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OPTIMAL CONTROL PROBLEMS WITH HIGHER ORDER CONSTRAINTS

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Abstract. We investigate the existence and properties of solutions for a higher order equation with only local conditions concerning smoothness or monotonicity of the nonlinearity. Next, we study the continuous dependence of solutions on functional parameters and apply this result to our optimal control problem.

1. Introduction

Recently research on continuous dependence on parameters of solutions for many BVPs and its applications to optimal control problems have been very active and enjoying the increasing interest (see e.g. [4], [6], [8], [9], [17], [18] and references therein). We also want to join in this discussion and consider the case when the constrains are given by a higher order differential equation with the multipoint boundary condition. Precisely, we study the following problem

$$J(u,z) = \int_0^1 F(t,u(t),z(t))dt \to \min \tag{1}$$

subject to

$$\begin{cases} -u^{(n)}(t) = \widetilde{f}(t,u(t),z(t)) \ in \ (0,1), \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = \sum_{i=1}^{m} b_i u^{(n-2)}(\alpha_i) - u^{(n-2)}(1) = 0, \end{cases} \tag{2}$$

where b_i and α_i are constants satisfying additional assumptions **(BC1)** and **(BC2)**. Let us note that for $n = 4$, the above equation is the generalization

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of the elastic beam equation

$$\frac{d^2}{dt^2}\left[E(t)I(t)\frac{d^2u}{dt^2}\right] + Q(t)u + \Phi(t) = 0,$$

in the case when Young's modulus of elasticity - E - and the moment of inertia of cross section of the beam - I - are constants.
Our first task is to investigate the existence and behavior of solutions of (2) for a given parameter z. Thus, we look for positive classical solutions for the following multipoint BVP

(3) $$-u^{(n)}(t) = f(t, u(t)) \ in \ (0,1)$$

(4) $$u(0) = u'(0) = ... = u^{(n-2)}(0) = \sum_{i=1}^{m} b_i u^{(n-2)}(\alpha_i) - u^{(n-2)}(1) = 0$$

with fixed numbers $m, n \in \mathbf{N} := \{1, 2, ...\}$. We investigate the existence of positive solutions of (3)-(4) and their properties provided that

- **(f1)** there exists $0 < d$ such that $f : [0,1] \times [0,d] \to [0,\infty)$, $f \in C([0,1] \times [0,d])$, $\int_0^1 f(t,0)dt > 0$;
- **(BC1)** $m > 1$, $n \geq 3$, $\alpha_i \in (0,1)$ for all $i \in \{1,...,m\}$, $0 < \alpha_1 < \alpha_2 < ... < \alpha_m < 1$;
- **(BC2)** $b_i > 0$ for all $i \in \{1,...,m\}$, $\sum\limits_{i=1}^{m} b_i = 1$.

Next we shall discuss the continuous (in some sense) dependence of solutions on functional parameters. Precisely, we shall show that the sequence $\{u_k\}_{k\in N}$ of solutions of our problem (corresponding to the sequence of parameters $\{z_k\}_{k\in N}$) tends uniformly (up to a subsequence) in $[0,1]$ to u_0 provided that the sequence of parameters is convergent to z_0 in $[0,1]$. In the last section, we apply this result to obtain the main theorem concerning the existence of an optimal process for the optimal control problem (1)-(2). Analogous problems have been usually discussed for the second order ordinary or partial differential equation.

Boundary value problems appear in many mathematical models of physical and biological phenomenons, for example in the investigation of effects of soil settlement (see e.g. [3]), in the considerations concerning deformations of structures (see e.g. [16], [20]) or in the theory of elasticity (see e.g. [1], [5], [11], [19], [21]). The case of second order ODEs with various multipoint boundary conditions was widely discussed e.g. in [2], [10], [12] (with boundary conditions (4)) or in [13], [14], [15] (with other boundary conditions).
The special case of (3)-(4) was studied in [7], where the authors present the

existence and nonexistence results for the nonlinear eigenvalue problem

$$u^{(n)}(t) + \lambda g(t)\mathbf{f}(u(t)) = 0 \text{ in } (0,1) \tag{5}$$

with boundary conditions (4), for continuous functions $\mathbf{f} : [0,\infty) \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$, $\lambda > 0$, b_i satisfying **(BC2)** and $\frac{1}{2} \le \alpha_1 < \alpha_2 < ... < \alpha_m < 1$. The main tool applied in [7] is the Krasnosielski fixed point theorem and the approach is based on the following conditions

$$(Af_\infty)^{-1} < \lambda < (BF_0)^{-1} \tag{6}$$

(see Theorem 3.1, [7]) or

$$(Af_0)^{-1} < \lambda < (BF_\infty)^{-1} \tag{7}$$

(see Theorem 3.2, [7]), where

$$F_0 := \limsup_{x\to 0^+} \tfrac{\mathbf{f}(x)}{x}, \quad f_0 := \liminf_{x\to 0^+} \tfrac{\mathbf{f}(x)}{x},$$

$$F_\infty := \limsup_{x\to +\infty} \tfrac{\mathbf{f}(x)}{x}, \quad f_\infty := \liminf_{x\to +\infty} \tfrac{\mathbf{f}(x)}{x},$$

$A := \int_0^1 G_n(1,s)s^{n-1}g(s)ds$ and $B := \int_0^1 G_n(1,s)s^{n-2}g(s)ds$ (G_n is given in Section 2). In [7] the authors also proved that if

$$\lambda B\mathbf{f}(x) < x \text{ for all } x \in (0,+\infty) \text{ or } \lambda A\mathbf{f}(x) > x \text{ for all } x \in (0,+\infty) \tag{8}$$

then (5)-(4) does not possess any positive solutions (see Theorem 3.3, [7]).

In this paper we discuss a more general when nonlinearity f (in (3)) satisfies only local assumptions concerning its smoothness and growth. Our approach allows us to consider both sublinear and superlinear cases simultaneously, because we need information about the value of nonlinearity f in $[0,d]$ only, while the previous papers (e. g. [2], [7]) were based on the behavior of $f(t,\cdot)$ at zero and at infinity (see e.g. (6) or (7)). We also show that these assumptions are still sufficient to prove that solutions depend continuously (in some sense) on parameters.

2. The existence of solutions for the higher order problem

In this section we apply the Schauder fixed point theorem to obtain the existence of positive classical solutions of (3)-(4). Thus we star with the definition of the following operator

$$Tu(t) := \int_0^1 G_n(t,s)f(s,u(s))ds,$$

where $G_n : [0,1]\times[0,1] \to [0,+\infty)$ is Green's function (associated with (3)) given by the iteration process (see e.g. [2])

$$G_2(t,s) := \frac{t}{1-\sum_{i=1}^{m} b_i\alpha_i}\left[(1-s)-\sum_{i=1}^{m} b_i(\alpha_i - s)\chi_{[0,\alpha_i]}(s)\right]-(t-s)\chi_{[0,t]}(s),$$

where χ is the characteristic function, and for $3 \le k \le n$

$$G_k(t,s) := \int_0^t G_{k-1}(v,s)dv.$$

We can we prove by induction an auxiliary estimation on G_n.

Lemma 1. *For all $2 \le k \le n$, we have*

$$0 \le G_k(t,s) \le c(1-s)\frac{t^{k-1}}{(k-1)!} \text{ in } [0,1]\times[0,1]$$

where $c := \frac{1}{1-\max\limits_{i\in\{1,\dots,m\}}\alpha_i}$.

Proof. It is easy to note that for $k=2$, $G_2(t,s) \le ct(1-s)$ and $G_2(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$ (see [2]). Assuming that $0 \le G_l(t,s) \le c(1-s)\frac{t^{l-1}}{(l-1)!}$ for $2 \le l \le k$, we obtain

$$G_{k+1}(t,s) := \int_0^t G_k(v,s)dv \le c(1-s)\frac{t^k}{k!}.$$

Finally, we get the required estimation. □

In the sequel we will investigate the existence and behavior of solutions of our problem. To this end we recall Lemma 2.1 from [7] concerning properties of functions satisfying boundary conditions (4).

Lemma 2 (Lemma 2.1 from [7])**.** *If $u \in C^n([0,1])$ satisfies (4) and $u^{(n)}(t) \le 0$ for all $t \in (0,1)$, then*

$$u^{(l)}(t) \ge 0 \tag{9}$$

for all $t \in [0,1]$ and $l \in \{0,1\}$ (with $u^{(0)} = u$).

Remark 1. *Taking into account the proof of Lemma 2.1 from [7] we can state that (9) holds for $l \in \{0,1,...,n-2\}$.*

Now we prove a stronger version of the previous lemma and give some further properties of solutions of (3)-(4).

Lemma 3. *If $u \in C^n([0,1])$ satisfies (4) and $u^{(n)}(t) \leq 0$ for all $t \in (0,1)$ and $u^{(n)}$ is not identically equal to zero in $(0,1)$, then*

$$u^{(l)}(t) > 0$$

for all $(0,1]$ and $l \in \{0,1,...,n-2\}$.

Proof. Fix $l \in \{0,1,...,n-2\}$ and suppose that there exists $t_0 \in (0,1]$ such that $u^{(l)}(t_0) = 0$. We show that this assumption leads to the contradiction with the fact that $u^{(n)}$ is not identically equal to zero.

Indeed, if $l \leq n-3$, then by Remark 1, $u^{(l+1)} \geq 0$ in $[0,1]$, namely, $u^{(l)}$ is nondecreasing in $[0,1]$. Thus, by the boundary condition $u^{(l)}(0) = 0$, we have $u^{(l)}(t) = 0$ for all $t \in [0,t_0]$. Moreover, $h(t) := u^{(n-2)}(t) = 0$ for all $t \in [0,t_0]$. (If we take $l = n-2$, we obtain the last equality immediately.) Since $h'' \leq 0$ in $[0,1]$, h' is nonincreasing in $(0,1)$. By the definition of h and the above reasoning, we infer that $h'(t) \leq 0$ in $(0,1)$, and in consequence, $h(t)$ is also nonincreasing in $(0,1)$. Thus $h(t) \leq 0$ in $(0,1)$. Finally, taking into account Remark 1, we obtain $h \equiv 0$ in $(0,1)$ and further $u^{(n)}(t) = h''(t) = 0$ in $(0,1)$, which is a contradiction. □

Lemma 4. *Assume that* **(BC1)**, **(BC2)** *and* **(f1)** *hold. If $u \in C^n([0,1])$ satisfies (3)-(4) and $||u||_\infty := \max_{t\in[0,1]} |u(t)| \leq d$, then for all $l \in \{0,1,...,n-2\}$*

$$0 \leq u^{(l)}(t) \leq t^{n-l-1} A_l \int_0^1 (1-s) \max_{u\in[0,d]} f(s,u)ds \text{ for all } t \in [0,1],$$

where $A_l := c\frac{1}{(n-l-1)!}$, and

$$|u^{(n-1)}(t)| \leq c \int_0^1 \max_{u\in[0,d]} f(s,u)ds \text{ for all } t \in [0,1]. \tag{10}$$

Proof. We start our proof with the observation that for all $l \in \{0,1,...,n-2\}$ the following chain of relations holds

$$\begin{aligned} u^{(l)}(t) = (Tu)^{(l)}(t) &= \int_0^1 \frac{\partial^l}{\partial t^l} G_n(t,s) f(s,u(s))ds \\ &= \int_0^1 G_{n-l}(t,s) f(s,u(s))ds \leq c\left(\frac{t^{n-l-1}}{(n-l-1)!}\right) \int_0^1 (1-s) \max_{u\in[0,d]} f(s,u)ds, \end{aligned}$$

where the last inequality follows from Lemma 1. To prove (10) it suffices to note that

$$u^{(n-1)}(t) = \frac{1}{1-\sum\limits_{i=1}^{m} b_i\alpha_i}\left[\int_0^1 (1-s)f(s,u(s))ds - \sum_{i=1}^{m}\int_0^{\alpha_i} b_i(\alpha_i - s)f(s,u(s))ds\right] -$$
$$-\int_0^t f(s,u(s))ds,$$

which ends the proof. □

Now we are in position to show the following result.

Theorem 1. *If* **(BC1), (BC2)** *and* **(f1)** *hold and we assume the additional estimation*

$$\textbf{(f2)} \qquad c\frac{1}{(n-1)!}\int_0^1 (1-s)\max_{u\in[0,d]} f(s,u)ds \le d,$$

then problem (3)-(4) *possesses at least one classical solution* $u_0 \in X$, *with*

$$X := \left\{u \in C([0,1]), 0 \le u(t) \le dt^{n-1} \text{ for all } t \in [0,1]\right\}.$$

Proof. We prove that T maps the convex set X into X, (i.e. $TX \subset X$). To this effect, we fix $u \in X$. The definition of T implies that $\widetilde{u} := Tu$ belongs to $C^n(0,1)$ and satisfies (4). Taking into account **(f1)-(f2)** and Lemma 1, we get for all $t \in [0,1]$

$$0 \le \widetilde{u}(t) = \int_0^1 G_n(t,s)f(s,u(s))ds \le c\frac{t^{n-1}}{(n-1)!}\int_0^1 (1-s)\max_{u\in[0,d]} f(s,u)ds \le t^{n-1}d.$$

Thus, $0 \le \widetilde{u}(t) \le dt^{n-1}$ for all $t \in [0,1]$ and finally, $\widetilde{u} \in X$. Moreover, by the standard reasoning, one can show that $T : X \to C([0,1])$ is continuous and compact. Hence, Schauder's fixed point theorem yields the existence of at least one fixed point $u_0 \in X$ of T. By Lemma 3, we state that u_0 is positive in $(0,1)$. □

Example 1. *Let us consider the problem*

$$-u^{(4)}(t) = \frac{\widetilde{a}_1(t)\,(u(t))^2\,(u(t)+1)\,(u(t)+2)}{(9-u(t))\,(u(t)+3)} + \ln\left[u(t) + \widetilde{b}_1(t)e\right] \text{ in } (0,1)$$

$$u(0) = u'(0) = u''(0) = \sum_{i=1}^{2} b_i u''(\alpha_i) - u''(1) = 0,$$

where $b_1 = b_2 = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{3}{4}$, $\widetilde{a}_1(t) = \sin t$, $\widetilde{b}_1(t) = 2 + 3t^4$.

We show that nonlinearity

$$f_1(t,u) := \frac{\widetilde{a}_1(t)u^2\,(u+1)\,(u+2)}{(9-u)\,(u+3)} + \ln\left[u + \widetilde{b}_1(t)e\right]$$

satisfies assumptions **(f1)-(f2)** with $d = 3$. Indeed, since f_1 is nondecreasing in $[0,3]$ we get

$$\max_{u\in[0,3]} f_1(t,u) = f_1(t,3) \ for \ all \ t \in [0,1].$$

Moreover, f_1 is continuous in $[0,1] \times [0,3]$ and $f_1(t,0)$ is not identically equal to zero. Thus, **(f1)** holds. To show that f_1 satisfies **(f2)** we have to calculate $c := \frac{1}{1-\max\limits_{i\in\{1,\dots,m\}}\alpha_i} = 4$ and

$$c\frac{1}{(n-1)!}\int_0^1 (1-s)\max_{u\in[0,3]} f_1(t,u)ds \le$$

$$\le \frac{4}{3!}\int_0^1 (1-s)\left(\frac{\widetilde{a}_1(s)d^2\,(d+1)\,(d+2)}{(9-d)\,(d+3)} + \ln\left[d + \widetilde{b}_1(s)e\right]\right)ds \le$$

$$\le \frac{4}{3!}\int_0^1 (1-s)\,(5+\ln 18)\,ds = \frac{2}{3!}\,(5+\ln 18) \le 2.6.$$

Finally, Theorem 1 leads to the existences of positive classical solution $\overline{u}$ of our problem such that $0 \le \overline{u}(t) \le 3t^{n-1}$ for all $t \in [0,1]$, with $n = 4$.

Example 2. *Let us consider the problem*

$$-u^{(7)}(t) = 2\widetilde{b}_2(t)e^{\frac{(u(t))^2}{(u(t)+4)}} + \frac{(u(t))^3\widetilde{a}_2(t)}{(5-u(t))\,(u(t)+1)} \ in \ (0,1)$$

$$u(0) = u'(0) = u''(0) = u'''(0) = u^{(4)}(0) = u^{(5)}(0) =$$
$$= \sum_{i=1}^{3} b_i u^{(5)}(\alpha_i) - u^{(5)}(1) = 0,$$

where $b_1 = \frac{1}{4}$, $b_2 = \frac{1}{3}$, $b_3 = \frac{5}{12}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{2}{3}$, $\widetilde{a}_2(t) = 3\left(t^2 + \sin^2 t\right)$, $\widetilde{b}_2(t) = 2t^6 + 3t^4$.

It is easy to check that **(f1)** holds for

$$f_2(t,u) = 2\widetilde{b}_2(t)e^{\frac{u^2}{(u+4)}} + \frac{u^3\widetilde{a}_2(t)}{(5-u)\,(u+1)}$$

with $d = 1$. Moreover we have

$$c\frac{1}{(n-1)!}\int_0^1 (1-s)\max_{u\in[0,1]} f_2(t,u)ds$$

$$\leq \frac{3}{6!}\left(10e^{\frac{1}{5}}+\frac{3}{4}\right)\int_0^1 (1-s)\,ds \leq 0.03$$

which gives **(f2).** By Theorem 1 we get the existence of at least one classical positive solution $\overline{u}$ of our problem such that $0 \leq \overline{u}(t) \leq t^{n-1}$ for all $t \in [0,1]$, with $n = 7$.

Remark 2. *It is worth noting that in both examples the nonlinearities do not satisfy growth conditions* (*with respect to the second variable*) *similar to* (6) *and* (7). *Indeed* (see Ex. 1)

$$\lim_{u\to 0^+}\frac{f_1(t,u)}{u} = \lim_{u\to 0^+}\left[\frac{\widetilde{a}_1(t)u\,(u+1)\,(u+2)}{(9-u)\,(u+3)} + \frac{\ln\left[u+\widetilde{b}_1(t)e\right]}{u}\right] = +\infty$$

and

$$\lim_{u\to +\infty}\frac{f_1(t,u)}{u} = \lim_{u\to +\infty}\left[\frac{\widetilde{a}_1(t)u\,(u+1)\,(u+2)}{(9-u)\,(u+3)} + \frac{\ln\left[u+\widetilde{b}_1(t)e\right]}{u}\right] = -\infty.$$

Similarly we get (see Ex. 2)

$$\lim_{u\to 0^+}\frac{f_2(t,u)}{u} = \lim_{u\to 0^+}\left[2\widetilde{b}_2(t)\frac{e^{\frac{u^2}{(u+4)}}}{u} + \frac{u^2\widetilde{a}_2(t)}{(5-u)\,(u+1)}\right] = +\infty$$

and

$$\lim_{u\to +\infty}\frac{f_2(t,u)}{u} = \lim_{u\to +\infty}\left[2\widetilde{b}_2(t)\frac{e^{\frac{u^2}{(u+4)}}}{u} + \frac{u^2\widetilde{a}_2(t)}{(5-u)\,(u+1)}\right] = +\infty.$$

Theorem 2. *If* **(BC1)-(BC2)** *and* **(f1)** *hold then for each $d > 0$ there exists a positive integer n_0 such that problem* (3)-(4) *of order grater than or equal to n_0 possesses at least one classical solution $\overline{u}$ satisfying the estimation*

$$0 < \overline{u}(t) \leq dt^{n-1} \text{ for all } t \in [0,1].$$

Proof. Fix $d > 0$ and take n_0 satisfying the inequality

$$c\frac{M}{d} \leq (n_0 - 1)!,$$

with $M := \max_{(s,u)\in[0,1]\times[0,d]} f(s,u)$. Then for each $n \geq n_0$, f satisfies **(f2)** with constant d. Thus Theorem 1 yields the solvability of (3)-(4) in X. $\square$

Remark 3. *It is worth comparing the above result with the nonexistence result presented in* [7]. *Let us consider the nonlinearity* $f(s,u) = \lambda g(s)\mathbf{f}(u)$, *where* g *and* $\mathbf{f}$ *satisfy assumption of paper* [7] *(presented in Section 1) . Let us note that in our case (*f *satisfies* **(f1)-(f2)**) *conditions* (8) *do not hold. Indeed, since* $\mathbf{f}$ *is continuous in* $[0,1]\times[0,d]$ *and* $\mathbf{f}(0) > 0$ *we have* $\lim_{u\to 0^+} \frac{u}{\mathbf{f}(u)} = 0$, *which is a contradiction to the condition*

$$\lambda B\mathbf{f}(x) < x \text{ for all } x \in (0,+\infty)$$

as $\lambda > 0$. *On the other hand we can derive that the second condition of* (8) *and* **(f2)** *exclude each other. Indeed, let us assume that function* f *satisfies* **(f1)**, $d > 0$ *and consider* n_0 *such that*

$$\frac{cM}{(n_0-1)!} \leq d,$$

where $M := \max_{(s,u)\in[0,1]\times[0,d]} f(s,u) = \lambda \max_{u\in[0,d]} \mathbf{f}(u) \max_{s\in[0,1]} g(s)$. *Then we can state that for* $n \geq n_0$, *function* f *satisfies* **(f2)**:

$$\frac{c}{(n-1)!}\int_0^1 (1-s)\max_{u\in[0,d]} f(s,u)ds \leq \frac{cM}{(n_0-1)!} \leq d. \tag{11}$$

If we suppose now that $\lambda A\mathbf{f}(x) > x$ *for all* $x \in (0,+\infty)$, *with*

$$A := \int_0^1 G_n(1,s)s^{n-1}g(s)ds,$$

then, using Lemma 1 and assertion (11), *we have for all* $n \geq n_0$

$$d < \lambda A\mathbf{f}(d) = \lambda\mathbf{f}(d)\int_0^1 G_n(1,s)s^{n-1}g(s)ds \leq$$

$$\leq \frac{c}{(n-1)!}\int_0^1 (1-s)s^{n-1}\lambda g(s)\mathbf{f}(d)ds \leq \frac{cM}{(n_0-1)!} \leq d$$

which is a contradiction. $\square$

3. Continuous dependence on parameters

Let us consider the case when the nonlinearity depends on functional parameter $z{:}(0,1) \to R^s$, $z \in Z \subset C([0,1])$, $s \in \mathbf{N}$, namely,

$$(12) \qquad \begin{cases} -u^{(n)}(t) = \widetilde{f}(t, u(t), z(t)) \text{ on } (0,1) \\ \qquad \text{and } u \text{ satisfies } (4) \end{cases}$$

with $\widetilde{f} : [0,1] \times [0,d] \times R^s \to R$. We have to assume conditions which guarantee that for each $z \in Z$ function $[0,1] \times [0,d] \ni (t,x) \longmapsto \widetilde{f}(t,x,z(t))$ satisfies hypothesis **(f1), (f2)** and there exists X_z satisfying the inclusion $TX_z \subset X_z$. Thus, we assume that

- **(f1z)** *there exists* $d > 0$ *such that* $\widetilde{f} : [0,1] \times [0,d] \times R^s \to [0,\infty)$, $\widetilde{f} \in C([0,1] \times [0,d] \times R^s)$ *and for all* $z \in Z$, $\int_0^1 \widetilde{f}(t,0,z(t))dt > 0$*;*
- **(f2z)** *for each* $z \in Z$ *there exists* $0 < d_z < d$ *such that*

$$c\frac{1}{(n-1)!}\int_0^1 \max_{u \in [0,d_z]} \widetilde{f}(t,u,z(t))dt \le d_z;$$

- **(f3z)** *there exists* $\varphi > 0$ *such that for each* $z \in Z$

$$\max_{(t,u) \in [0,1] \times [0,d_z]} \widetilde{f}(t,u,z(t)) \le \varphi.$$

Let

$$X_z := \left\{u \in C([0,1]), 0 \le u(t) \le d_z t^{n-1} \text{ on } [0,1]\right\}.$$

Theorem 3. *Let us denote by* $\overline{u}_k \in X_{z_k}$ *the solution of* (12) *corresponding to* $z = z_k$, $k \in \mathbf{N}$. *If* **(BC1)-(BC2)** *and* **(f1z)-(f3z)** *hold and the sequence of parameters* $\{z_k\}_{k=1}^{\infty} \subset Z$ *tends pointwisely to* $z_0 \in Z$ *in* $[0,1]$, *then there exists a subsequence, still denoted by* $\{\overline{u}_k\}_{k=1}^{\infty}$, *converging uniformly to certain* $\overline{u}_0 \in C^n([0,1])$ *which is the positive solution of* (12) *with* $z = z_0$ *such that* $0 < \overline{u}_0(t) \le dt^{n-1}$ *for all* $t \in [0,1]$.

Proof. Taking into account Lemma 4, assumptions **(f2z)** and **(f3z)** we get the estimations

$$(13) \qquad |\overline{u}_k^{(l)}(t)| \le c\varphi \text{ on } [0,1], \text{ for all } l \in \{0,1,...n-1\} \text{ and all } k \in N,$$

and

$$|\overline{u}_k^{(n)}(t)| \le \varphi \ in \ (0,1).$$

Thus $\{\overline{u}_k^{(n)}\}_{k=1}^{\infty}$ is bounded in $L^2(0,1)$ and (up to a subsequence) tends weakly to a certain $v_n \in L^2(0,1)$. Hence, by (13), we can derive that

$\{\overline{u}_k^{(n-1)}\}_{k=1}^{\infty}$ (up to a subsequence) is weakly convergent in $H^1(0,1)$ to certain v_{n-1} such that $v_n = v'_{n-1}$. Applying the Rellich-Kondrashov theorem and the Sobolev embedding theorem we obtain

$$\overline{u}_k^{(n-1)} \rightrightarrows v_{n-1} \text{ in } [0,1] \text{ and } v_{n-1} \in C([0,1]),$$

(where " $\rightrightarrows$ " denotes the uniform convergence). Analogously, we can prove that $\{\overline{u}_k^{(n-2)}\}_{k=1}^{\infty}$ (up to a subsequence) is weakly convergent in $H^1(0,1)$ to certain v_{n-2} such that $v_{n-1} = v'_{n-2}$. Applying again above theorems we can derive that

$$\overline{u}_k^{(n-2)} \rightrightarrows v_{n-2} \text{ in } [0,1] \text{ and } v_{n-2} \in C([0,1]).$$

Iterating this process we infer that for each $l \in \{1,...,n\}$, there exists v_{n-l} such that $\{\overline{u}_k^{(n-l)}\}_{k=1}^{\infty}$ (up to a subsequence) is weakly convergent in $H^1(0,1)$ to v_{n-l} and $v_{n-(l-1)} = v'_{n-l}$. Therefore, we obtain

$$\overline{u}_k^{(n-l)} \rightrightarrows v_{n-l} \text{ in } [0,1] \text{ and } v_{n-l} \in C([0,1]) \text{ for all } l \in \{1,...,n\}.$$

Thus, taking $\overline{u}_0 := v_0$ we conclude that $\overline{u}_0 \in C^{(n-1)}([0,1])$ with $\overline{u}_0^{(n)} = v_n \in L^2(0,1)$, and for each $0 \le l \le n-1$, $\{\overline{u}_k^{(l)}\}_{k\in N}$ (up to a subsequence) converges uniformly to $\overline{u}_0^{(l)}$ in $[0,1]$. We also get $0 \le \overline{u}_0$ on $[0,1]$, $\overline{u}_0^{(l)}(0) = 0$ for all $0 \le l \le n-2$ and

$$\sum_{i=1}^{m} b_i \overline{u}_0^{(n-2)}(\alpha_i) - \overline{u}_0^{(n-2)}(1) = 0.$$

Since $\overline{u}_k \in X_{z_k}$ for each $k \in N$, we have $0 \le \overline{u}_k(t) \le dt^{n-1}$ on $[0,1]$, which implies the estimation $0 \le \overline{u}_0(t) \le dt^{n-1}$ on $[0,1]$.
Moreover for each $k \in \mathbf{N}$, $\overline{u}_k$ is the solution of (12) corresponding to the parameter z_k, namely

$$u_k^{(n)}(t) = -\widetilde{f}(t, \overline{u}_k(t), z_k(t)) \text{ in } (0,1).$$

Now we get that for all $h \in H_0^1(0,1)$,

$$\int_0^1 \overline{u}_0^{(n-1)}(t)h'(t)dt = \lim_{k\to\infty} \int_0^1 \overline{u}_k^{(n-1)}(t)h'(t)dt = \lim_{k\to\infty} \int_0^1 -\overline{u}_k^{(n)}(t)h(t)dt = \tag{14}$$
$$= \lim_{k\to\infty} \int_0^1 \widetilde{f}(t, \overline{u}_k(t), z_k(t))h(t)dt.$$

On the other hand we see that for $t \in (0,1)$

$$\lim_{k\to\infty} \widetilde{f}(t, u_k(t), z_k(t)) = \widetilde{f}(t, \overline{u}_0(t), z_0(t))$$

and, by **(f3z)**

$$0 \leq \widetilde{f}(t, u_k(t), z_k(t)) \leq \varphi \text{ in } (0,1).$$

Therefore we can apply the Lebesgue dominated convergence theorem and conclude that

$$\lim_{k\to\infty} \int_0^1 \widetilde{f}(t, \overline{u}_k(t), z_k(t))h(t)dt = \int_0^1 \widetilde{f}(t, \overline{u}_0(t), z_0(t))h(t)dt. \tag{15}$$

Assertions (14) and (15) give

$$\int_0^1 \overline{u}_0^{(n-1)}(t)h'(t)dt = \int_0^1 \widetilde{f}(t, \overline{u}_0(t), z_0(t))h(t)dt$$

for all $h \in H_0^1(0,1)$, and further, the du Bois-Reymond Lemma leads to the following equality

$$-\overline{u}_0^{(n)}(t) = \widetilde{f}(t, \overline{u}_0(t), z_0(t)) \text{ a. e. in } (0,1).$$

Since $\widetilde{f}$ is continuous, $\overline{u}_0^{(n)} \in C([0,1])$. Concluding, $\overline{u}_0$ is the classical positive solution of (12) with $z = z_0$ and satisfies the estimation $\overline{u}_0(t) \leq dt^{n-1}$ for all $t \in [0,1]$. □

4. Main result for optimal control problem

The last section is devoted to the sufficient conditions for the optimal control problem (1)-(2). We deal with the existence of an optimal pair (u_0, z_0) in the set $X\widetilde{Z}$ defined as

$$X\widetilde{Z} := \Big\{(u,z) \in C^n([0,1]) \times \widetilde{Z},\ 0 \leq u(t) \leq dt^{n-1} \text{ on } [0,1]$$
$$\text{and } u \text{ is a solution of (2) corresponding to parameter } z\Big\},$$

where $\widetilde{Z} := \{\ z : [0,1] \to K,\ z$ satisfies the Lipschitz condition with a fixed constant $L\}$, $L > 0$, K is a compact subset of R^s, and d is given in **(f1z)**. Suppose that **(BC1), (BC2), (f1z)**, **(f2z)** and **(f3z) -** with $Z = \widetilde{Z}$ - hold. Assume additionally the following conditions concerning the cost functional (1):

- **(F1)** *$F : [0,1] \times [0,d] \times R^s \to R$ is measurable with respect to the first variable for all $(u,z) \in [0,d] \times R^s$ and $F(t,\cdot,\cdot)$ is continuous in $[0,d] \times R^s$ for $t \in [0,1]$;*
- **(F2)** *there exists $\alpha \in L^1((0,1), R_+)$ such that for all $z \in K$ and $u \in [0,d]$*

$$|F(t,u,z)| \leq \alpha(t) \text{ a. e. in } [0,1].$$

Theorem 4. *Under the above assumptions, there exists* $(u_0, z_0) \in X\widetilde{Z}$ *such that*

$$J(u_0, z_0) = \min_{(u,z)\in X\widetilde{Z}} J(u, z).$$

Proof. Let us consider a minimizing sequence $\{(u_k, z_k)\}_{k\in N} \subset X\widetilde{Z}$ of functional $J : X\widetilde{Z} \to R$. Taking into account the fact that $\{z_k\}_{k\in N}$ is bounded and equicontinuous in $[0, 1]$, (by the definition of $\widetilde{Z}$) we can apply the Arzela-Ascoli theorem and obtain the existence of a subsequence, still denoted by $\{z_k\}_{k\in N}$, tending uniformly to $z_0 \in \widetilde{Z}$. Therefore, Theorem 3 leads to existence of a subsequence of $\{u_k\}_{k\in N}$, again denoted by $\{u_k\}_{k\in N}$, converging uniformly to $u_0 \in X$, where u_0 is a positive solution of (2) corresponding to parameter z_0 and such that $0 \le u_0(t) \le dt^{n-1}$. Due to the pointwise convergence of $\{(u_k, z_k)\}_{k\in N}$ to $(u_0, z_0) \in X\widetilde{Z}$ in $[0, 1]$ and assumption **(F1),** we infer that

$$\lim_{k\to\infty} F(t, u_k(t), z_k(t)) = F(t, u_0(t), z_0(t)) \text{ in } [0, 1]$$

and, by **(F2),**

$$|F(t, u_k(t), z_k(t))| \le \alpha(t) \text{ in } [0, 1].$$

Now, applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \int_0^1 F(t, u_k(t), z_k(t))dt = \int_0^1 F(t, u_0(t), z_0(t))dt.$$

Summarizing,

$$\min_{(u,z)\in X\widetilde{Z}} J(u, z) = \liminf_{k\to\infty} J(u_k, z_k) = J(u_0, z_0).$$

which ends the proof. □

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