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SOME REMARKS ON NOWHERE MONOTONE FUNCTIONS

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Abstract. We prove that for each set E of the first category the typical (in the sense of Baire) function in $C[0,1]$ is nowhere monotone and one-to-one on E.

A function defined on the closed interval I is said to be nowhere monotone on I if it is monotone on no subinterval of I.

The properties of continuous nowhere monotone functions were investigated by well-known mathematicians as K. Padmavally, S. Marcus, K.M. Garg, A.M. Bruckner and others. J.B. Brown, U.B. Darji and E.P. Larsen investigated some kinds of monotonicity on no interval and at no point.

We also considered the continuous nowhere monotone functions and investigated how large a set can be on which a continuous nowhere monotone function is one-to-one. In [2] the first named author proved that if $f : [a,b] \to \mathbb{R}$ is a continuous nowhere monotone function then each set E having the Baire property such that $f_{|E}$ is one-to-one is of the first category. For the Lebesgue measure the situation is more complicated. For each $\eta \in [0,1)$ we constructed a continuous nowhere monotone function which is one-to-one on some set of measure η and every measurable set on which this function is one-to-one has measure not greater than η.

In [3] we considered the σ-ideal of microscopic sets, which is situated between the countable sets and the sets of Hausdorff dimension zero and proved that a typical function in $C[0,1]$ (in the sense of Baire) is nowhere monotone and one-to-one except on some microscopic set.

In this note we answer the question of Z. Kominek if for each set of the first category there exists a continuous nowhere monotone function which is one-to-one on this set.

Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{N}$ the set of all positive integers.

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Let $f : [a, b] \to \mathbb{R}$ be a continuous and non-constant function. Let us put

$$B_f = \{y \in f([a, b]) : f^{-1}(\{y\}) \text{ is of the first category}\}$$

and

$$C_f = f^{-1}(B_f).$$

We have the following results:

Theorem 1 ([2], Theorem 8)**.** *If a function $f : [a, b] \to \mathbb{R}$ is continuous and the set C_f is of the first category then each set E such that $f_{|E}$ is one-to-one is of the first category.*

Theorem 2 ([2], Theorem 9)**.** *If the set C_f is not of the first category then there exists a set $E \subset [a, b]$ such that E is not of the first category and $f_{|E}$ is one-to-one.*

The proof of the last theorem requires the continuum hypothesis CH (or some weaker condition like Martins Axiom MA).

Theorem 3 ([2], Theorem 6)**.** *If $f : [a, b] \to \mathbb{R}$ is a continuous and nowhere monotone function, then each set E having the Baire property and such that $f_{|E}$ is one-to-one is of the first category.*

Now let f be a continuous nowhere monotone function defined on $[a, b]$. K.M. Garg in [G] proved that for such a function the set $f^{-1}(\{y\})$ is nowhere dense for each y. Consequently, $C_f = [a, b]$ and from Theorem 2 we have the following:

Theorem 4 (CH)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous and nowhere monotone function. There exists a set $E \subset [a, b]$ such that E is not of the first category and $f_{|E}$ is one-to-one.*

The set E in the last theorem does not have the Baire property because as it was observed in Theorem 3 each set having the Baire property on which a continuous nowhere monotone function is one-to-one is of the first category.

Z. Kominek asked if for each set of the first category there exists a continuous and nowhere monotone function which is one-to-one on this set. We shall prove that the answer is positive. Moreover, using Cantor-Bendixson Theorem we shall prove that for each set E of the first category the set of all continuous and nowhere monotone functions which are one-to-one on E is residual in $C[0, 1]$ For this purpose we need the following lemma.

Lemma 1 ([3], Lemma 10)**.** *Let $K \subset [a, b]$ be an arbitrary non-empty perfect nowhere dense set. The set of all continuous functions which are one-to-one on K is residual in $C[a, b]$.*

Theorem 5. *For each set $E \subset [0,1]$ of the first category the set of all continuous nowhere monotone functions which are one-to-one on E is residual in $C[0,1]$.*

Proof. Let E be a set of the first category. Then $E \subset \bigcup_{n=1}^{\infty} P_n$, where P_n is closed and nowhere dense for each $n \in \mathbb{N}$. Using Cantor-Bendixson Theorem we obtain that $P_n = F_n \cup N_n$, where F_n is perfect nowhere dense and N_n is a countable set, for each $n \in \mathbb{N}$. Put

$$N = \bigcup_{n=1}^{\infty} N_n = \{x_1, x_2, ...\}.$$

Let n be a positive integer. We have two cases:

(1) If $x_n \in F_n$ then put $L_n = F_n$.
(2) If $x_n \notin F_n$ then put $L_n = F_n \cup C_n$, where C_n is a Cantor-like set such that $x_n \in C_n$.

Let us consider $K_n = \bigcup_{i=1}^{n} L_i$ for $n \in \mathbb{N}$. Since each K_n is non-empty, perfect nowhere dense set, then from Lemma 1 we obtain that for any $n \in \mathbb{N}$ the set Ω_n of all continuous functions which are one-to-one on K_n is a residual in $C[0,1]$. Obviously, the intersection $\bigcap_{n=1}^{\infty} \Omega_n$ is also residual set in $C[0,1]$. Moreover, each function from this set is one-to-one on E. Indeed, let $f \in \bigcap_{n=1}^{\infty} \Omega_n$, $x_1, x_2 \in E$ and $x_1 \neq x_2$. Obviously $E \subset \bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} (F_n \cup N_n) \subset \bigcup_{n=1}^{\infty} L_n = \bigcup_{n=1}^{\infty} K_n$. Since the sequence $\{K_n\}_{n \in \mathbb{N}}$ is ascending, so there exists n_0 such that $x_1, x_2 \in K_{n_0}$. Consequently, $f(x_1) \neq f(x_2)$ because $f \in \Omega_{n_0}$.

On the other hand it is well-known that the set of all continuous nowhere monotone functions is also residual in $C[0,1]$, since a typical (in the sense of Baire) continuous function is nowhere differentiable. Consequently, a typical continuous function is nowhere monotone and one-to-one on E. □

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