Folia Mathematica
Vol. 16, No. 1, pp. 25–30

Acta Universitatis Lodziensis

ON DIFFERENTIABILITY OF ABSOLUTELY MONOTONE SET-VALUED FUNCTIONS

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Abstract. We prove that the existence of all Hukuhara derivatives $H^{(k)}(t)$ on $[0, b)$ containing zero is equivalent to the absolutely monotonicity of a set-valued function H on $[0, b)$ with nonempty convex compact values.

1. Let A and B be two subsets of a real vector space X. We define the *sum* of A and B by the formula

$$A + B := \{a + b : a \in A, b \in B\}.$$

Let X be a real normed vector space and let $cc(X)$ denote the family of all nonempty compact convex subsets of X. A set $C \in cc(X)$ is the *Hukuhara difference* of $A \in cc(X)$ and $B \in cc(X)$ if

$$A = B + C$$

(see [2]). If the difference $A-B$ exists, then it is unique. It is a consequence of the following lemma.

Lemma 1 (cf. [4]). *Let A, B and C be subsets of a topological vector space such that*

$$A + B \subset C + B.$$

If C is convex closed and B is nonempty bounded, then

$$A \subset C.$$

Now, let $-\infty < a < b \leq +\infty$ and let $H : [a, b) \to cc(X)$. We define the *p-th differences* $\Delta_s^p H(t)$ by the reccurence

$$\Delta_s^0 H(t) = H(t),$$

$$\Delta_s^{p+1} H(t) = \Delta_s^p H(t + s) - \Delta_s^p H(t)$$

for every nonnegative integer p, $t \in [a, b)$, $s > 0$ such that $t + (p + 1)s < b$.

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Key words and phrases: absolutely monotone set-valued functions, Hukuhara derivatives.

AMS subject classifications: 26A48, 26A51, 26E25.

A set-valued function is said to be *absolutely monotone* if there exist all differences $\Delta_s^p H(t)$ and each contains zero.

Example 1. *Assume that $A \in cc(X)$, $0 \in A$ and $h : [a,b) \to [0,\infty)$. Then $H(t) = h(t)A$ is absolutely monotone set-valued function if and only if h is an absolutely monotone real function.*

Example 2. *Assume that $f : [a,b) \to [0,\infty)$ and $g : [a,b) \to [0,\infty)$ are such that $f(t) \le 0 \le g(t)$ and $H(t) = [f(t), g(t)]$ for $t \in [a,b)$. Then the set-valued function H is absolutely monotone if and only if $-f$ and g are absolutely monotone.*

In [5] the following theorem was proved.

Theorem 1. *A set-valued function $H : [0,b) \to cc(X)$ is absolutely monotone if and only if there exist sets $A_n \in cc(X)$, $n = 0, 1, ...$ containing zero such that*

$$H(t) = \sum_{n=0}^{\infty} t^n A_n \tag{1}$$

for $t \in [0,b)$.

The convergence of the series in (1) is the convergence in $cc(X)$ with respect to the Hausdorff metric derived from the norm in X.

Assume that $H : [a,b) \to cc(X)$ is a set-valued function such that the differences $H(s) - H(t)$ exist if $t, s \in [a,b)$ whenever $t < s$. The Hukuhara derivative of H at t is defined by the formula

$$H'(t) = \lim_{s \to t+} \frac{H(s) - H(t)}{s - t} = \lim_{s \to t-} \frac{H(t) - H(s)}{t - s},$$

whenever both limits exist with respect to the Hausdorff metric d in $cc(X)$ derived from the norm in X. Moreover

$$H'(a) = \lim_{s \to a+} \frac{H(s) - H(a)}{s - a}.$$

Lemma 2 (cf. Lemma 5 in [6])**.** *Let $F, G : [\alpha, \beta] \to cc(X)$ be two differentiable set-valued functions such that $F'(t) = G'(t)$ for $t \in [\alpha, \beta]$ and $F(\alpha) = G(\alpha)$, then $F(t) = G(t)$ for $t \in [\alpha, \beta]$.*

A set-valued function $F : [\alpha, \beta] \to cc(X)$ is called *increasing* if $F(x) \subset F(y)$ for all $x, y \in [\alpha, \beta]$ such that $x < y$.

We say that a set-valued function $F : [\alpha, \beta] \to cc(X)$ is *concave* if

$$F((1-\lambda)x + \lambda y) \subset (1-\lambda)F(x) + \lambda F(y)$$

for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.

The Riemann type integral for set-valued functions was introduced by A. Dinghas in [1] (see also [2]). We have the following two lemmas.

Lemma 3 (cf. Theorem 2.6 in [3])**.** *If $G : [\alpha, \beta] \to cc(X)$ is an increasing set-valued function, $D \in cc(X)$ and*

$$F(s) = D + \int_{\alpha}^{s} G(u)du,$$

then F is a concave multifunction for which there exist all differences $F(s) - F(t)$ whenever $\alpha \le t < s \le \beta$.

Lemma 4 (cf. Theorem 1.3 in [3])**.** *If $F : [\alpha, \beta] \to cc(X)$ is integrable, α', β', a, b are real numbers such that $\alpha' < \beta'$, $a\alpha' + b = \alpha$, $a\beta' + b = \beta$, then*

$$\int_{\alpha}^{\beta} F(t)dt = a \int_{\alpha'}^{\beta'} F(au + b)du.$$

Our main result characterizes absolutely monotone set-valued functions.

Theorem 2. *Let X be a Banach space. A set-valued function $H : [0, b) \to cc(X)$ is absolutely monotone if and only if there exist the $k-th$ derivatives $H^{(k)}(t)$ containing zero for $k = 1, 2, ...$ and $t \in [0, b)$.*

Proof. Let $H : [0, b) \to cc(X)$ be an absolutely monotone set-valued function. It is of the form (1) by Theorem 1. Fix $t \in [0, b)$ and $c \in (t, b)$. Define

$$G(s) = \frac{H(s) - H(t)}{s - t}$$

for $s \in (t, c]$. Theorem 1 yields

$$G(s) = \sum_{n=1}^{\infty} (s^{n-1} + s^{n-2}t + ... + t^{n-1})A_n$$

for $s \in (t, c]$. It is clear that

$$\|(s^{n-1} + s^{n-2}t + ... + t^{n-1})A_n\| = (s^{n-1} + s^{n-2}t + ... + t^{n-1})\|A_n\| < \\ < nc^{n-1}\|A_n\|$$

for $s \in (t, c]$ and $n = 1, 2, ...$ (the norm $\|A\|$ of a set A is defined as $\sup\{\|a\| : a \in A\}$). Let

$$\lambda := \limsup_{n \to \infty} \sqrt[n]{\|A_n\|}.$$

Fix $d \in (c, b)$. The convergence of the series $\sum_{n=0}^{\infty} d^n A_n$ implies that there exists a constant $M > 0$ such that

$$d^n \|A_n\| \le M, n = 0, 1, 2, ...$$

Therefore

$$\lambda d \leq 1.$$

We see that

$$\limsup_{n\to\infty} \sqrt[n]{nc^{n-1}\|A_n\|} = c\lambda < d\lambda \leq 1.$$

By completeness of X the series

$$\sum_{n=0}^{\infty} nc^{n-1}A_n$$

is absolutely convergent and the series

$$\sum_{n=1}^{\infty}(s^{n-1} + s^{n-2}t + ... + t^{n-1})A_n$$

is absolutely and uniformly convergent for $s \in (t, c]$. The set-valued function $G(s)$ is continuous on $(t, c]$ with respect to Hausdorff metric in $cc(X)$. Since every function $s \mapsto (s^{n-1} + s^{n-2}t + ... + t^{n-1})A_n$ has the limit $nt^{n-1}A_n$ at $s = t$, the set-valued function G may be extended to the continuous (with respect to Hausdorff metric in $cc(X)$ derived from the norm in X) function on $[t, c]$ and

$$H'_+(t) = \lim_{s\to t+} \frac{H(s) - H(t)}{s - t} = \lim_{s\to t+} \sum_{n=0}^{\infty}(s^{n-1} + s^{n-2}t + ... + t^{n-1})A_n =$$

$$= \lim_{s\to t+} G(s) = G(t) = \sum_{n=0}^{\infty} nt^{n-1}A_n.$$

The proof of the formula

$$H'_-(t) = \lim_{s\to t-} \frac{H(s) - H(t)}{s - t} = \sum_{n=0}^{\infty} nt^{n-1}A_n$$

is similar.

We see that every absolutely monotone set-valued function $H : [0, b) \to cc(X)$ is differentiable and its derivative H' is absolutely monotone set-valued function. It is obvious that $0 \in H'(t)$. It follows by induction that $H^{(k)}(t)$ exists for every $k = 1, 2, \dots$ and $t \in [0, b)$ and $0 \in H^{(k)}(t)$ for the same k and t.

Conversely, suppose that there exist all derivatives $H^{(k)}(t)$ and

$$0 \in H^{(k)}(t) \tag{2}$$

for $k = 0, 1, ...,$ $t \in [0, b)$. They are continuous. Therefore these derivatives are integrable in Riemann sense and the set-valued function $s \mapsto$

$\int_0^s H^{(k+1)}(u)du$ is differentiable in $[0, b)$ and

$$\frac{d}{ds}\left(\int_0^s H^{(k+1)}(u)du\right) = H^{(k+1)}(s)$$

(see [2] p. 216). This implies that for any $t \in [0, b)$ the set-valued function $\Phi(s) := H^{(k)}(t) + \int_t^s H^{(k+1)}(u)du$ is differentiable in $[t, b)$ and

$$\frac{d}{ds}\Phi(s) = H^{(k+1)}(s) = \frac{d}{ds}H^{(k)}(s).$$

Since $\Phi(t) = H^{(k)}(t)$, Lemma 2 shows that $\Phi(s) = H^{(k)}(s)$ for all $s \in [t, b)$. Consequently there exist the Hukuhara differences

$$H^{(k)}(s) - H^{(k)}(t) = \int_t^s H^{(k+1)}(u)du \tag{3}$$

for $0 \le t < s < b$. From $0 \in H^{(k+1)}(u)$ we have

$$0 \in H^{(k)}(s) - H^{(k)}(t)$$

and hence

$$H^{(k)}(t) \subset H^{(k)}(s)$$

for $0 \le t < s < b$ and $k = 0, 1, \ldots$ Thus all $H^{(k)}$ are concave set-valued functions according to Lemma 3.

There exists the difference $\Delta_h^1 H(t)$ for $0 \le t < b$ and $0 < h < b - t$. By formula (3)

$$\Delta_h^1 H(t) = \int_t^{t+h} H'(u)du \tag{4}$$

for the same t and h. Since $0 \in H'(u)$ for every u, we have that

$$0 \in \Delta_h^1 H(t)$$

whenever $0 \le t < t + h < b$. We conclude from (4) and Lemma 4 that

$$\begin{aligned}\Delta_h^2 H(t) = \Delta_h^1 H(t+h) - \Delta_h^1 H(t) &= \int_{t+h}^{t+2h} H'(u)du - \int_t^{t+h} H'(u)du = \\ &= \int_t^{t+h} H'(u+h)du - \int_t^{t+h} H'(u)du = \\ &= \int_t^{t+h} (\Delta_h^1 H'(u) + H'(u))du - \int_t^{t+h} H'(u)du = \\ &= \int_t^{t+h} \Delta_h^1 H'(u)du\end{aligned}$$

for t,h such that $0 \le t < t+2h < b$. By (2) we have

$$0 \in \int_t^{t+h} \left(\int_u^{u+h} H''(v)dv \right) du = \int_t^{t+h} \Delta_h^1 H'(u)du = \Delta_h^2 H(t).$$

Now assume that there exist differences $\Delta_h^1 H(t), ..., \Delta_h^k H(t)$, $0 \in \Delta_h^i H(t)$, $i = 1, \dots, k$ and

$$\Delta_h^k H(t) = \int_t^{t+h} \Delta_h^{k-1} H'(u)du$$

for t and h such that $t \in [0,b)$ and $t + kh < b$. Suppose that $t \in [0,b)$ and $t + (k+1)h < b$. On account of the assumption we have

$$\Delta_h^k H(t+h) = \int_{t+h}^{t+2h} \Delta_h^{k-1} H'(u)du = \int_t^{t+h} \Delta_h^{k-1} H'(u+h)du.$$

Therefore

$$\begin{aligned}\Delta_h^{k+1} H(t) &= \Delta_h^k H(t+h) - \Delta_h^k H'(t) = \\ &= \int_t^{t+h} \Delta_h^k H'(u+h)du - \int_t^{t+h} \Delta_h^k H'(u)du = \\ &= \int_t^{t+h} \Delta_h^{k+1} H'(u)du.\end{aligned}$$

By induction we can prove that

$$\Delta_h^{k+1} H(t) = \int_t^{t+h} \left(\int_{u_1}^{u_1+h} \dots \left(\int_{u_{k-1}}^{u_{k-1}+h} \left(\int_{u_k}^{u_k+h} H^{(k+1)}(v)dv \right) du_k \right) \dots \right) du_1.$$

This implies that

$$0 \in \Delta_h^{k+1} H(t)$$

for $k = 0, 1, ..., t \in [0,b)$ and $0 < h$ such that $t + (k+1)h < b$. Thus H is absolutely monotone. □

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