously to $\dot{x}_{B_{21}} = \dot{x}_{B_{12}}$ we obtain that $\dot{x}_{B_{23}} = \dot{x}_{B_{32}}$. The orbits in all four quadrants are connected.

For some $T_B = t_{end} - t_{start} > 0$ and $a \in R$, we obtain $\dot{x}_B(a) = \dot{x}_B(a+T_B)$ and $\dot{\tilde{p}}_B(\dot{x}_B(a)) = \dot{\tilde{p}}_B(\dot{x}_B(a+T_B))$.

Along an Euler-Lagrange curve, we obtain: $\dot{\tilde{p}}_B(\dot{x}_B(a)) = \dot{\tilde{p}}_B(\dot{x}_B(a + T_B)) \Leftrightarrow \dot{\hat{p}}_B\,(x_B(a)) = \dot{\hat{p}}_B\,(x_B(a+T_B)) \Leftrightarrow x_B(a) = x_B(a+T_B)$.

Theorem 2. *Let L_B be the Lagrange function: $L_B\,(x,\dot{x}) = g(\dot{x}) - \frac{1}{2}x^2$ for $L_B\,(x,\dot{x}) \in R^2$ fulfilling our assumptions. Then are all the Euler-Lagrange solutions periodic.*

Proof. We have $\dot{x}_B(a) = \dot{x}_B(a+T_B)$ and $x_B(a) = x_B(a+T_B)$ for $a \in R$. That proves to us that the Euler-Lagrange equation (10):

$$\begin{pmatrix} \ddot{\tilde{p}}_B(\dot{x}_B(t)) \\ \ddot{x}_B(t) \end{pmatrix} = \begin{pmatrix} -\dot{x}_B(t) \\ \dot{\hat{p}}_B(x(t))\frac{d\dot{x}^*}{d\tilde{p}}(\tilde{p}(\dot{x}(t))) \end{pmatrix},$$

for L_B along an Euler-Lagrange curve, has periodic solutions in $x(t)$ and $\dot{x}(t)$. □

References

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