Folia Mathematica
Vol. 16, No. 1, pp. 45–55

Acta Universitatis Lodziensis

CONDITIONS OF SEPARATION FOR QUASI-PSEUDOMETRICS

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Abstract. For a quasi-pseudometric d on X and for $A \subseteq X$, the concept of an A-modification of d is defined. Conditions of separation from T_0 to $T_{3\frac{1}{2}}$ for d and its A-modifications are investigated.

1. Preliminaries

1.1. **Axioms of set theory.** First, to prove a theorem of a theory, we need to establish the set of axioms of this theory. For simplicity, let us assume the version of ZF studied in [9], although all the axioms of ZF are only hypothetical assumptions and they are not statements that assert absolute truth. Moreover, it seems that there are not enough axioms of ZF to give complete proofs to the theorems of this paper. Therefore, we assume a "suitable interpretation" of ZF.

Let us recall that an *ordinal number* in the sense of Zermelo-von Neumann is a set α such that each element of α is a subset of α, every non-empty subset B of α has an element disjoint from B and, for any two different elements x, y of α, either $x \in y$ or $y \in x$ (cf. [1]). We will use exactly this notion of an ordinal number partly in order to show that the axiom of foundation introduced by von Neumann and Zermelo, included in ZF, is not needed to prove the theorems of this article.

As usual, the symbol ω stands for the collection of all finite ordinal numbers of Zermelo-von Neumann. In ZF, the collection ω both exists and is a set of elements, while in $ZF - Inf$ (cf. [9]), it is unprovable that ω is a set even if one assumes that ω exists.

Let $\mathbb{R}$ be a fixed field of all real numbers in the sense of Hilbert (cf.[6], [7]) such that ω is the set of all non-negative integers of $\mathbb{R}$. No replacement

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Key words and phrases: quasi-pseudometrizable topologies, A-modifications of quasi-pseudometrics, separation properties, bitopological spaces, axioms of set theory.

AMS subject classifications: Primary: 54D10, 54D15, 54E55. Secondary: 03E30, 03E10.

scheme is needed to prove that such a field $\mathbb{R}$ exists in ZF. All intervals of real numbers considered below will be in this fixed $\mathbb{R}$.

To summarize, in the present work, instead of the whole ZF, it is better to assume ZF with the axioms of foundation and replacement deleted. A modification of ZF which leads to more precise results than ZF is discussed in [21].

1.2. **Introduction to quasi-pseudometrics.** Mimicking Wilson [22], Kelly [8], Fletcher-Lindgren [4], Künzi and others (cf. e.g. [10]), by a *quasi-pseudometric* on a set X let us understand a function $d : X \times X \to [0; +\infty)$ such that, for all x, y, z in X, the following conditions hold:

(i) $d(x, x) = 0$;
(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

If, in addition, d satifsies the following

(iii) $d(x, y) = 0 \Rightarrow x = y$,

then d is called a *quasi-metric* on X.

A *non-archimedean quasi-pseudometric* on X is a function $d : X \times X \to [0; +\infty)$ such that, for all x, y, z in X, d satisfies (i) and the following strong triangle inequality:

(iv) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Some conflicting terminology concerning so called non-symmetric topology has appeared in the literature. For instance, Rutten, Seda and Smyth use the term "a generalized ultrametric" for a non-archimedean quasi-pseudometric (cf.[18]–[20]), while Priess-Crampe and Ribenboim a non-archimedean metric called an ultrametric and they generalized this notion to ultrametrics taking values in partially ordered sets (cf. [14], [15]).

For a quasi-pseudometric d on X the conjugate d^{-1} of d is defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, while $d^* = \max\{d, d^{-1}\}$ is the pseudometric on X associated with d. Of course, if d is non-archimedean, both d^{-1} and d^* are non-archimedean. The topology on X induced by the quasi-pseudometric d is the topology $\tau(d)$ having the collection of all d-balls

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$$

as a base for the open sets ($x \in X, \epsilon \in (0; +\infty)$). The topology $\tau(d^*)$ is the smallest topology on X containing $\tau(d) \cup \tau(d^{-1})$. A topological space (X, τ) is called *(non-archimedeanly) quasi-(pseudo)metrizable* exactly when the topology τ is induced by some (non-archimedean) quasi-(pseudo)metric on X. All other topological notions needed but not defined in this article can be found in [3] and [4].

2. Conditions of separation

In what follows, let d be a quasi-pseudometric on X. If P is a property of the topological space $(X, \tau(d))$, we shall say that d has the property P. For instance, d is Hausdorff (resp. regular, completely regular and so on) only when $(X, \tau(d))$ is Hausdorff (resp. regular, completely regular and so on).

2.1. **T_0 and T_1.** For completeness of the present work, it makes sense to notice that $(X, \tau(d))$ is a T_0-space if and only if d^* is a metric on X, which holds if and only if, for every pair x, y of points of X, the condition $d(x, y) = d(y, x) = 0$ implies that $x = y$. The space $(X, \tau(d))$ is T_1 if and only if d is a quasi-metric on X. There exist in ZF quasi-pseudometrizable T_0-spaces which are not T_1. For instance, the quasi-pseudometric defined by Seda in Examples 2.1.(4) of [19] on the interval $[0; 1]$ induces a T_0-topology which is not T_1. A simpler T_0 but not T_1 quasi-pseudometric is described in Example 4 given below. Of course, the antidiscrete topology on X is pseudometrizable but not T_0 when X consists of at least two distinct points.

2.2. **Hausdorff's condition of separation.** Kelly was probably the first to give, in ZF, an example of a quasi-metric on $\mathbb{R}^2$ inducing a non-Hausdorff topology (cf. 5.4 of [8]). Later, Patty found a non-Hausdorff quasi-metric on a countable subset of $\mathbb{R}$ (cf. Counter-example 2.6 of [13]). Since the book "Quasi-Uniform Spaces" by Fletcher and Lindgren appeared in print (cf. [4]), more mathematicians have realized that one of the consequences of Corollary 7.1 of this book is that the co-finite topology on a countable infinite set is non-archimedeanly quasi-metrizable in ZFC because it has a σ-point-finite base in ZFC. It is, however, impossible to decide whether the relatively simple minimal T_1-topology on an arbitrary countable set is quasi-metrizable in ZF or it is not. To explain this situation, we need to look at some more precise concepts that have their roots in the outstanding work of Cantor and Dedekind.

It is said that a set X is *equipollent* to a set Y when there exists a one-to-one mapping from X onto Y. A set X is *infinite* (in the sense of Dedekind) exactly when X is equipollent to some of its proper subsets. A set is *finite* if and only if it is not infinite. A set X is *countable* exactly when every infinite subset of X is equipollent to X. A *cardinal number* (in the sense of von Neumann) is an ordinal number α of Zermelo-von Neumann such that no element of α is equipollent to α. A set X has its cardinal number (*the cardinal number of all elements of* X) exactly when there exists a cardinal number which is equipollent to X. It is shown in [21] that, without the axioms of choice and the replacement scheme, there is no way to prove that every set has its cardinal number. Perhaps, too huge chaos among elements of a set makes it impossible for every cardinal number to be the cardinal number of this set. After having seen the ideas of this paper, David Fremlin thought that every infinite countable set in the sense of the present work was of size ω. However, the following reasoning of the author of this article shows that it seems impossible to prove that all infinite countable sets are of cardinality ω in ZF.

Suppose that X is an infinite countable set and that it is possible to establish an element of X and denote this chosen element by x_0. Since $X \setminus \{x_0\}$ is an infinite subset of the countable set X, there is a one-to-one mapping f of X onto $X \setminus \{x_0\}$. Let $x_1 = f(x_0)$. Suppose that, for $n \in \omega$, we have already defined an element x_n of X. Let $x_{n+1} = f(x_n)$. In this way, by induction, we would get an infinite set $Y = \{f(x_n) : n \in \omega\}$ equipollent to X, so X would be of size ω if it were possible to choose some element x_0 of X. However, if there is no relation that well-orders X, we might be unable to establish an element of X. We call a set X *uncountable* if it is not countable, which holds if and only if X contains an infinite subset which is not equipollent to X. It is interesting that if X were a finite set without its cardinality, then the set $X \cup \omega$ would be uncountable in ZF.

Let us notice that the minimal T_1-topology on a set X is the collection which consists of the empty set and of all sets of the form $X \setminus A$ where A runs over the family of all those finite subsets of X that have their cardinal numbers. The co-finite topology on X is the smallest among all those topologies on X that contain all the sets $V \subseteq X$ such that $X \setminus V$ is finite. It is not obvious whether the co-finite topology and the minimal T_1-topology on the same set are always identical in ZF. Of course, the minimal T_1-topology and the co-finite topology on ω coincide. If a countable set X has its cardinal number, the minimal T_1-topology on X is non-archimedeanly quasi-metrizable as the following example shows:

Example 1. *We define a non-archimedean quasi-metric d on ω by putting*

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ \dfrac{1}{2^{y+1}} & \text{if} \quad x \neq y. \end{cases}$$

for $x, y \in \omega$. It is easily seen that, for arbitrary $x, n \in \omega$, the following equality holds:

$$B_d\left(x, \frac{1}{2^n}\right) = \{x\} \cup (\omega \setminus n).$$

Since the collection $\{\{x\} \cup (\omega \setminus n) : n \in \omega\}$ is a neighbourhood base for x in the co-finite topology on ω, the quasi-metric d induces the minimal T_1-topology on ω. If one follows the proof of Ribeiro's quasi-metrization theorem (cf.

[16], [17]*) given on page 489 of* [5] *to find a quasi-metric generating the co-finite topology on* ω*, one can come just to the function* d*.*

Participants of a faculty seminar at Łódź University posed the question: is there a quasi-metric which induces a non-Hausdorff T_1-topology on $\mathbb{R}$? The following example answers this question.

Example 2. *Consider once again the quasi-metric* d *on* ω *defined in Example 1, and denote by* d_1 *the discrete metric on* $\mathbb{R} \setminus \omega$*. Let us apply the sum operation to* d *and* d_1 *(cf. 4.2.1 of* [3]*) to get a non-archimedean quasi-metric* ρ *on* $\mathbb{R}$ *such that* $(\mathbb{R}, \tau(\rho))$ *is a non-Hausdorff* T_1*-space. Then:*

- $\rho(x, y) = 1$ *if either* $x \in \mathbb{R} \setminus \omega$ *and* $y \in \omega$ *or* $x \in \omega$ *and* $y \in \mathbb{R} \setminus \omega$*,*
- ρ *restricted to* $\omega \times \omega$ *is equal to* d*,*
- ρ *restricted to* $(\mathbb{R} \setminus \omega) \times (\mathbb{R} \setminus \omega)$ *is equal to* d_1*.*

Let us give one simple necessary and sufficient condition for a quasi-metric to satisfy Hausdorff's condition of separation.

Proposition 1. *A quasi-metric* d *on a set* X *induces a Hausdorff topology on* X *if and only if, for each pair x,y of distinct points of* X*, there exists a positive real number* ϵ *such that* $\max\{d(x, z), d(y, z)\} \geq \epsilon$ *for each* $z \in X$*.*

2.3. **The** A**-modification of a quasi-pseudometric.** It occurs that the following deformation of the quasi-pseudometric d on X plays an important role in the study of quasi-pseudometrizable spaces (cf. e.g.[11], [12]).

Definition 1. *For a subset* A *of* X*, the* A-modification of the quasi-pseudometric d *on* X *is the function* d_A *defined on* $X \times X$ *by:*

$$d_A(x, y) = \begin{cases} d(x, y) + 1 & \text{when } y \in A \text{ and } x \notin A, \\ d(x, y) & \text{otherwise.} \end{cases}$$

It is a simple exercise to check that, for every subset A of X and for every quasi-pseudometric d on X, the A-modification of d is a quasi-pseudometric on X which is non-archimedean if d is non-archimedean. The following example shows that for some not non-archimedean d and a suitable A, the quasi-pseudometric d_A is non-archimedean.

Example 3. *Let* $X = \{0, 1, 2\}$*,* $A = \{0, 2\}$ *and* $d(x, y) = |x - y|$ *for all* $x, y \in X$*. Since* $d(0, 2) > \max\{d(0, 1), d(1, 2)\}$*, the metric* d *is not non-archimedean. To see that* d_A *is non-archimedean, we observe that* $1 = d_A(0, 1) = d_A(2, 1) < d_A(0, 2) = d_A(1, 0) = d_A(1, 2) = d_A(2, 0) = 2$*.*

Let us turn our attention to some connections between conditions of separation for d and d_A. To begin, let us establish some more or less obvious facts.

Proposition 2. *For every quasi-pseudometric d on X and for every $A \subseteq X$, the following properties hold:*

(i) *for any $x \in$ and $\epsilon \in (0;1]$, we have:*

$$B_{d_A}(x,\epsilon) = \begin{cases} B_d(x,\epsilon) & \text{if } x \in A, \\ B_d(x,\epsilon) \setminus A & \text{if } x \notin A; \end{cases}$$

(ii) *the topology $\tau(d_A)$ is the smallest among all those topologies on X that contain $\tau(d) \cup \{X \setminus A\}$;*

(iii) *if d is T_i, then d_A is T_i for $i \in \{0,1,2\}$;*

(iv) *if d_A is T_1, then d is T_0;*

(v) *if A is co-dense in $(X, \tau(d))$ and d_A is Hausdorff, then d is Hausdorff, too.*

Proof. Property (i) is deduced from the definition of d_A, while (ii) follows from (i) and implies (iii). Assume that d_A is T_1 and consider any pair x, y of points of X such that $d(x,y) = d(y,x) = 0$. If $d_A(x,y) \neq 0$, then $d_A(x,y) = 1$ and $y \in A$, so that $d_A(y,x) = d(y,x) = 0$. In consequence, $x = y$ because d_A is a quasi-metric. This proves that (iv) holds.

Now, assume that A is a co-dense set in $(X, \tau(d))$ and the quasi-pseudometric d_A is Hausdorff. Consider any pair x, y of distinct points of X. There exists a pair U, V of disjoint members of $\tau(d_A)$ such that $x \in U$ and $y \in V$. It follows from (ii) that there exist sets $U_1, V_1 \in \tau(d_A)$ and there exist sets $U_2, V_2 \in \tau(d)$, such that $x \in U_1 \subseteq U$, $y \in V_1 \subseteq V$, either $U_1 = U_2$ or $U_1 = U_2 \setminus A$ and either $V_1 = V_2$ or $V_1 = V_2 \setminus A$. Since the sets U and V are disjoint, we have $U_2 \cap V_2 \subseteq A$. Therefore, the sets U_2 and V_2 are disjoint because the interior of A with respect to $\tau(d)$ is empty. Of course, $x \in U_2$ and $y \in V_2$, which completes the proof of (v). □

It is hidden in property (iii) of the proposition above that if d is a quasi-metric, then d_A is a quasi-metric. That this implication is true, one may deduce directly from the inequality $d \leq d_A$. Robert Plebaniak suspected that the converse implication might be false. The following example, offered by the author of this article, shows that Robert Plebaniak was right.

Example 4. *Let $X = \{0,1\}$, $d(0,0) = d(1,1) = d(1,0) = 0$ and $d(0,1) = 1$, while $A = \{0\}$. Then $d_A(0,1) = d_A(1,0) = 1$, so d_A is the discrete metric on X. Of course, d is not a quasi-metric. Observe that $\tau(d) = \{\emptyset, \{0\}, X\}$ is not T_1, while $\tau(d_A)$ is Hausdorff and even discrete.*

2.4. **When d_A is irregular.** Perhaps, the first explanation that a quasi-metrizable Hausdorff space need not be regular appeared in print in Patty's article [13]. To avoid confusion, we do not assume that a regular or completely regular space is necessarily T_1. Let us present in terms of d and

A a newly discovered criterion for d_A to induce an irregular topology even when d is a regular quasi-pseudometric on X. To begin with, let us consider a topological space (X, τ) and a subset A of X. Denote by $\tau[A]$ the smallest topology on X that contains $\tau \cup \{X \setminus A\}$. Then

$$\tau[A] = \{U \cup (V \setminus A) : U, V \in \tau\}.$$

Definition 2. *We say that A* brings the irregularity *of $\tau[A]$ if*

$$(X \setminus A) \cup int_\tau A \notin \tau.$$

Remark 1. *Let $E(A, \tau) = A \setminus int_\tau A$. Since $E(A, \tau) = X \setminus [(X \setminus A) \cup int_\tau A]$, we can observe that A brings the irregularity of $\tau[A]$ if and only if $E(A, \tau)$ is not closed in (X, τ).*

Theorem 1. *Suppose that a subset A of a topological space (X, τ) brings the irregularity of $\tau[A]$. Then the topological space $(X, \tau[A])$ is not regular.*

Proof. Of course, $\mathrm{cl}_\tau E(A, \tau) \subseteq X \setminus \mathrm{int}_\tau A$ and the set $E(A, \tau)$ is not closed in (X, τ). Therefore, there exists $x_0 \in [\mathrm{cl}_\tau E(A, \tau)] \setminus A$. The set $X \setminus A$ is an open neighbourhood of x_0 in $(X, \tau[A])$. Suppose that the space $(X, \tau[A])$ is regular. There exists $U \in \tau$ such that $x_0 \in U \setminus A \subseteq \mathrm{cl}_{\tau[A]}(U \setminus A) \subseteq X \setminus A$. There exists $y_0 \in U \cap E(A, \tau)$. Let us consider an arbitrary $V \in \tau$ such that $y_0 \in V$. Since $y_0 \notin \mathrm{int}_\tau A$, the set $(V \cap U) \setminus A$ is non-empty. Moreover, the collection $\{V \in \tau : y_0 \in V\}$ is a neighbourhood base for y_0 in $\tau[A]$. In consequence, $y_0 \in [\mathrm{cl}_{\tau[A]}(U \setminus A)] \cap A$, which is impossible. The contradiction obtained proves that $\tau[A]$ is not regular. □

Theorem 2. *Let τ be a regular topology on X and let A be a subset of X such that $\tau[A]$ is not regular. Then A brings the irregularity of $\tau[A]$.*

Proof. Suppose that A does not bring the irregularity of $\tau[A]$. Then the set $E(A, \tau)$ is closed in (X, τ). We shall show that $\tau[A]$ is regular. To do this, let us consider any $x \in X$ and any $G \in \tau[A]$ such that $x \in G$. Suppose first that $x \in A$. Since τ is regular, there exists $V \in \tau$ such that $x \in V \subseteq \mathrm{cl}_\tau V \subseteq G$. Then $\mathrm{cl}_{\tau[A]} V \subseteq G$.

Now, assume that $x \notin A$. There exists $U \in \tau$ such that $x \in U \setminus A \subseteq G$. The set $U \setminus E(A, \tau)$ is an open neighbourhood of x in (X, τ). From the regularity of τ, we infer that there exists $W \in \tau$ such that $x \in W \subseteq \mathrm{cl}_\tau W \subseteq U \setminus E(A, \tau)$. The following inclusions hold:

$$\mathrm{cl}_{\tau[A]}(W \setminus A) \subseteq X \setminus \mathrm{int}_\tau A$$

and

$$\mathrm{cl}_{\tau[A]}(W \setminus A) \subseteq \mathrm{cl}_\tau W \subseteq (U \setminus A) \cup \mathrm{int}_\tau A.$$

Hence $\mathrm{cl}_{\tau[A]}(W \setminus A) \subseteq U \setminus A$, which completes the proof that $\tau[A]$ is regular. □

Corollary 1. *Let τ be a regular topology on X and let $A \subseteq X$. Then the topology $\tau[A]$ is regular if and only if A does not bring the irregularity of $\tau[A]$.*

Remark 2. *Let d be a quasi-pseudometric on X and let $A \subseteq X$. It follows from Proposition 2(ii) that $\tau(d_A) = \tau(d)[A]$, therefore we can say that A* brings the irregularity of d_A *when $E(A, d) = A \setminus int_{\tau(d)} A$ is not closed in $(X, \tau(d))$.*

As immediate consequences of the above remark and of Theorems 1–2, we get the following:

Corollary 2. *If d is a quasi-pseudometric on X and a subset A of X brings the irregularity of d_A, then the topology on X induced by d_A is not regular.*

Corollary 3. *If d is a regular quasi-pseudometric on X and A is a subset of X such that the quasi-pseudometric d_A is not regular, then A brings the irregularity of d_A.*

Corollary 4. *If d is a regular quasi-pseudometric on X and $A \subseteq X$, then the quasi-pseudometric d_A is regular if and only if A does not bring the irregularity of d_A.*

Example 5. *Let $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ and let $A = \{\frac{1}{n} : n \in \omega \setminus \{0\}\}$. Since A brings the irregularity of d_A, the space $(\mathbb{R}, \tau(d_A))$ is not regular. Of course, this space is Hausdorff by Proposition 2. In 1.5.7 of* [3], *the topology $\tau(d_A)$ is introduced by a neighbourhood system without using any quasi-metric; in consequence, relatively few people know that the most frequently shown Hausdorff space which is not regular is quasi-metrizable in ZF.*

2.5. **When both d and d_A are completely regular.** When τ is a topology on X, then we say that a function $f : X \to \mathbb{R}$ is *τ-continuous at a point* $x \in X$ if, for each $\epsilon \in (0; +\infty)$, there exists $U \in \tau$ such that $x \in U$ and $f(U) \subseteq (f(x) - \epsilon; f(x) + \epsilon)$. Clearly, the function f is τ-continuous if f is *τ-continuous* at each point of X. Let $C(X, \tau)$ stand for the collection of all τ-continuous real functions on X.

Theorem 3. *Suppose that (X, τ) is a completely regular space and a subset A of X does not bring the irregularity of $\tau[A]$. Then the space $(X, \tau[A])$ is completely regular.*

Proof. Let F be a closed set in $(X, \tau[A])$ and let $x \in X \setminus F$. Suppose that $x \in A$. Then there exists $U \in \tau$ such that $x \in U \subseteq X \setminus F$. It follows from the complete regularity of τ that there exists $f \in C(X, \tau)$ such that

$f(x) = 0$ and $f(X \setminus U) \subseteq \{1\}$. Since $\tau \subseteq \tau[A]$, we have $f \in C(X, \tau[A])$ such that $f(x) = 0$ and $f(F) \subseteq \{1\}$.

Now, assume that $x \notin A$. Then there exists $V \in \tau$ such that $x \in V \setminus A \subseteq X \setminus F$. As the set $(X \setminus V) \cup E(A, \tau)$ does not contain x and it is closed in (X, τ), it follows from the complete regularity of τ that there exists $g \in C(X, \tau)$ such that $g(x) = 0$ and $g[(X \setminus V) \cup E(A, \tau)] \subseteq \{1\}$. We define a function h as follows:

$$h(t) = \begin{cases} 1 & \text{for } t \in \operatorname{int}_\tau A, \\ g(t) & \text{for } t \in X \setminus \operatorname{int}_\tau A. \end{cases}$$

To show that $\tau[A]$ is completely regular, it suffices to check that h is $\tau[A]$-continuous.

Clearly, the function h is $\tau[A]$-continuous at each point $t \in \operatorname{int}_\tau A$. Assume that $t_0 \notin \operatorname{int}_\tau A$. Then either $t_0 \in X \setminus A$ or $t_0 \in E(A, \tau)$. Let $\epsilon \in (0; +\infty)$. Consider the case when $t_0 \in X \setminus A$. Then $h(t_0) = g(t_0)$, so, by the τ-continuity of g, there exists $G \in \tau$ such that $t_0 \in G$ and $g(G) \subseteq (h(t_0) - \epsilon; h(t_0) + \epsilon)$. This, together with the facts that $t_0 \in G \setminus A \in \tau[A]$ and $h(G \setminus A) \subseteq g(G)$, implies that h is $\tau[A]$-continuous at t_0.

Finally, consider the case when $t_0 \in E(A, \tau)$. Then $h(t_0) = g(t_0) = 1$. There exists $W \in \tau$ such that $t_0 \in W$ and $g(W) \subseteq (1 - \epsilon; 1 + \epsilon)$. To complete the proof of the $\tau[A]$-continuity of h, notice that $h(W) \subseteq (1 - \epsilon; 1 + \epsilon)$. □

Corollary 5. *For every subset A of a completely regular space (X, τ), the space $(X, \tau[A])$ is completely regular if and only if it is regular.*

Corollary 6. *Let d be a completely regular quasi-pseudometric on X and let A be a subset of X. Then d_A is completely regular if and only if A does not bring the irregularity of d_A.*

Corollary 7. *For every completely regular quasi-pseudometric d on X and for every $A \subseteq X$, the quasi-pseudometric d_A is completely regular if and only if d_A is regular.*

Is there, in ZF, a Hausdorff regular quasi-metrizable space which is not completely regular?

No satisfactory answer to this question is known to the author. The non-completely regular but regular quasi-metric described by Deák in [2] is not constructed in ZF. Certainly, in the light of Corollary 7, the A-modification of a completely regular quasi-metric d does not induce a regular topology which is not completely regular. The famous not completely regular but regular Mysior's T_1-space (cf. 1.5.9 of [3]) is not quasi-pseudometrizable. For completeness of this work, let us recall that the square of the Sorgenfrey line can serve as an example in ZFC of a completely regular Hausdorff non-archimedeanly quasi-metrizable not normal space (cf. e.g. [4], [5]).

What other properties of a quasi-pseudometric d on X does the quasi-pseudometric d_A share when $A \subseteq X$ does not bring the irregularity of d_A? Many other relevant questions can be posed.

3. Some applications to bitopological spaces

Patty looked for examples of quasi-metrics ρ on X such that $\tau(\rho)$ is metrizable but $\tau(\rho^{-1})$ is not regular where $\rho^{-1}(x, y) = \rho(y, x)$ for $x, y \in X$ (cf. Counter-Examples 2.6-2.7 of [13]). We shall show how to apply the results of the previous section to produce some kind of such examples of bitopological spaces $(X, \tau(\rho), \tau(\rho^{-1}))$.

Theorem 4. *Let d be a metric on a set X and let A be a subset of X such that A brings the irregularity of d_A, while $B = X \setminus A$ does not bring the irregularity of d_B. Then, for $\rho = d_B$, the topological space $(X, \tau(\rho))$ is metrizable in ZFC, while the topological space $(X, \tau(\rho^{-1}))$ is not regular. Moreover, if $(X, \tau(d))$ is second countable, so are $(X, \tau(\rho))$ and $(X, \tau(\rho^{-1}))$.*

Proof. It is easy to check that $\rho^{-1} = d_A$. It follows from Corollary 2 that the topology $\tau(\rho^{-1})$ is not regular, while $(X, \tau(\rho))$ is a regular T_1-space. The space $(X, \tau(d))$ has a σ-locally finite open base in ZFC, and $\tau(\rho) = \tau(d)[B]$. Therefore, the space $(X, \tau(\rho))$ is metrizable in ZFC by the metrization theorem of Nagata-Smirnov. Finally, when $(X, \tau(d))$ is second countable, the second countability of both $(X, \tau(\rho))$ and $(X, \tau(\rho^{-1}))$ follows from Remark 2. □

Example 6. *Let us consider once again the set A and the metric d of Example 5. Since $B = \mathbb{R} \setminus A$ does not bring the irregularity of d_B, in the light of Theorem 4, the space $(\mathbb{R}, \tau(d_B))$ is metrizable in ZFC, while the space $(\mathbb{R}, \tau({d_B}^{-1}))$ is not regular.*

It might be interesting to carefully extend, if possible, some results written in ZFC to theorems of ZF.

Acknowledgement. The author thanks Sally Robison and David Fremlin for their helpful comments on the English language that improved the text. The author is also grateful to Gary Gruenhage, Franz-Viktor Kuhlmann and Salvador Romaguera for their kind help in pointing out some important papers relevant to this work.

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