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THE EXISTENCE OF SOLUTIONS FOR DIRICHLET PROBLEM WITH NONCONVEX NONLINEARITY

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Abstract. We consider the existence of solutions for a nonlinear Dirichlet problem

$$\Delta x(y) + x(y) = F_x(y, x(y))$$

a.e. $x \mid \Omega = 0$ where $\Omega \subset R^k$ and F is nonconvex. We introduce a dual variational method with the aid of which the existence result is obtained.

1. Introduction

The aim of the paper is to prove the existence of solutions to the following Dirichlet problem

$$\Delta x(y) + x(y) = F_x(y, x(y)),\ x : \Omega \to R \tag{1}$$

where Ω is a locally Lipschitz.

The aim of the paper is to study a certain type of nonlinear, superlinear elliptic problems, i.e. to prove that the solution exists and investigate duality relations between various action functionals. On the nonlinearity we impose some general nonrestrictive conditions.

As concerns the existence of solutions, the similar problems are considered in [3], [4], [6], [8], [9], [10], [13], [14] by other methods depending on a topological argument or a different variational approach. The dual variational method allows one to consider the growth condition that cannot be treated by the classical variational method and also by dual least action principle see for example. The main difference between our result and some other results cited is that we do not assume the convexity of F. Once this assumption is lacking other methods cannot be applied.

Instead we require that set Ω is locally lipschitz subset of R^k, functional F satisfies:

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(A1) $F : \Omega \times R \to R$ is measurable with respect to the first variable and finite,

(A2) $x \mapsto \frac{1}{2}|x|^2 - F(y,x)$ is convex and lower semicontinous for a.e. $y \in \Omega$,

(A3) there exists a Carathedory function K, $K : \Omega \times R \to R$, e.g.measurable with respect to the first variable and continous with respect to the second one and such that for all $x \in R$ and a.e. $y \in \Omega$

$$\tfrac{1}{2}|x|^2 - F(y,x) \geq K(y,x).$$

Thorought the paper we assume that (A1), (A2), (A3) hold.

Following function satisfies above conditions:

$$F(y,x) = -e^x f(y) + \tfrac{1}{4}x^2,$$

where $f(y) \in L^\infty$, $f(y) > 0$.

2. Duality results

In order to prove the existence of solutions we have first to develop a duality theory i.e. the theory which relates the critical points and the critical values to the action functional

$$J(x) = \int_\Omega \left\{ \tfrac{1}{2}|\nabla x(y)|^2 - \tfrac{1}{2}|x(y)|^2 + F(y, x(y)) \right\} dy \tag{2}$$

i.e. such a functional for which equation (1) is Euler-Lagrange equation and dual functionals. We will use two dual functionals which will be defined below. In the sequel we will use the following sets:

$A = \{\nabla x \in L^2(\Omega, R^n) : x \in W_0^{1,2}(\Omega, R)\}$,

$B = \left\{p \in L^2(\Omega, R^n) : \text{there exists } \operatorname{div} p \in L^2(\Omega, R)\right\}$,

$C = \left\{p \in W^{1,2}(\Omega, R^n) : \operatorname{div} p(y) = 0 \quad \text{for a.e. } y \in \Omega\right\}$,

$D = \left\{\operatorname{div} p \in L^2(\Omega, R) : p \in B\right\}$.

Let

$$G(y,v) = \sup_{x \in R} \left\{ \langle x, v \rangle - \tfrac{1}{2}|x|^2 + F(y,x) \right\}, \tag{3}$$

for a.e. $y \in R$, $v \in R$ denote the Fenchel-Young dual of a convex function $x \mapsto \frac{1}{2}|x|^2 - F(y,x)$. Hence G itself is convex and lower semicontinous for a.e. $y \in \Omega$.

Definition 1. *The dual functional $J_D : B + C \to R$ is given by*

$$J_D(p+v) = \int_\Omega \left\{ -\tfrac{1}{2}|p(y) + v(y)|^2 + G(y, -\operatorname{div}(p(y))) \right\} dy. \tag{4}$$

Definition 2. *The Clarke dual functional is defined by*

$$J_C(x, \operatorname{div} p) = \int_\Omega \left\{ \langle x(y), \operatorname{div} p(y)) \rangle + G(y, -\operatorname{div} p(y)) + \tfrac{1}{2}|\nabla x(y)|^2 \right\} dy$$

where $x \in W_0^{1,2}(\Omega, R)$*,* $\operatorname{div} p \in L^2(\Omega, R)$.

In order to avoid the calculations of Fenchel-Young transform with respect to a dual space we introduce a perturbation function at $J_x : L^\infty(\Omega, R) \to R$ by the formula

$$J_x(g) = \int_\Omega \left\{ -\tfrac{1}{2}|\nabla x(y)|^2 + \tfrac{1}{2}|x(y) + g(y)|^2 + F(y, x(y) + g(y)) \right\} dy. \tag{5}$$

and for $x \in W_0^{1,2}(\Omega, R)$ we consider a type of Fenchel-Young dual

$$J_x^\#(p) = \sup_{g \in L^2(\Omega,R)} \int_\Omega \{ \langle -\operatorname{div} p(y), g(y) \rangle - F(y, x(y) + g(y)) + \\ + \tfrac{1}{2}|\nabla x(y)|^2 - \tfrac{1}{2}|x(y) + g(y)|^2 \} dy.$$

Using (3) we obtain

$$J_x^\#(p) = \int_\Omega \left\{ G(y, -\operatorname{div} p(y)) + \tfrac{1}{2}|\nabla x(y)|^2 - \langle -\operatorname{div} p(y), x(y) \rangle \right\} dy. \tag{6}$$

Using (6),(4) we get for any $p \in B$:

$$\sup_{x \in W_0^{1,2}(\Omega,R)} (-J_x^\#(-p)) =$$
$$= \sup_{\nabla x \in A} \int_\Omega \left\{ -G(y, -\operatorname{div} p(y)) - \tfrac{1}{2}|\nabla x(y)|^2 + \langle p(y), \nabla x(y) \rangle \right\} dy =$$
$$= \inf_{v \in C} \int_\Omega \left\{ \tfrac{1}{2}|p(y) + v(y)|^2 - G(y, -\operatorname{div} p(y)) \right\} dy = \inf_{v \in C} -J_D(p + v).$$

Let us introduce the second dual by the formula $J_x^{\#\#} : L^2(\Omega, R) \to R$, where $x \in W_0^{1,2}(\Omega, R)$:

$$J_x^{\#\#}(g) = \sup_{h \in D} \int_\Omega \left\{ \langle g(y), h(y) \rangle + \langle x(y), h(y) \rangle - \tfrac{1}{2}|\nabla x(y)|^2 - G(y, h(y)) \right\} dy.$$

Observe that for any $x \in W_0^{1,2}(\Omega, R)$ we have $J_x^{\#\#}(0) = -J(x)$. Indeed by density of D w $L^2(\Omega, R)$ we get:

$$J_x^{\#\#}(0) = \sup_{h \in L^2(\Omega,R)} \int_\Omega \left\{ \langle x(y), h(y) \rangle - \tfrac{1}{2}|\nabla x(y)|^2 - G(y, h(y)) \right\} dy =$$

$$= \int_\Omega \left\{ -\tfrac{1}{2}|\nabla x(y)|^2 + \tfrac{1}{2}|x(y)|^2 - F(y, x(y)) \right\} dy = -J(x).$$

Now we may prove two duality principles which relate the critical values to the action and the dual functionals.

Theorem 1. *Duality Principle I*

$$\inf_{x \in W_0^{1,2}(\Omega,R)} J(x) = \inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p+v).$$

Proof. By definitions of C and D we get:

$$\sup_{x \in W_0^{1,2}(\Omega,R)} J_x^{\#\#}(0) =$$

$$= \sup_{x \in W_0^{1,2}(\Omega,R)} \sup_{\operatorname{div} p \in D} \int_\Omega \left\{ \langle x(y), -\operatorname{div} p(y) \rangle - G(y, -\operatorname{div} p(y)) - \tfrac{1}{2}|\nabla x(y)|^2 \right\} dy =$$

$$= \sup_{\operatorname{div} p \in D} \sup_{x \in W_0^{1,2}(\Omega,R)} (-J_x^{\#}(-p)) = \sup_{\operatorname{div} p \in D} \inf_{v \in C} (-J_D(p+v)) =$$

$$= - \inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p+v).$$

From the above and equality $J_x^{\#\#}(0) = -J(x)$ we obtain:

$$\inf_{x \in W_0^{1,2}(\Omega,R)} J(x) = \inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p+v).$$

□

Theorem 2. *Duality Principle II*

$$\inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p+v) = \inf_{x \in W_0^{1,2}(\Omega,R),\, \operatorname{div} p \in D} J_C(x, \operatorname{div} p).$$

Proof. For $\operatorname{div} p \in D$ we have:

$$\inf_{x \in W_0^{1,2}(\Omega,R)} J_C(x, \operatorname{div} p) =$$

$$= \inf_{x \in W_0^{1,2}(\Omega,R)} \int_\Omega \left\{ \langle x(y), \operatorname{div} p(y)) \rangle + \tfrac{1}{2}|\nabla x(y)|^2 \right\} dy + \int_\Omega G(y, -\operatorname{div} p(y)) dy =$$

$$= - \sup_{\nabla x \in A} \int_\Omega \left\{ \langle \nabla x(y), p(y) + v_p \rangle - \tfrac{1}{2}|\nabla x(y)|^2 \right\} dy + \int_\Omega G(y, -\operatorname{div} p(y)) dy.$$

The first term is a Fenchel-Young conjugate of a function $f(y) \mapsto \frac{1}{2}\int\limits_{\Omega} |f(y)|^2 dy$ calculated at $p + v_p$ where

$$v_p = \operatorname{argmin}_{v\in C} \int\limits_{\Omega} |p(y) + v|^2 dy$$

This conjugate reads $\frac{1}{2}\int\limits_{\Omega} |p(y) + v_p|^2 dy$. Finally

$$\inf_{x\in W_0^{1,2}(\Omega,R)} J_C(x, \operatorname{div} p) = J_D(p + v_p) = \sup_{v\in C} J_D(p + v)$$

and therefore

$$\inf_{x\in W_0^{1,2}(\Omega,R), \operatorname{div} p\in L^2(\Omega,R)} J_C(x, \operatorname{div} p) = \inf_{\operatorname{div} p\in D} \sup_{v\in C} J_D(p + v).$$

□

Now we prove the lemma which allows one to estimate from below the functional G from the definition of J_D.

Lemma 1. *Let $G : \Omega \times R \to R$, be given by*

$$G(y, v) = \sup_{x\in R} \left(\langle x, v\rangle - \tfrac{1}{2}|x|^2 + F(y, x)\right),\ y \in \Omega, v \in R$$

and let F be measurable with respect to the first variable and finite. Then for any $v \in R$ and a.e. $y \in \Omega$

$$G(y, v) \geq F^{**}(y, v) + \tfrac{1}{2}|v|^2$$

where F^{**} denotes the second conjugate of $F(y, \cdot)$.

Proof. From Fenchel-Young inequality we have

$$F^*(y, z) \geq \langle x, z\rangle - F(y, x)$$

for any $y \in \Omega, x, z \in R$.

$$F^*(y, z) + \tfrac{1}{2}|z - x|^2 \geq \tfrac{1}{2}|x|^2 - F(y, x)$$

for any $y \in \Omega, x, z \in R$.

Hence

$$\inf_{z\in R} \left\{F^*(y, z) + \tfrac{1}{2}|z - x|^2\right\} \geq \tfrac{1}{2}|x|^2 - F(y, x).$$

Denoting $g(x) = \frac{1}{2}|x|^2$ we get

$$\inf_{z} \{F^*(y, z) + g(x - z)\} \geq \tfrac{1}{2}|x|^2 - F(y, x)$$

for $z \in R$.

Denoting $(f \oplus g)(u) = \inf_{x \in X}(f(x) + g(u-x))$ dla $u \in X$ we get

$$(F_y^* \oplus g)(x) \geq \tfrac{1}{2}|x|^2 - F(y,x)$$

for any $x \in R$.

From properties of conjugate we have for any $y \in \Omega, v \in R$

$$(F_y^* \oplus g)^*(v) \leq G(y,v).$$

So

$$F^{**}(y,v) + g^*(v) \leq G(y,v).$$

Because $g^*(v) = \frac{1}{2}|v|^2$ it follows $y \in \Omega, v \in R$:

$$G(y,v) \geq F^{**}(y,v) + \tfrac{1}{2}|v|^2.$$

□

3. Variational principles

In this section we formulate and prove the variational principles which relate the critical points to the action and the dual action functional.

Theorem 3. *The pair* $(\bar{x}, \operatorname{div} \bar{p})$ *minimizes* J_C *on* $W_0^{1,2}(\Omega, R) \times D$ *iff* $\operatorname{div} \bar{p}$ *minimizes* $\sup_{v \in C} J_D(p+v)$ *on* D, $\bar{x}$ *minimizes* J *na* $W_0^{1,2}(\Omega, R)$ *and at least one of the following conditions hold:*

$$J(\bar{x}) - J_{\bar{x}}^*(-\bar{p} - \bar{v}) = 0 \tag{7}$$

$$J_D(\bar{p} + \bar{v}) - J_{\bar{x}}^*(-\bar{p} - \bar{v}) = 0 \tag{8}$$

here $J_{\bar{x}}^*$ *is Fenchel-Young conjugate of* $J_{\bar{x}}$ *and* $\bar{v} = arg\,sup\, J_D(\bar{p} + v)$ *for* $v \in C$.

Proof. Assume first that $(\bar{x}, \operatorname{div} \bar{p})$ minimizes J_C on $W_0^{1,2}(\Omega, R) \times D$. By (1) and (2) we get

$$J_C(\bar{x}, \operatorname{div} \bar{p}) = \inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p+v) \tag{9}$$

and

$$\inf_{x \in W_0^{1,2}(\Omega,R)} J_C(x, \operatorname{div} \bar{p}) = \sup_{v \in C} J_D(\bar{p} + v). \tag{10}$$

Since by $\inf_{x \in W_0^{1,2}(\Omega,R)} J_C(x, \operatorname{div} \bar{p}) \leq J_C(\bar{x}, \operatorname{div} \bar{p})$, it follows from (9) and (10) that

$$J_D(\bar{p} + \bar{v}) = \inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p+v).$$

Similiary for $x \in W_0^{1,2}(\Omega, R)$.

$$\inf_{\operatorname{div} p \in D} J_C(x, \operatorname{div} p) =$$

$$= \inf_{\operatorname{div} p \in D} \int_\Omega \{\langle x(y), \operatorname{div} p(y)\rangle + G(y, -\operatorname{div} p(y))\}\, dy + \tfrac{1}{2}\int_\Omega |\nabla x(y)|^2 dy =$$

$$= -\sup_{\operatorname{div} p \in D} \int_\Omega \{\langle x(y), -\operatorname{div} p(y)\rangle - G(y, -\operatorname{div} p(y))\}\, dy + \tfrac{1}{2}\int_\Omega |\nabla x(y)|^2 dy.$$

By the assumptions on F and definition of G we have

$$\inf_{\operatorname{div} p \in D} J_C(x, \operatorname{div} p) = \int_\Omega \left\{F(y, x(y)) - \tfrac{1}{2}|x(y)|^2\right\} dy + \tfrac{1}{2}\int_\Omega |\nabla x(y)|^2 dy = J(x).$$

It follows

$$\inf_{\operatorname{div} p \in D} J_C(\bar{x}, \operatorname{div} p) = J(\bar{x}). \tag{11}$$

By duality principle and the assumption we have:

$$J_C(\bar{x}, \operatorname{div} \bar{p}) = \inf_{x \in W_0^{1,2}(\Omega, R)} J(x). \tag{12}$$

Since $\inf_{\operatorname{div} p \in D} J_C(\bar{x}, \operatorname{div} p) \le J_C(\bar{x}, \operatorname{div} \bar{p})$ then (11) i (12) follows that

$$J(\bar{x}) = \inf_{x \in W_0^{1,2}(\Omega, R)} J(x).$$

In a consequence

$$J(\bar{x}) = J_C(\bar{x}, \operatorname{div} \bar{p}) = J_D(\bar{p} + \bar{v}).$$

Since $J_C(\bar{x}, \operatorname{div} \bar{p}) = J^*_{\bar{x}}(-\bar{p} - \bar{v})$ and by the above we have (7) and (8).

Let us assume that $\bar{x}$ minimizes J on $W_0^{1,2}(\Omega, R)$, $\operatorname{div} \bar{p}$ minimizes $\sup_{v \in C} J_D(p + v)$ on D, $J(\bar{x}) = J_D(\bar{p} + \bar{v})$ and (7) holds.

Since by (7) it follows that $J(\bar{x}) = J^*_{\bar{x}}(-\bar{p} - \bar{v}) > -\infty$ and $J_C(\bar{x}, \operatorname{div} \bar{p}) = J^*_{\bar{x}}(-\bar{p} - \bar{v})$ we have that $J_C(\bar{x}, \operatorname{div} \bar{p})$ is finite. By duality principle we have

$$J(\bar{x}) = \inf_{x \in W_0^{1,2}(\Omega, R), \operatorname{div} p \in D} J_C(x, \operatorname{div} p)$$

From the above it follows

$$\inf_{x \in W_0^{1,2}(\Omega, R), \operatorname{div} p \in D} J_C(x, \operatorname{div} p) = J_C(\bar{x}, \operatorname{div} \bar{p}).$$

Assuming (8) we have that

$$J_D(\bar{p} + \bar{v}) = J(\bar{x}) = J^*_{\bar{x}}(-\bar{p} - \bar{v}) = J_C(\bar{x}, \operatorname{div} \bar{p})$$

is finite. On the other hand

$$J_D(-\bar{p}-\bar{v}) = \inf_{x\in W_0^{1,2}(\Omega,R),\operatorname{div} p\in D} J_C(x,\operatorname{div} p)$$

and the lemma is proved. □

Theorem 4. *Let $\bar{x} \in W_0^{1,2}(\Omega, R)$ be such that*

$$-\infty < J(\bar{x}) = \inf_{x\in W_0^{1,2}(\Omega,R)} J(x) < \infty.$$

Then there exists $\bar{p} = \tilde{p} + \bar{v} \in B + C$ such that:

$$J_D(\bar{p}) = \inf_{\operatorname{div} p\in D} \sup_{v\in C} J_D(p+v).$$

Moreover

$$J_{\bar{x}}^{\#}(-\bar{p}) + J_{\bar{x}}(0) = 0,$$
$$J_{\bar{x}}^{\#}(-\bar{p}) - J_D(\bar{p}) = 0.$$

Proof. By remarks following relation (3) we see that functional

$$M(z) = -\int_\Omega \{\langle \bar{x}(y), -z(y)\rangle - G(y,-z(y))\}\, dy \quad , \; z \in L^2(\Omega,R)$$

is convex and lower semicontinous. By (3) we have

$$G(y,-z(y)) = \sup_{x\in L^2(\Omega,R)} \left\{\langle x(y), -z(y)\rangle - \tfrac{1}{2}|x(y)|^2 + F(y,x(y))\right\},$$

so by (A3) we obtain that M is coercive.

Since $L^2(\Omega,R)$ is reflexive it follows functional M attains infimum at a certain point $\bar{z} \in L^2(\Omega,R)$. On the other hand we have

$$\sup_{z\in L^2(\Omega,R)} (-M(z)) = \sup_{z\in L^2(\Omega,R)} \int_\Omega \{\langle \bar{x}(y), -z(y)\rangle - G(y,-z(y))\}\, dy =$$

$$= \int_\Omega \left\{F(y,\bar{x}(y)) - \tfrac{1}{2}|\bar{x}|^2\right\} dy.$$

By the above it follows

$$\begin{aligned}&\int_\Omega \left\{F(y,\bar{x}(y)) - \tfrac{1}{2}|\bar{x}|^2 - \tfrac{1}{2}|\nabla\bar{x}(y)|^2)\right\} dy + \\ &+ \int_\Omega \left\{\langle \bar{x}(y), \bar{z}(y)\rangle + G(y,-\bar{z}(y)) + \tfrac{1}{2}|\nabla\bar{x}(y)|^2\right\} dy = 0.\end{aligned} \tag{13}$$

Let $\tilde{p} \in B$ be such that $\operatorname{div} \tilde{p} = \bar{z}$. We may put

$$\tilde{p}(y) = \left(\int_{\alpha_1}^{y_1} \bar{z}(t, y_2, ..., y_n) dt, \underbrace{0, ..., 0}_{n-1} \right),$$

where $y = (y_1, ..., y_n)$, and $\alpha = (\alpha_1, ..., \alpha_n)$ is fixed in region Ω). Define $\tilde{p}_v = \tilde{p} + v$, where $v \in C$. We have $\operatorname{div} \tilde{p}_v = \operatorname{div} \tilde{p} = \bar{z}$. In a consequence by definitions of $J_{\bar{x}}^{\#}$ and $J_{\bar{x}}$, and by (13) we have

$$J_{\bar{x}}^{\#}(-\tilde{p}_v) + J_{\bar{x}}(0) = 0. \tag{14}$$

Moreover $J_{\bar{x}}^{*}(-\operatorname{div} \tilde{p}_v) = J_{\bar{x}}^{\#}(-\tilde{p}_v)$. Indeed

$$J_{\bar{x}}^{*}(-\operatorname{div} \tilde{p}_v) =$$

$$= \sup_{g \in L^2(\Omega, R)} \int_{\Omega} \left\{ \langle -\operatorname{div} \tilde{p}_v(y), g(y) \rangle + \tfrac{1}{2} |\nabla \bar{x}(y)|^2 - F(y, \bar{x}(y) + g(y)) \right\} dy =$$

$$= \int_{\Omega} \left\{ \tfrac{1}{2} |\nabla \bar{x}(y)|^2 + \langle -\operatorname{div} \tilde{p}_v(y), \bar{x}(y) \rangle + G(y, -\operatorname{div} \tilde{p}_v(y)) \right\} dy = J_{\bar{x}}^{\#}(-\tilde{p}_v).$$

Observe now that since C is a closed subspace of a reflexive space it itself is reflexive. Hence the mapping

$$C \ni v \mapsto -\int_{\Omega} \left\{ -\tfrac{1}{2} |\tilde{p}(y) + v(y)|^2 + G(y, -\operatorname{div} \tilde{p}(y)) \right\} dy$$

is convex, l.s.c. and coercive. It attains its infimum at certain $\bar{v} \in C$. Since $J_{\bar{x}}(0) = -J(\bar{x})$, and by (14) we have

$$-J(\bar{x}) = -J_{\bar{x}}^{\#}(-\tilde{p}_v) \le \sup_{x \in W_0^{1,2}(\Omega, R)} (-J_{\bar{x}}^{\#}(-\tilde{p}_v)) = -\sup_{v \in C} J_D(\tilde{p} + v) = -J_D(\bar{p}),$$

where $\bar{p} = \tilde{p} + \bar{v}$. Since $\operatorname{div} \bar{p} = \operatorname{div} \tilde{p}$, z (14) we have

$$J_{\bar{x}}^{\#}(-\bar{p}) + J_{\bar{x}}(0) = 0.$$

By the above and the duality principle we have

$$J_D(\bar{p}) = \inf_{\operatorname{div} p \in D} \sup_{v \in C} J_D(p + v).$$

Now the equality $J_{\bar{x}}^{\#}(-\bar{p}) - J_D(\bar{p}) = 0$ is a consequence of the equality $J_D(\bar{p}) = J(\bar{x}) = -J_{\bar{x}}(0)$.

□

Now we provide a certain version of the above results which is valid for minimizing sequences. We present the dual version of the so called

ε-variational principle. In order to do so we introduce the perturbation functional defined by

$$J_{D_p}(h) = \int_\Omega \left\{ \tfrac{1}{2}|p(y) + h(y)|^2 - G(y, -\operatorname{div} p(y)) \right\} dy, \tag{15}$$

for any $p \in B$.

Theorem 5. *Let $\{p_n\}_{n\in N} \subset B$ be a minimizing sequence J_D on B and let $-\infty < \inf_{n\in N} J_D(p_n) < \infty$, $n \in N$. Let $n \in N$ element $x_n \in W_0^{1,2}(\Omega, R)$, be such that*

$$\nabla x_n = p_n$$

then $\{x_n\}_{n\in N} \subset W_0^{1,2}(\Omega, R)$ is a minimizing sequence for J on $W_0^{1,2}(\Omega, R)$ i.e.

$$\inf_{n\in N} J(x_n) = \inf_{x\in W_0^{1,2}(\Omega,R)} J(x),$$

Moreover for any $n \in N$

$$J_{D_{p_n}}(0) + J^{\#}_{x_n}(-p_n) = 0 \tag{16}$$

and for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for any $n > n_0$

$$J^{\#}_{x_n}(-p_n) - J(x_n) \le \varepsilon$$

$$J_D(-p_n) - J(x_n) \le \varepsilon.$$

Proof. We have

$$\inf_{\operatorname{div} p\in D)} J_D(p) = \inf_{n\in N} J_D(p_n) = c > -\infty.$$

By the definition of minimum we have that for $\forall\varepsilon > 0$ $\exists n_0 \in N$ such that for any $n > n_0$ $\varepsilon > J_D(p_n) - c$.
Since x_n is chosen so that

$$\nabla x_n = p_n,$$

we have

$$\int_\Omega \left\{ \langle \nabla x_n(y), p_n(y)\rangle - \tfrac{1}{2}|\nabla x_n(y)|^2 \right\} dy = \int_\Omega \tfrac{1}{2}|p_n(y)|^2 dy.$$

In a consequence

$$\int_\Omega \left\{ \langle \nabla x_n(y), p_n(y)\rangle - \tfrac{1}{2}|\nabla x_n(y)|^2 - G(y, -\operatorname{div} p_n(y)) \right\} dy =$$

$$= \int_\Omega \left\{ \tfrac{1}{2}|p_n(y)|^2 - G(y, -\operatorname{div} p_n(y)) \right\} dy,$$

so (16) hold. For any $n \in N$ we have

$$J^*_{D_{p_n}}(\nabla x_n) =$$
$$= \sup_{h \in L^2(\Omega, R^n} \int_\Omega \left\{ \langle \nabla x_n(y), h(y) \rangle - \tfrac{1}{2}|p_n(y) + h(y)|^2 - G(y, -\operatorname{div} p_n(y)) \right\} dy =$$
$$= \int_\Omega \left\{ \langle x_n(y), \operatorname{div} p_n(y) \rangle - \tfrac{1}{2}|\nabla x_n(y)|^2 - G(y, -\operatorname{div} p_n(y)) \right\} dy = J^{\#}_{x_n}(-p_n),$$

where $J^*_{D_{p_n}}$ is a F-Y conjugate of $J_{D_{p_n}}$. By the above and by the properties of the subdifferential we have $\nabla x_n \in \partial J_{D_p}(0)$.

We show now that $\{x_n\}_{n \in N} \subset W^{1,2}_0(\Omega, R)$ is minimizing for J on $W^{1,2}_0(\Omega, R)$.

By the definition it follows that $J_{D_{p_n}}(0) = -J_D(p_n)$. By (16),(6) we get that $\forall \varepsilon > 0\ \exists n_0 \in N\ \forall n > n_0$ we have

$$c + \varepsilon > J_D(p_n) \geq J^{\#}_{x_n}(-p_n) \geq \inf_{p \in L^2(\Omega, R)} (J^{\#}_{x_n}(-p)) =$$
$$= - \sup_{p \in L^2(\Omega, R^n} \int_\Omega \left\{ \langle -\operatorname{div} p(y), x_n(y) \rangle - \tfrac{1}{2}|\nabla x_n(y)|^2 - G(y, -\operatorname{div} p(y)) \right\} dy =$$
$$= - \int_\Omega \left\{ G(y, x_n(y)) - \tfrac{1}{2}|\nabla x_n(y)|^2 \right\} dy = J(x_n).$$

So $c = \inf_{n \in N} J(x_n)$, by (1) we have

$$\inf_{n \in N} J(x_n) = c = \inf_{n \in N} J_D(p_n) = \inf_{p \in L^2(\Omega, R)} J_D(p) = \inf_{x \in W^{1,2}(\omega, R)} J(x).$$

To prove the last assertion we use the definition of c. Since $J^{\#}_{x_n}(-p_n) \leq c + \varepsilon\ \forall n > n_0$ we get

$$J^{\#}_{x_n}(-p_n) - J(x_n) \leq c + \varepsilon - c = \varepsilon.$$

By the above for any $n > n_0$ we have

$$J_D(p_n) - J(x_n) \leq c + \varepsilon - c = \varepsilon$$

and

$$J(x_n) - J_D(p_n) \leq c + \varepsilon - c = \varepsilon.$$

□

Now we may get the following

Corollary 1. *If $\{p_n\}_{n \in N} \subset B$ is a minimizing sequence for J_D on B and $-\infty < \inf_{n \in N} J_D(p_n) < \infty$, $n \in N$ there exists sequence $\{x_n\}_{n \in N} \subset W^{1,2}_0(\Omega, R)$ minimizing functional J on $W^{1,2}_0(\Omega, R)$, such that $\forall n \in N$*

$$\nabla x_n(y) = p_n(y)$$

a.e. $y \in \Omega$.

4. Existence of solutions

To the end of the paper we consider the sequences $\{x_k\}$ i $\{p_k\}$ ($x_k \in C_0^1(\bar{\Omega})$, $p_k \in C^1(\bar{\Omega})$, $k \in N$) such that $\{x_k, \operatorname{div} p_k\} \in W_0^{1,2}(\Omega, R) \times D$ is minimizing sequence for J_C.

We assume that

$$\|x_k\|_{L^\infty} \leq C_1 \|\nabla x_k\|_{L^2}, \tag{17}$$

for a certain $C_1 \leq 1$ ($C_1 = C_1(\Omega)$) and for each $k \in N$, moreover

$$\operatorname{vol} \Omega \leq 1. \tag{18}$$

Theorem 6. *There exist a pair $(\bar{x}, \operatorname{div} \bar{p})$ minimizing J_C on $W_0^{1,2}(\Omega, R) \times D$, i.e.*

$$J_C(\bar{x}, \operatorname{div} \bar{p}) = \inf_{x \in W_0^{1,2}(\Omega, R), \operatorname{div} p \in D} J_C(x, \operatorname{div} p).$$

Proof. Let $\{x_k, \operatorname{div} p_k\} \in W_0^{1,2}(\Omega, R) \times D$ be a minimizing sequence to J_C. We show that sequence is bounded on $W_0^{1,2}(\Omega, R) \times D$. From (1)

$$G(y, v) \geq F^{**}(y, v) + \tfrac{1}{2}|v|^2, \qquad y \in \Omega, v \in R.$$

For each $k \in N$ we have:

$$J_C(x_k, \operatorname{div} p_k) \geq$$
$$\geq \int_\Omega \left\{ \langle x_k(y), \operatorname{div} p_k(y) \rangle + F^{**}(y, -\operatorname{div} p_k(y)) + \tfrac{1}{2}|\operatorname{div} p_k(y)|^2 + \tfrac{1}{2}|\nabla x_k(y)|^2 \right\} dy$$

for any $k \in N$

$$\begin{aligned} J_C(x_k, \operatorname{div} p_k) \geq & \int_\Omega \langle x_k(y), \operatorname{div} p_k(y) \rangle dy + \int_\Omega F^{**}(y, -\operatorname{div} p_k(y)) dy + \\ & + \tfrac{1}{2} \left(\int_\Omega |\operatorname{div} p_k(y)|^2 dy + \int_\Omega |\nabla x_k(y)|^2 dy \right). \end{aligned} \tag{19}$$

Now we consider the set

$$\Phi_b = \{(x, \operatorname{div} p) : \quad J_C(x, \operatorname{div} p) \leq b\} \tag{20}$$

where $b \in R$ is sufficiently large for $\Phi_b \neq \emptyset$ (for example $b > J_C(x_1, \operatorname{div} p_1)$).

Applying Holder inequality for the first integral on the right hand side of (19) we have:

$$
\begin{aligned}
&b \geq J_C(x_k, \operatorname{div} p_k) \geq \\
&\geq -\|x_k\|_{L^\infty} \|\operatorname{div} p_k\|_{L^1} + \tfrac{1}{2}\left(\|\operatorname{div} p_k\|^2_{L^2} + \|\nabla x_k\|^2_{L^2}\right) + \\
&+ \int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy.
\end{aligned} \tag{21}
$$

Using (17) the above inequality takes the form:

$$
\begin{aligned}
&b \geq \tfrac{1}{2}\left(-2\|x_k\|_{L^\infty} \|\operatorname{div} p_k\|_{L^1} + \|\operatorname{div} p_k\|^2_{L^2} + \|x_k\|^2_{L^\infty}\right) + \\
&+ \int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy.
\end{aligned}
$$

From [5] we have:

$$
\|\operatorname{div} p_k\|_{L^1} \leq \sqrt{\operatorname{vol}\Omega}\, \|\operatorname{div} p_k\|_{L^2}. \tag{22}
$$

Denoting $V = 1/\operatorname{vol}\Omega$, we have:

$$
\begin{aligned}
&b \geq \tfrac{1}{2}\left(-2\|x_k\|_{L^\infty} \|\operatorname{div} p_k\|_{L^1} + V\|\operatorname{div} p_k\|^2_{L^1} + \|x_k\|^2_{L^\infty}\right) + \\
&+ \int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy.
\end{aligned}
$$

Observe that $f(x, y) = -2xy + Vy^2 + x^2$ is nonnegative provided $x, y \geq 0$, $V \geq 1$ hence:

$$
-2\|x_k\|_{L^\infty} \|\operatorname{div} p_k\|_{L^1} + V\|\operatorname{div} p_k\|^2_{L^1} + \|x_k\|^2_{L^\infty} \geq 0.
$$

From (21) we get the following estimate which is uniform with respect to k:

$$
b \geq J_C(x_k, \operatorname{div} p_k) \geq \int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy. \tag{23}
$$

By definition of the conjugate we have

$$
F^{**}(y, q) \geq \langle v, q\rangle - F^*(y, v)
$$

so

$$
F^{**}(y, q) \geq \sup_{v \in [-r, r]} \{\langle v, q\rangle - F^*(y, v)\}
$$

where $r > 0$ is a certain constant.

By properties of the supremum:

$$
F^{**}(y, q) \geq \sup_{v \in [-r, r]} \langle v, q\rangle - \sup_{v \in [-r, r]} F^*(y, v) \geq r|q| - \sup_{v \in [-r, r]} F^*(y, v)
$$

by the above

$$\int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy \geq r \int_\Omega |\operatorname{div} p_k(y)|dy - \int_\Omega \sup_{v\in[-r,r]} F^*(y,v)dy. \tag{24}$$

Let $K = \int_\Omega \sup_{v\in[-r,r]} F^*(y,v)dy$. By the assumption it follows that K is finite.

By inequalities (23),(24) we get:

$$J_C(x_k, -\operatorname{div} p_k) \geq r\|\operatorname{div} p_k(y)\|_{L^1} - K. \tag{25}$$

By (25) we infer that the sequence is bounded $\{\|\operatorname{div} p_k(y)\|_{L^1}\}_{k\in N}$ i.e. there exists a constant C:

$$\|\operatorname{div} p_k(y)\|_{L^1} \leq C$$

for any $k \in N$.

By (21) and (17) we have for any $k \in N$:

$$\begin{aligned} b &\geq J_C(x_k, \operatorname{div} p_k) \geq \\ &\geq -\|\nabla x_k\|_{L^2}\|\operatorname{div} p_k\|_{L^1} + \tfrac{1}{2}\left(\|\operatorname{div} p_k\|^2_{L^2} + \|\nabla x_k\|^2_{L^2}\right) + \\ &+ \int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy. \end{aligned}$$

By (22) we have for any $k \in N$:

$$\begin{aligned} b &\geq -\|\nabla x_k\|_{L^2}\|\operatorname{div} p_k\|_{L^1} + \tfrac{1}{2}\left(V\|\operatorname{div} p_k\|^2_{L^1} + \|\nabla x_k\|^2_{L^2}\right) + \\ &+ \int_\Omega \{F^{**}(y, -\operatorname{div} p_k(y))\}\, dy. \end{aligned}$$

Since the integral $\int_\Omega F^{**}(y, -\operatorname{div} p_k(y))dy$ is bounded (uniformly in $k \in N$) and sequence $\{\|\operatorname{div} p_k(y)\|_{L^1}\}_{k\in N}$ is bounded, it follows that:

$$\{\|\nabla x_k\|_{L^2}\}_{k\in N}$$

by corollary (1) is bounded:

$$\{\|p_k(y)\|_{L^2}\}_{k\in N} \tag{26}$$

by inequalty (17) it follows that:

$$\{\|x_k\|_{L^\infty}\}_{k\in N}. \tag{27}$$

Hence (21) we get the boundeness of the sequence:

$$\{\|\operatorname{div} p_k\|_{L^2}\}_{k\in N}. \tag{28}$$

By reflexivity of $L^2(\Omega, R^n)$ and $L^2(\Omega, R)$ and by (26) we may choose a subsequence $\{p_{k_i}\}$ from $\{p_k\}$ which is weakly convergent to a certain $p_0 \in L^2(\Omega, R^n)$ and such that $\operatorname{div} p_{k_i} \rightharpoonup z$, where $z \in L_2(\Omega, R)$ ($\rightharpoonup$ denotes the weak convergence in $L^2(\Omega, R)$). The resulting subsequence $\{p_{k_i}\}$ we denote by $\{p_k\}$.

Now we show that $\operatorname{div} p_0$ exists and $\operatorname{div} p_0 = z$. Indeed by weak convergence of $\{p_k\}$ to p_0 we have for arbitrary $h \in L^2(\Omega, R^k)$:

$$\lim_{k\to\infty} \int_\Omega \langle p_k(y), h(y)\rangle dy = \int_\Omega \langle p_0(y), h(y)\rangle dy. \tag{29}$$

Using the weak convergence of $\{\operatorname{div} p_k\}_{k\in N}$ to z in $L^2(\Omega, R)$ we have for any $g \in L^2(\Omega, R)$:

$$\lim_{k\to\infty} \int_\Omega \langle \operatorname{div} p_k(y), g(y)\rangle dy = \int_\Omega \langle z(y), g(y)\rangle dy. \tag{30}$$

By (29) and (30) we obtain for any $h \in C_0^\infty(\Omega, R)$:

$$\int_\Omega \langle p_0(y), \nabla h(y)\rangle dy = \lim_{k\to\infty} \int_\Omega \langle p_k(y), \nabla h(y)\rangle dy =$$

$$= -\lim_{k\to\infty} \int_\Omega \langle \operatorname{div} p_k(y), h(y)\rangle dy = -\int_\Omega \langle z(y), h(y)\rangle dy.$$

Hence for all $h \in C_0^\infty(\Omega, R)$:

$$\int_\Omega (\langle p_0(y), \nabla h(y)\rangle + \langle z(y), h(y)\rangle) dy = 0.$$

Thus by Eulera-Lagrange lemma it follows that $\operatorname{div} p_0(y) = z(y)$ for a.e. $y \in \Omega$.

In order to show that J_C attains an infimium on $L^2(\Omega, R^n) \times L^2(\Omega, R^n)$ it suffices to show that

$$\liminf_{k\to\infty} J_C(x_k, \operatorname{div} p_k) \geq J_C(\bar{x}, \operatorname{div} \bar{p}). \tag{31}$$

By boundness of $\{\|\nabla x_k(y)\|_{L_2}\}_{k\in N}$ and Poincare inequality it follows that $\{x_k\}_{k\in N}$ is also bounded in L^2. By reflexivity of $W^{1,2}(\Omega, R)$ we have the existence of a weak limit $\bar{x}$. By Aubin's lemma[2] it follows that $\{x_k\}_{k\in N}$ is strongly convergent on $L^2(\Omega, R^n)$ to $\bar{x} \in L^2(\Omega, R)$.

Since $\{x_k\}_{k\in N}$ is strongly convergent on $L^2(\Omega, R)$ and $\{\operatorname{div} p_k\}_{k\in N}$ is weakly convergent on $L^2(\Omega, R)$ it follows

$$\lim_{k\to\infty} \int_\Omega \{\langle x_k(y), \operatorname{div} p_k(y)\rangle\} dy = \int_\Omega \{\langle \bar{x}(y), \operatorname{div} \bar{p}(y)\rangle\} dy. \tag{32}$$

Since G is convex and lower semicontinous it follows that $L^2(\Omega, R) \ni \operatorname{div} p \mapsto \int_\Omega G(y, -\operatorname{div} p(y)dy$ is convex and lower semicontinous so it is weakly lower semicontinous.Hence

$$\liminf_{k\to\infty} \int_\Omega \{G(y, -\operatorname{div} p_k(y)\}\, dy \geq \int_\Omega \{G(y, -\operatorname{div} \bar{p}(y)\}\, dy. \tag{33}$$

Since $L^2(\Omega, R^n) \ni \nabla x \mapsto \frac{1}{2}\int_\Omega |\nabla x(y)|^2 dy$

$$\liminf_{k\to\infty} \frac{1}{2} \int_\Omega |\nabla x_k(y)|^2 dy \geq \frac{1}{2} \int_\Omega |\nabla x(y)|^2 dy. \tag{34}$$

Hence by (32), (33), (34) we have

$$\liminf_{k\to\infty} J_C(x_k, \operatorname{div} p_k) =$$

$$= \liminf_{k\to\infty} \int_\Omega \left\{\langle x_k(y), \operatorname{div} p_k(y))\rangle + G(y, -\operatorname{div} p_k(y)) + \tfrac{1}{2}|\nabla x_k(y)|^2\right\} dy \geq$$

$$\geq \int_\Omega \left\{\langle \bar{x}(y), \operatorname{div} \bar{p}(y))\rangle + G(y, -\operatorname{div} \bar{p}(y)) + \tfrac{1}{2}|\nabla \bar{x}(y)|^2\right\} dy = J_C(\bar{x}, \operatorname{div} \bar{p}).$$

□

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