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SOME TRIGONOMETRIC IDENTITIES RELATED TO POWERS OF COSINE AND SINE FUNCTIONS

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Abstract. Some decomposition of certain basic symmetric functions of n-th power of cosine and sine functions is presented here. Next the applications of these decompositions to generating many new binomial and trigonometric identities are discussed.

1. Introduction

In this paper we wish to investigate different trigonometric identities connected with the following trigonometric identities previously presented in [8]:

$$\sin^{2n}(x) + \cos^{2n}\left(x - \tfrac{\pi}{6}\right) + \cos^{2n}\left(x + \tfrac{\pi}{6}\right) + \cos^{2n}(x) + $$
$$+ \cos^{2n}\left(x - \tfrac{\pi}{3}\right) + \cos^{2n}\left(x + \tfrac{\pi}{3}\right) \equiv \text{const} \;\Leftrightarrow\; n = 1, 2, \ldots, 5. \tag{1}$$

It is natural to relate these purpose to decompositions of the following four symmetric functions of the n-th powers of cosine and sine functions:

$$C_n^+(x,\varphi) := \cos^n(x+\varphi) + \cos^n(x-\varphi), \tag{2}$$

$$C_n^-(x,\varphi) := \cos^n(x-\varphi) - \cos^n(x+\varphi), \tag{3}$$

$$S_n^+(x,\varphi) := \sin^n(x+\varphi) + \sin^n(x-\varphi), \tag{4}$$

$$S_n^-(x,\varphi) := \sin^n(x+\varphi) - \sin^n(x-\varphi). \tag{5}$$

for $n \in \mathbb{N}$. The term "decomposition" means here the trigonometric polynomial form of these functions. We note that some other decomposition of the functions (2)–(5) using the Chebyshev polynomial of the second kind in the separate paper [7] is discussed.

The basic decompositions of functions (2)–(5) are presented in Section 2 and in two next sections are applied to generating of trigonometric identities, starting from the simplest forms (see Section 3 and Lemma 8) to

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main identities shown in Lemma 10 and Corollaries 8, 10, 11, 12 and 14, respectively.

Moreover at the end of the Section 2 an interesting application of these decompositions to obtained some nontrivial binomial identities is also given.

2. The basic decomposition

Lemma 1 (see [3]). *The following two classical identities hold*

$$2^{n-2}\,C_n^{+}(x,\varphi)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\cos\big((n-2k)\varphi\big)\cos\big((n-2k)x\big)-\tag{6}$$
$$-\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n-1}{2}\right\rfloor\right)\binom{n}{\lfloor\frac{n}{2}\rfloor}$$

and

$$2^{n-2}\,C_n^{-}(x,\varphi)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{k}\sin\big((n-2k)\varphi\big)\sin\big((n-2k)x\big).\tag{7}$$

The proof of the above identities by induction follows and will be omitted here.

The sequence of the following identities can easily be deduced from identities (6) and (7) (the formulas (12)–(14) are known see [1], the formula (15) is probably new):

$$\begin{aligned}2^{2n-2}\,S_{2n}^{+}(x,\varphi)&=\\&=\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k}\cos\big(2(n-k)\varphi\big)\cos\big(2(n-k)x\big)-\frac{1}{2}\binom{2n}{n}\\&=\sum_{k=0}^{n-1}(-1)^{n-k}\binom{2n}{k}\cos\big(2(n-k)\varphi\big)\cos\big(2(n-k)x\big)+\frac{1}{2}\binom{2n}{n};\end{aligned}\tag{8}$$

$$\begin{aligned}2^{2n-3}\,S_{2n-1}^{+}(x,\varphi)&=\\&=\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{2n-1}{k}\cos\big((2n-2k-1)\varphi\big)\sin\big((2n-2k-1)x\big);\end{aligned}\tag{9}$$

$$\begin{aligned}2^{2n-3}\,S_{2n-1}^{-}(x,\varphi)&=\\&=\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{2n-1}{k}\sin\big((2n-2k-1)\varphi\big)\cos\big((2n-2k-1)x\big);\end{aligned}\tag{10}$$

$$2^{2n-2}\, S^{-}_{2n}(x,\varphi) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n}{k} \sin\bigl(2(n-k)\varphi\bigr) \sin\bigl(2(n-k)x\bigr); \tag{11}$$

$$2^{2n-2}\, \sin^{2n-1}(x) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n-1}{k} \sin\bigl((2n-2k-1)x\bigr); \tag{12}$$

$$\begin{aligned} 2^{2n-1}\, \sin^{2n}(x) &= 2^{2n-2}\, S^{+}_{2n}(x,0) = \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{k} \cos\bigl(2(n-k)x\bigr) - \frac{1}{2}\binom{2n}{n}; \end{aligned} \tag{13}$$

$$\begin{aligned} 2^{n-1}\, \cos^{n}(x) &= 2^{n-2}\, C^{+}_{n}(x,0) = \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \cos\bigl((n-2k)x\bigr) - \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor\right) \binom{n}{\lfloor n/2 \rfloor}; \end{aligned} \tag{14}$$

$$\begin{aligned} 2^{2n-2}\, C^{+}_{2n}\bigl(x+\tfrac{\pi}{8}, \tfrac{\pi}{4}\bigr) &= \cos^{2n}\bigl(x-\tfrac{\pi}{8}\bigr) + \cos^{2n}\bigl(x+\tfrac{3}{8}\pi\bigr) = \\ &= \sum_{k=0}^{n} \binom{2n}{k} \cos\bigl((n-k)\tfrac{\pi}{2}\bigr) \cos\bigl(2(n-k)x + (n-k)\tfrac{\pi}{4}\bigr) - \frac{1}{2}\binom{2n}{n} = \\ &= \frac{1}{2}\binom{2n}{n} - \binom{2n}{n-2}\cos\bigl(4x+\tfrac{\pi}{2}\bigr) + \binom{2n}{n-4}\cos\bigl(8x+\pi\bigr) - \\ &\qquad - \binom{2n}{n-6}\cos\bigl(12x+\tfrac{3}{2}\pi\bigr) + \binom{2n}{n-8}\cos\bigl(16x+2\pi\bigr) - \dots \\ &= \frac{1}{2}\binom{2n}{n} + \binom{2n}{n-2}\sin(4x) - \binom{2n}{n-4}\cos(8x) + \\ &\qquad + \binom{2n}{n-6}\sin(12x) + \binom{2n}{n-8}\cos(16x) + \\ &\qquad + \binom{2n}{n-10}\sin(20x) - \binom{2n}{n-12}\cos(24x) + \\ &\qquad + \binom{2n}{n-14}\sin(28x) + \binom{2n}{n-16}\cos(32x) + \dots \end{aligned} \tag{15}$$

In the Tables 1 and 2 below the first seven decompositions of the type (6)–(11) of every function C^{+}_{n}, C^{-}_{n}, S^{+}_{n} and S^{-}_{n} are presented.

Table. 1. The first seven decompositions of every function C_n^+ and C_n^-

n	$C_n^+(x,\varphi)$	$C_n^-(x,\varphi)$
1	$2\cos(\varphi)\cos(x)$	$2\sin(\varphi)\sin(x)$
2	$1+\cos(2\varphi)\cos(2x)$	$\sin(2\varphi)\sin(2x)$
3	$\frac{1}{2}\big(3\cos(\varphi)\cos(x)+\cos(3\varphi)\cos(3x)\big)$	$\frac{1}{2}\big(3\sin(\varphi)\sin(x)+\sin(3\varphi)\sin(3x)\big)$
4	$\frac{1}{4}\big(3+4\cos(2\varphi)\cos(2x)+\cos(4\varphi)\cos(4x)\big)$	$\frac{1}{4}\big(4\sin(2\varphi)\sin(2x)+\sin(4\varphi)\sin(4x)\big)$
5	$\frac{1}{8}\big(10\cos(\varphi)\cos(x)+5\cos(3\varphi)\cos(3x)+$ $+\cos(5\varphi)\cos(5x)\big)$	$\frac{1}{8}\big(10\sin(\varphi)\sin(x)+5\sin(3\varphi)\sin(3x)+$ $+\sin(5\varphi)\sin(5x)\big)$
6	$\frac{1}{16}\big(10+15\cos(2\varphi)\cos(2x)+$ $+6\cos(4\varphi)\cos(4x)+\cos(6\varphi)\cos(6x)\big)$	$\frac{1}{16}\big(15\sin(2\varphi)\sin(2x)+$ $+6\sin(4\varphi)\sin(4x)+\sin(6\varphi)\sin(6x)\big)$
7	$\frac{1}{32}\big(35\cos(\varphi)\cos(x)+21\cos(3\varphi)\cos(3x)+$ $+7\cos(5\varphi)\cos(5x)+\cos(7\varphi)\cos(7x)\big)$	$\frac{1}{32}\big(35\sin(\varphi)\sin(x)+21\sin(3\varphi)\sin(3x)+$ $+7\sin(5\varphi)\sin(5x)+\sin(7\varphi)\sin(7x)\big)$

Table. 2. The first seven decompositions of every function S_n^+ and S_n^-

n	$S_n^+(x,\varphi)$	$S_n^-(x,\varphi)$
1	$2\cos(\varphi)\sin(x)$	$2\sin(\varphi)\cos(x)$
2	$1-\cos(2\varphi)\cos(2x)$	$\sin(2\varphi)\sin(2x)$
3	$\frac{1}{2}\big(3\cos(\varphi)\sin(x)-\cos(3\varphi)\sin(3x)\big)$	$\frac{1}{2}\big(3\sin(\varphi)\cos(x)-\sin(3\varphi)\cos(3x)\big)$
4	$\frac{1}{4}\big(3-4\cos(2\varphi)\cos(2x)+\cos(4\varphi)\cos(4x)\big)$	$\frac{1}{4}\big(4\sin(2\varphi)\sin(2x)-\sin(4\varphi)\sin(4x)\big)$
5	$\frac{1}{8}\big(10\cos(\varphi)\sin(x)-5\cos(3\varphi)\sin(3x)+$ $+\cos(5\varphi)\sin(5x)\big)$	$\frac{1}{8}\big(10\sin(\varphi)\cos(x)-5\sin(3\varphi)\cos(3x)+$ $+\sin(5\varphi)\cos(5x)\big)$
6	$\frac{1}{16}\big(10-15\cos(2\varphi)\cos(2x)+$ $+6\cos(4\varphi)\cos(4x)-\cos(6\varphi)\cos(6x)\big)$	$\frac{1}{16}\big(15\sin(2\varphi)\sin(2x)-$ $-6\sin(4\varphi)\sin(4x)+\sin(6\varphi)\sin(6x)\big)$
7	$\frac{1}{32}\big(35\cos(\varphi)\sin(x)-21\cos(3\varphi)\sin(3x)+$ $+7\cos(5\varphi)\sin(5x)-\cos(7\varphi)\sin(7x)\big)$	$\frac{1}{32}\big(35\sin(\varphi)\cos(x)-21\sin(3\varphi)\cos(3x)+$ $+7\sin(5\varphi)\cos(5x)-\sin(7\varphi)\cos(7x)\big)$

At the end of this section it will be given the application some of above decompositions to generating an interesting sets of binomial identities (see also [2, 4]).

Corollary 1. *Since for $x,\varphi\to 0$ we have*

$$C_n^-(x,\varphi)=\left(1-\frac{(x-\varphi)^2}{2!}+\ldots\right)^n-\left(1-\frac{(x+\varphi)^2}{2!}+\ldots\right)^n\approx$$
$$\approx\frac{n}{2}\Big((x+\varphi)^2-(x-\varphi)^2\Big)=2\,n\,x\,\varphi,$$

so by (6) *the following formula hold:*

$$\lim_{x,\varphi\to 0} 2^{n-2}\frac{C_n^-(x,\varphi)}{x\,\varphi} = \lim_{x,\varphi\to 0} \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{k} \frac{\sin\big((n-2k)\,\varphi\big)}{\varphi}\,\frac{\sin\big((n-2k)\,x\big)}{x},$$

i.e.,

$$n\,2^{n-1} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{k} \big(n-2\,k\big)^2.$$

Corollary 2. *From* (12) *it can be easily derived the formula*

$$2^{2n}\left(\frac{\sin(x)}{x}\right)^{2n+1} = \\ = \sum_{r=1}^{\infty}\sum_{k=0}^{n}(-1)^{n-k+r}\binom{2\,n+1}{k}\frac{(2\,n-2\,k+1)^{2r-1}}{(2\,r-1)!}\,x^{2(r-n-1)},$$

hence we get the sequence of the following binomial identities:

$$\sum_{k=0}^{n}(-1)^k\binom{2\,n+1}{k}\big(2\,n-2\,k+1\big)^{2r-1} = 0$$

for $r = 1, 2, \ldots, n$,

$$\sum_{k=0}^{n}(-1)^k\binom{2\,n+1}{k}\big(2\,n-2\,k+1\big)^{2n+1} = 2^{2n}\,(2\,n+1)!,$$

$$\sum_{k=0}^{n}(-1)^k\binom{2\,n+1}{k}\big(2\,n-2\,k+1\big)^{2n+3} = \frac{1}{3}\,(2\,n+1)\,2^{2n-1}\,(2\,n+3)!,$$

$$\sum_{k=0}^{n}(-1)^k\binom{2\,n+1}{k}\big(2\,n-2\,k+1\big)^{2n+5} = \frac{10\,n+3}{360}\,(2\,n+1)\,2^{2n}\,(2\,n+5)!,$$

etc., since we have:

$$\left(\frac{\sin(x)}{x}\right)^{n} = 1 - \frac{n}{6}\,x^2 + \frac{n\,(5\,n-2)}{360}\,x^4 + \ldots \tag{16}$$

Similarly, from (13) *we deduce the formula*

$$2^{2n-1}\left(\frac{\sin(x)}{x}\right)^{2n} + \frac{1}{2}\binom{2\,n}{n}\,x^{-2n} = \\ = \sum_{r=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k+r}\binom{2\,n}{k}\frac{\big(2\,(n-k)\big)^{2r}}{(2\,r)!}\,x^{2(r-n)},$$

which by (16) *implies the second sequence of the following binomial identities:*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{2\,n}{k}=\frac{1}{2}\binom{2\,n}{n},$$
$$\sum_{k=0}^{n-1}(-1)^{k}\binom{2\,n}{k}(n-k)^{2r}=0,$$

for $r=1,2,\ldots,n-1$,

$$\sum_{k=0}^{n-1}(-1)^{k}\binom{2\,n}{k}\big(n-k\big)^{2n}=\frac{1}{2}\,(2\,n)!,$$
$$\sum_{k=0}^{n-1}(-1)^{k}\binom{2\,n}{k}\big(n-k\big)^{2n+2}=\frac{n}{24}\,(2\,n+2)!,$$
$$\sum_{k=0}^{n-1}(-1)^{k}\binom{2\,n}{k}\big(n-k\big)^{2n+4}=\frac{n\,(5\,n-1)}{2880}\,(2\,n+4)!,$$

etc.

Corollary 3. *From* (14) *we obtain*

$$2^{n-1}\left(1-\frac{n}{2}\,x^2+\frac{n\,(3\,n-2)}{24}\,x^4+\ldots\right)=$$
$$=\frac{1}{2}\left(\left\lfloor\frac{n-1}{2}\right\rfloor-\left\lfloor\frac{n}{2}\right\rfloor\right)\binom{n}{\lfloor n/2\rfloor}+\sum_{r=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{2r}\binom{n}{k}\frac{(n-2\,k)^{2r}}{(2\,r)!}\,x^{2r},$$

which implies:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\big(n-2\,k\big)^{2}=n\,2^{n-1},$$
$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\big(n-2\,k\big)^{4}=n\,(3\,n-2)\,2^{n-1},$$

etc.

Corollary 4. *We have the formula* (*see* [4, 6])*:*

$$P_n(x)=2^{-n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{k}\binom{n-k}{k}\binom{2\,n-2\,k}{n-k}\,x^{n-2k},$$

where $P_n(x)$ denotes the n-th Legendre polynomial. Hence, by (14) *for even $n \in \mathbb{N}$, we get*

$$\int_0^{\pi} P_{2n}\big(\cos\varphi\big)\, d\varphi = \pi\, 2^{-4n} \sum_{k=0}^{n} (-4)^k \binom{2n-k}{k}\binom{4n-2k}{2n-k}\binom{2n-2k}{n-k}$$

$$= \pi\, 2^{-4n} \sum_{k=0}^{n} (-4)^k \frac{\big(4n-2k\big)!}{k!\,\big((n-k)!\big)^2\,(2n-k)!}. \tag{17}$$

On the other hand, we have ([6])*:*

$$P_{2n}\big(\cos\varphi\big) = 2^{-4n}\binom{2n}{n}^2 + \sum_{k=0}^{n-1} a_{k,n}\, \cos\big(2n-2k\big)$$

for some $a_{k,n} \in \mathbb{Q}$, which by (17) *implies the identity:*

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} (-4)^k \frac{\big(4n-2k\big)!}{k!\,\big((n-k)!\big)^2\,(2n-k)!}$$

$$= \sum_{k=0}^{n} (-4)^k \binom{2n-k}{k}\binom{4n-2k}{2n-k}\binom{2n-2k}{n-k}.$$

3. Simple trigonometric identities

We mention only the simple form of trigonometric identities to discuss here. Simultaneously this approach leads in Section 4 to direct our considerations to only symmetric trigonometric identities with respect to the phase translations.

Lemma 2. *Fix $a, \varphi, \psi \in \mathbb{R}$. Then*

$$f_{a,\varphi,\psi}(x) := a \left\{ \begin{array}{c} \sin^2 x \\ \cos^2 x \end{array} \right\} + \sin^2(x+\varphi) + \sin^2(x+\psi) \equiv \text{const}$$

($f_{a,\varphi,\psi}(x)$ is independent of x under this values a, φ, ψ) iff either:

$$\varphi - \psi = (2k+1)\tfrac{\pi}{2} \quad \textit{and} \quad a = 0$$

or

$$\varphi + \psi = k\,\pi \quad \textit{and} \quad a = \left\{ \begin{array}{r} -2\cos(2\varphi) \\ 2\cos(2\varphi) \end{array} \right\}.$$

Proof. First, we note that:

$$f_{a,\varphi,\psi}(x) \equiv \text{const} \implies f'_{a,\varphi,\psi}(x) \equiv 0 \implies f'_{a,\varphi,\psi}(0) = 0.$$

Hence, we get:

$$\sin(2\varphi)+\sin(2\psi)=0 \Longleftrightarrow \sin(\varphi+\psi)\,\cos(\varphi-\psi)=0 \Longleftrightarrow$$
$$\Longleftrightarrow \varphi+\psi=k\,\pi \quad \text{or} \quad \varphi-\psi=(2k+1)\,\tfrac{\pi}{2} \quad \text{for some } k\in\mathbb{Z}.$$

If $\varphi-\psi=(2k+1)\,\frac{\pi}{2}$ then:

$$f_{a,\varphi,\psi}(x):=a\left\{\begin{array}{c}\sin^2 x\\ \cos^2 x\end{array}\right\}+1$$

which implies $a=0$. If $\varphi+\psi=k\,\pi$ then, we have:

$$f_{a,\varphi,\psi}(x)=\left\{\begin{array}{c}a\,\sin^2 x+S_2^+(x,\varphi)\\ a\,\cos^2 x+S_2^+(x,\varphi)\end{array}\right\}=\left\{\begin{array}{c}\big(2\cos(2\varphi)+a\big)\,\sin^2 x+2\sin^2(\varphi)\\ \big(a-2\cos(2\varphi)\big)\,\cos^2 x+2\cos^2(\varphi)\end{array}\right\}$$

and the proof is completed. □

Corollary 5. *We have:*

$$a\left\{\begin{array}{c}\sin^2 x\\ \cos^2 x\end{array}\right\}+\cos^2(x+\varphi)+\cos^2(x+\psi)\equiv \mathrm{const}$$

iff either:

$$\varphi-\psi=(2k+1)\tfrac{\pi}{2} \quad \textit{and} \quad a=0$$

or

$$\varphi+\psi=k\,\pi \quad \textit{and} \quad a=\left\{\begin{array}{r}-2\cos(2\varphi)\\ 2\cos(2\varphi)\end{array}\right\}.$$

Proof. Set $\varphi:=\varphi+\frac{\pi}{2}$ and $\psi:=\psi+\frac{\pi}{2}$ in the Lemma 2. □

Lemma 3. *We have:*

1) $a\,\cos^4(x)+S_4^+(x,\varphi)\equiv \mathrm{const}$

   iff either $a=1$ *and* $\varphi=\pm\frac{\pi}{6}+k\,\pi,\quad k\in\mathbb{Z}$
   or $a=-2$ *and* $\varphi=\pm\frac{\pi}{2}+k\,\pi,\quad k\in\mathbb{Z}$*;*

2) $a\,\sin^4(x)+S_4^+(x,\varphi)\equiv \mathrm{const}$

   iff either $a=1$ *and* $\varphi=\pm\frac{\pi}{3}+k\,\pi,\quad k\in\mathbb{Z}$
   or $a=-2$ *and* $\varphi=k\,\pi,\quad k\in\mathbb{Z}$*.*

Proof. The assertion from the following decomposition follows:

1)

$$a\,\cos^4(x)+S_4^+(x,\varphi)=\Big(\big(4\cos^2(\varphi)-3\big)\big(4\cos^2(\varphi)-1\big)+a-1\Big)\cos^4(x)-$$
$$-4\cos^2(\varphi)\big(4\cos^2(\varphi)-3\big)\cos^2(x)+2\cos^4(\varphi).$$

2)

$$a\,\sin^4(x)+S_4^+(x,\varphi)=\Big(\big(4\cos^2(\varphi)-3\big)\big(4\cos^2(\varphi)-1\big)+a-1\Big)\sin^4(x)+$$
$$+4\sin^2(\varphi)\big(4\cos^2(\varphi)-1\big)\sin^2(x)+2\sin^4(\varphi).$$

□

Let us set:

$$g_{a,\varphi,\psi}(x):=a\,\cos^3(x)+\cos^3(x+\varphi)+\cos^3(x+\psi).$$

Lemma 4. *We have:*

1) $g_{a,\varphi,\psi}(x)\equiv\text{const}$ (*is independent on* x) $\Longrightarrow$

$$\Longrightarrow \begin{array}{l} \textit{either}\ \ \varphi+\psi=2\,k\,\pi \quad \textit{for some}\ k\in\mathbb{Z} \\ \textit{or}\ \ a=0\ \textit{and}\ \varphi-\psi=(2k+1)\,\pi \quad \textit{for some}\ k\in\mathbb{Z}. \end{array}$$

2) $g_{a,\varphi,-\varphi}(x)\equiv\text{const}$ $\Longleftrightarrow$

$$\Longleftrightarrow \begin{array}{l} \textit{either}\ a=2\,(-1)^{l+1}\ \textit{and}\ \varphi=l\,\pi,\quad l\in\mathbb{Z}, \\ \textit{or}\ a=0\ \textit{and}\ \varphi=(2\,l+1)\,\frac{\pi}{2},\quad l\in\mathbb{Z}. \end{array}$$

Proof. 1) We have:

$$g_{a,\varphi,\psi}\big(\tfrac{\pi}{2}\big)=g_{a,\varphi,\psi}\big(-\tfrac{\pi}{2}\big)\Leftrightarrow\sin^3(\varphi)+\sin^3(\psi)=0\Leftrightarrow$$
$$\Leftrightarrow\sin(\varphi)+\sin(\psi)=0\Leftrightarrow\sin\Big(\frac{\varphi+\psi}{2}\Big)\cos\Big(\frac{\varphi-\psi}{2}\Big)=0\Leftrightarrow$$
$$\Leftrightarrow\varphi+\psi=2\,k\,\pi\quad\vee\quad\varphi-\psi=(2\,k+1)\,\pi,\quad k\in\mathbb{Z}$$

and

$$g_{a,\varphi,\psi}\big(-\varphi\big)=g_{a,\varphi,\psi}\big(-\psi\big)\Leftrightarrow 2\,a\,\Big(\cos^3(\varphi)-\cos^3(\psi)\Big)=0\Leftrightarrow$$
$$\Leftrightarrow a=0\quad\vee\quad\cos(\varphi)-\cos(\psi)=0\quad\Leftrightarrow$$
$$\Leftrightarrow\quad a=0\quad\vee\quad\sin\Big(\frac{\varphi+\psi}{2}\Big)\sin\Big(\frac{\varphi-\psi}{2}\Big)=0\quad\Leftrightarrow$$
$$\Leftrightarrow a=0\quad\vee\quad\varphi+\psi=2\,l\,\pi\quad\vee\quad\varphi-\psi=2\,l\,\pi,\quad l\in\mathbb{Z}.$$

Hence, we conclude that either $\varphi+\psi=2\,k\,\pi$, $k\in\mathbb{Z}$, or $\varphi-\psi=(2\,k+1)\,\pi$, $k\in\mathbb{Z}$ and $a=0$. In the second case $g_{a,\varphi,\psi}(x)\equiv 0$.

2) By (6) we get:

$$g_{a,\varphi,-\varphi}(x)=C_3^+(x,\varphi)+\tfrac{a}{2}\,C_3^+(x,0)=$$
$$=\tfrac{1}{4}\,\big(2\,\cos(3\,\varphi)+a\big)\,\cos(3\,x)+\tfrac{3}{4}\,\big(2\,\cos(\varphi)+a\big)\,\cos(x),$$

which is independent on x iff

$$\begin{cases} 2\cos(3\varphi)+a=0 \\ 2\cos(\varphi)+a=0 \end{cases} \Leftrightarrow \begin{cases} a=-2\cos(\varphi) \\ \cos(3\varphi)-\cos(\varphi)=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=-2\cos(\varphi) \\ \sin(\varphi)\,\sin(2\varphi)=0 \end{cases} \Leftrightarrow \begin{cases} a=-2\cos(\varphi) \\ \sin(2\varphi)=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=-2\cos(\varphi) \\ \varphi=k\,\frac{\pi}{2} \text{ for some } k\in\mathbb{Z} \end{cases} \Leftrightarrow \begin{array}{l} \text{either } a=2\,(-1)^{l+1} \text{ and } \varphi=l\,\pi,\ l\in\mathbb{Z}, \\ \text{or } a=0 \text{ and } \varphi=(2\,l+1)\,\frac{\pi}{2},\ l\in\mathbb{Z}. \end{array}$$

□

Now, let us set:

$$\begin{aligned} h_{a,\varphi}(x) &:= 2\,a\,\cos^3(x)+\cos^3(x-\varphi)+\cos^3(x+\varphi) \\ &= a\,C_3^+(x,0)+C_3^+(x,\varphi). \end{aligned}$$

Lemma 5. *We have:*

1) $$\frac{h_{a,\varphi}(x)}{\cos(x)} \equiv \text{const} \quad\Leftrightarrow\quad \frac{h_{a,\varphi}(x)}{\cos(x)} \equiv 6\,\sin^2(\varphi)\,\cos(\varphi) \quad\Leftrightarrow$$
$$\Leftrightarrow\quad a+T_3\big(\cos(\varphi)\big)=0 \quad\Leftrightarrow\quad a=-\cos(3\varphi);$$

2) $$2\,t(a) := \sqrt[3]{a+\sqrt{a^2-1}}+\sqrt[3]{a-\sqrt{a^2-1}} \quad\Rightarrow\quad T_3\big(t(a)\big)=a;$$

3) $t(a)=\cos(\varphi)$ *for some* $a,\varphi\in\mathbb{R} \;\Leftrightarrow\; |a|=1 \;\Leftrightarrow$
$a=(-1)^{k+1}$ *and* $\varphi=k\,\pi,\ k\in\mathbb{Z}$,

where $T_3(x)$ *denotes the third Chebyshev polynomial of the first kind* (*see* [5]).

Proof. 1) It follows from (6) for $n=3$:

$$\frac{h_{a,\varphi}(x)}{\cos(x)} = \tfrac{1}{2}\big(\cos(3\varphi)+a\big)\frac{\cos(3x)}{\cos(x)}+\tfrac{3}{2}\big(\cos(\varphi)+a\big) \equiv \text{const} \quad\Leftrightarrow$$

$$\Leftrightarrow\quad a=-\cos(3\varphi) \quad\text{and}\quad \frac{h_{a,\varphi}(x)}{\cos(x)} \equiv \tfrac{3}{2}\big(\cos(\varphi)-\cos(3\varphi)\big),$$

which implies 1).
2) We have

$$\begin{aligned} 2\,T_3\big(t(a)\big) &= \big(a_++a_-\big)^3-3\,\big(a_++a_-\big)=\big(a_++a_-\big)\Big[\big(a_++a_-\big)^2-3\Big]= \\ &= \big(a_++a_-\big)\Big[\big(a_+\big)^2-1+\big(a_-\big)^2\Big]=\big(a_+\big)^3+\big(a_-\big)^3=2\,a. \end{aligned}$$

where $a_+ := \sqrt[3]{a+\sqrt{a^2-1}}$ and $a_- := \sqrt[3]{a-\sqrt{a^2-1}}$.

3) We note that $t(a)$ is an odd function. Moreover we have:

$$t'(a) = \frac{1}{6\sqrt{a^2-1}} \left(\sqrt[3]{a+\sqrt{a^2-1}} - \sqrt[3]{a-\sqrt{a^2-1}} \right).$$

Hence, we deduce that $t(a)$ is a decreasing function on $(-\infty, -1]$ and consequently an increasing one on $[1, \infty)$. Should also be noticed, that $t(1) = 1$. □

Lemma 6. *We have:*

$$\frac{C_5^+(x,\varphi)}{\cos(x)} \equiv \text{const} \quad \Longleftrightarrow \quad \varphi = (2\,k+1)\,\tfrac{\pi}{2}, \quad k \in \mathbb{Z}.$$

Proof. Since

$$C_5^+(x,\varphi) = 2\,\cos(5\varphi)\,\cos^5(x) + 5\,\sin(\varphi)\,\sin(4\varphi)\,\cos^3(x) + \\ + 10\,\sin^4(\varphi)\,\cos(\varphi)\,\cos(x),$$

so we obtain:

$$\frac{C_5^+(x,\varphi)}{\cos(x)} \equiv \text{const} \Longleftrightarrow \left\{ \begin{array}{l} \cos(5\varphi) = 0 \\ \sin(4\varphi) = 0 \end{array} \right. \Longleftrightarrow \varphi = (2\,k+1)\,\frac{\pi}{2}, \quad k \in \mathbb{Z}.$$

□

Now, let us set:

$$F_{a,\varphi,\psi}(x) := a\,\cos^6(x) + \cos^6(x+\varphi) + \cos^6(x+\psi).$$

Lemma 7. *We have:*

$$F_{a,\varphi,\psi}(x) \equiv \text{const} \quad \Longleftrightarrow \\ \Longleftrightarrow \quad a = -2 \quad \textit{and} \quad \varphi = k\,\pi \quad \textit{and} \quad \psi = l\,\pi \quad \textit{for some} \quad k, l \in \mathbb{Z}.$$

Proof. If $F_{a,\varphi,\psi}(x) \equiv \text{const}$ then $F'_{a,\varphi,\psi}(x) \equiv 0$, i.e.:

$$a\,\cos^5(x)\,\sin(x) + \cos^5(x+\varphi)\,\sin(x+\varphi) + \cos^5(x+\psi)\,\sin(x+\psi) \equiv 0.$$

Hence, for $x = 0$ and $x = \frac{\pi}{2}$, respectively, we get:

$$\cos^5(\varphi)\,\sin(\varphi) + \cos^5(\psi)\,\sin(\psi) = 0, \tag{18}$$

$$\sin^5(\varphi)\,\cos(\varphi) + \sin^5(\psi)\,\cos(\psi) = 0. \tag{19}$$

Subtracting (19) from (18), we obtain:

$$\sin(4\varphi) + \sin(4\psi) = 0,$$

i.e.

$$\sin\big(2(\varphi+\psi)\big)\cos\big(2(\varphi-\psi)\big) = 0$$

which implies that either

$$\varphi + \psi = k\,\tfrac{\pi}{2} \tag{20}$$

or

$$\varphi-\psi=\tfrac{\pi}{4}+k\,\tfrac{\pi}{2} \tag{21}$$

for some $k\in\mathbb{Z}$.

On the other hand, if we realize the operation: $\sin^4(\varphi)\cdot(18)-\cos^4(\varphi)\cdot(19)$, i.e.

$$\sin^4(\varphi)\,\cos^5(\psi)\,\sin(\psi)-\cos^4(\varphi)\,\sin^5(\psi)\,\cos(\psi)=0,$$

$$\sin(2\psi)\,\sin(\varphi+\psi)\,\sin(\varphi-\psi)\left(\sin^2(\varphi)\,\cos^2(\psi)+\cos^2(\varphi)\,\sin^2(\psi)\right)=0, \tag{22}$$

and operation: $\sin^4(\psi)\cdot(18)-\cos^4(\psi)\cdot(19)$, i.e.

$$\sin^4(\psi)\,\cos^5(\varphi)\,\sin(\varphi)-\cos^4(\psi)\,\sin^5(\varphi)\,\cos(\varphi)=0,$$

$$\sin(2\varphi)\,\sin(\varphi+\psi)\,\sin(\varphi-\psi)\left(\sin^2(\varphi)\,\cos^2(\psi)+\cos^2(\varphi)\,\sin^2(\psi)\right)=0 \tag{23}$$

and we assume that (21) holds, then from (22) and (23) we get $\sin(\varphi+\psi)=0$, i.e.:

$$\begin{cases}\varphi+\psi=l\,\pi,\\ \varphi-\psi=\tfrac{\pi}{4}+k\,\tfrac{\pi}{2}\end{cases}$$

for some $k,l\in\mathbb{Z}$, so:

$$\varphi=\tfrac{\pi}{8}+l\,\tfrac{\pi}{2}+k\,\tfrac{\pi}{4}\quad\text{and}\quad\psi=l\,\tfrac{\pi}{2}-k\,\tfrac{\pi}{4}-\tfrac{\pi}{8}.$$

Hence, we obtain:

$$\begin{aligned}F_{a,\varphi,\psi}(x)&=a\,\cos^6(x)+C_6^+\big(x+l\,\tfrac{\pi}{2},k\,\tfrac{\pi}{4}+\tfrac{\pi}{8}\big)=\\&=2^{-5}\,a\left(\frac{1}{2}\binom{6}{3}+\binom{6}{2}\cos(2x)+\binom{6}{1}\cos(4x)+\cos(6x)\right)+\\&+2^{-4}\Big(10+15\,\cos\big(\tfrac{\pi}{4}+k\,\tfrac{\pi}{2}\big)\,\cos\big(2\,x+l\,\pi\big)+\\&\qquad+\cos\big(\tfrac{3}{4}\pi+\tfrac{3}{2}\,k\,\pi\big)\,\cos\big(6\,x+3\,l\,\pi\big)\Big)=\\&=\frac{5(a+2)}{2^4}+15\left(2^{-5}\,a+(-1)^l\,2^{-4}\,\cos\big(\tfrac{\pi}{4}+k\,\tfrac{\pi}{2}\big)\right)\cos(2x)+\\&\qquad+\frac{3}{2^4}\,a\,\cos(4x)+\left(a\,2^{-5}+(-1)^l\,\cos\big(\tfrac{3}{4}\pi+\tfrac{3}{2}\,k\,\pi\big)\right)\cos(6x)\end{aligned}$$

which, from the linear independence of trigonometric system, implies $a=0$ and, in consequence, $F_{a,\varphi,\psi}(x)$ is not const, contrary to our assumptions.

If we suppose now that (20) holds, i.e. $\varphi+\psi=\frac{\pi}{2}+l\,\pi$, $l\in\mathbb{Z}$, then:

$$F_{a,\varphi,\psi}(x):=a\,\cos^6(x)+\cos^6(x+\varphi)+\sin^6(x-\varphi)$$

and

$$F_{a,\varphi,\psi}(\varphi)=F_{a,\varphi,\psi}(-\varphi)$$

i.e.:

$$\cos^6(2\varphi) = 1 + \sin^6(2\varphi) \implies \sin(2\varphi) = 0 \iff \varphi = k\,\tfrac{\pi}{2},\ k \in \mathbb{Z}.$$

Hence:

$$F_{a,\varphi,\psi}(x) = (a+1)\,\cos^6(x) + \sin^6(x) \not\equiv \text{const}\,.$$

Next let us assume that $\varphi + \psi = l\,\pi$, $l \in \mathbb{Z}$, then:

$$\begin{aligned} F_{a,\varphi,\psi}(x) &= a\,\cos^6(x) + C_6^+(x,\varphi) = \\ &= 2^{-5}\,\big(a + 2\,\cos(6\varphi)\big)\,\cos(6x) + 3\cdot 2^{-5}\,\big(a + 2\,\cos(4\varphi)\big)\,\cos(4x) + \\ &\qquad + 15\cdot 2^{-5}\,\big(a + 2\,\cos(2\varphi)\big)\,\cos(2x) + \ldots \end{aligned}$$

hence

$$F_{a,\varphi,\psi}(x) \equiv \text{const} \iff \begin{cases} a = -2\,\cos(2\varphi), \\ \cos(6\varphi) = \cos(4\varphi) = \cos(2\varphi). \end{cases} \tag{24}$$

We note that

$$\cos(t) = \cos(2t) = \cos(3t) \iff \begin{cases} 2\,(\cos(t)-1)\,(\cos(t)+\tfrac{1}{2}) = 0, \\ \cos(t)\,(\cos(t)-1)\,(\cos(t)+1) = 0, \end{cases}$$

which implies $\cos(t) = 1$. Thus, from (24) it follows that:

$$a = -2 \ \wedge\ \cos(2\varphi) = 1 \iff a = -2 \ \wedge\ \varphi = k\,\pi, \quad k \in \mathbb{Z}.$$

□

Now, let us set:

$$\begin{aligned} f_n(x) &:= \sin^{2n}(x) + \cos^{2n}\big(x - \tfrac{\pi}{6}\big) + \cos^{2n}\big(x + \tfrac{\pi}{6}\big), \\ g_n(x) &:= \cos^{2n}(x) + \cos^{2n}\big(x - \tfrac{\pi}{3}\big) + \cos^{2n}\big(x + \tfrac{\pi}{3}\big). \end{aligned}$$

We have the following identities:

n	$f_n(x)$	$g_n(x)$
1	$\frac{3}{2}$	$\frac{3}{2}$
2	$\frac{9}{8}$	$\frac{9}{8}$
3	$\frac{3}{32}\big(10 - \cos(6x)\big)$	$\frac{3}{32}\big(10 + \cos(6x)\big)$
4	$\frac{3}{128}\big(35 - 8\,\cos(6x)\big)$	$\frac{3}{128}\big(35 + 8\,\cos(6x)\big)$
5	$\frac{27}{512}\big(14 - 5\,\cos(6x)\big)$	$\frac{27}{512}\big(14 + 5\,\cos(6x)\big)$
6	$\frac{3}{2048}\big(462 - 220\,\cos(6x) + \cos(12x)\big)$	$\frac{3}{2048}\big(462 + 220\,\cos(6x) + \cos(12x)\big)$

n	$f_n(x)+g_n(x)$	n	$f_n(x)+g_n(x)$
1	3	7	$\frac{3}{2048}\big(858+7\,\cos(12x)\big)$
2	$\frac{9}{4}$	8	$\frac{45}{16384}\big(429+8\,\cos(12x)\big)$
3	$\frac{15}{8}$	9	$\frac{51}{32768}\big(715+24\,\cos(12x)\big)$
4	$\frac{105}{64}$	10	$\frac{969}{262144}\big(286+15\,\cos(12x)\big)$
5	$\frac{189}{128}$	11	$\frac{1197}{524288}\big(442+33\,\cos(12x)\big)$
6	$\frac{3}{1024}\big(462+\cos(12x)\big)$		

The form of $f_n(x)+g_n(x)$ for $n\ge 12$ is more complicated (there are at least three terms in the respective decomposition), for example we have:

$$f_{12}(x)+g_{12}(x)=\tfrac{3}{4194304}\Big(6118\big(221+22\cos(12\,x)\big)+\cos(24\,x)\Big).$$

Remark 1. *We have also*

$$\begin{aligned}&\big(C_2^+\big(x,\tfrac{\pi}{6}\big)\big)^n+\big(C_2^+\big(x,\tfrac{\pi}{3}\big)\big)^n-\\&\qquad-\big((-1)^n+1\big)\left(\tfrac12\,\cos(2\,x)\right)^n+n\,\big((-1)^n-1\big)\left(\tfrac12\,\cos(2\,x)\right)^{n-1}=\\&=\begin{cases}0, & \textit{for}\quad n=1,\\ 2, & \textit{for}\quad n=2,3,\\ 2+3\,\cos^2(2\,x), & \textit{for}\quad n=4,\\ 2+5\,\cos^2(2\,x), & \textit{for}\quad n=5.\end{cases}\end{aligned}$$

The next result indicates the direction in which attempts at generalizing certain results from Section 3 should follow.

Lemma 8. *Let $\varphi,\psi\in\mathbb{R}$ and*

$$\Theta_n(x)=C_n^+(x,0)+2\,C_n^+(x,\varphi)+2\,C_n^+(x,\psi),\quad n\in\mathbb{N},\ x\in\mathbb{R}.$$

If $\Theta_n(x)\equiv\mathrm{const}$ for some two different values of $n\in\mathbb{N}$ and at least for one odd value of $n\in\mathbb{N}$, then

$$\Theta_n(x)=C_n^+(x,0)+2\,C_n^+(x,\tfrac{2}{5}\,\pi)+2\,C_n^+(x,\tfrac{4}{5}\,\pi),\quad n\in\mathbb{N},\ x\in\mathbb{R}.$$

If $\Theta_n(x)\equiv\mathrm{const}$, for two different even values of $n\in\mathbb{N}$, then

$$\Theta_{2n}(x)=C_{2n}^+(x,0)+2\,C_{2n}^+(x,\tfrac{2}{5}\,\pi)+2\,C_{2n}^+(x,\tfrac{4}{5}\,\pi),\quad n\in\mathbb{N},\ x\in\mathbb{R}.$$

Proof. Directly from decomposition (6) it follows, that if

$$\Theta_k(x)\equiv\mathrm{const}\qquad\text{and}\qquad\Theta_l(x)\equiv\mathrm{const}$$

for $k,l\in\mathbb{N}$, $k<l$, so, in view of the linear independence of the trigonometric system one of the following three conditions holds:

$$\begin{cases}\cos(\varphi)+\cos(\psi)=-\tfrac12,\\ \cos(2\varphi)+\cos(2\psi)=-\tfrac12,\end{cases}\tag{25}$$

whenever $(-1)^k + (-1)^l = 0$; or

$$\left\{\begin{array}{l} \cos(\varphi) + \cos(\psi) = -\frac{1}{2}, \\ \cos(3\varphi) + \cos(3\psi) = -\frac{1}{2}, \end{array}\right. \tag{26}$$

whenever $(-1)^k + (-1)^l = -2$; or

$$\left\{\begin{array}{l} \cos(2\varphi) + \cos(2\psi) = -\frac{1}{2}, \\ \cos(4\varphi) + \cos(4\psi) = -\frac{1}{2}, \end{array}\right. \tag{27}$$

whenever $(-1)^k + (-1)^l = 2$.

Ad (25) The given system implies:

$$\left\{\begin{array}{l} \cos^2(\varphi) + 2\,\cos(\varphi)\cos(\psi) + \cos^2(\psi) = \frac{1}{4}, \\ \cos^2(\varphi) + \cos^2(\psi) = \frac{3}{4}, \end{array}\right.$$

i.e.

$$\cos(\varphi)\cos(\psi) = -\tfrac{1}{4}.$$

So, system (25) is equivalent to the following system:

$$\left\{\begin{array}{l} \cos(\varphi) + \cos(\psi) = -\frac{1}{2}, \\ \cos(\varphi)\cos(\psi) = -\frac{1}{4}. \end{array}\right. \tag{28}$$

The solutions $\cos\varphi$ and $\cos\psi$ of (28) form the set of the roots of polynomial

$$x^2 + \tfrac{1}{2}\,x - \tfrac{1}{4},$$

i.e. the set

$$\big\{\cos\varphi, \cos\psi\big\} = \big\{\tfrac{1}{4}\,(-1 \pm \sqrt{5})\big\} = \big\{\cos(\tfrac{2}{5}\pi), \cos(\tfrac{4}{5}\pi)\big\}. \tag{29}$$

Ad (26) We have:

$$\left\{\begin{array}{l} \cos(\varphi) + \cos(\psi) = -\frac{1}{2}, \\ T_3\big(\cos(\varphi)\big) + T_3\big(\cos(\psi)\big) = -\frac{1}{2}, \end{array}\right.$$

i.e.

$$\left\{\begin{array}{l} \cos(\varphi) + \cos(\psi) = -\frac{1}{2}, \\ 4\big(\cos^3(\varphi)) + \cos^3(\psi)\big) = -2. \end{array}\right.$$

By transforming the second equation we obtain, respectively:

$$4\,\big(\cos(\varphi) + \cos(\psi)\big)\,\Big(\big(\cos(\varphi) + \cos(\psi)\big)^2 - 3\cos(\varphi)\cos(\psi)\Big) = -2,$$

$$4\,\big(-\tfrac{1}{2}\big)\,\Big(\tfrac{1}{4} - 3\cos(\varphi)\cos(\psi)\Big) = -2,$$

$$\cos(\varphi)\cos(\psi) = -\tfrac{1}{4},$$

so the system (28) holds and the equalities (29) are satisfied.

Ad (27) Condition (27) from condition (25) for $\varphi := 2\varphi$ and $\psi := 2\psi$ follows. Hence, we get:

$$\big\{\cos(2\varphi), \cos(2\psi)\big\} = \big\{\cos(\tfrac{2}{5}\pi), \cos(\tfrac{4}{5}\pi)\big\}. \tag{30}$$

Now, the assertions of Lemma 8 from (29) and (30) follows. □

4. Some generalizations

Let us now step down to present the announced generalized trigonometric identities of the same nature as identity (1). Each one should be preceded by essential technical lemmas describing the values of some trigonometric sums.

Lemma 9. *Let $n, r \in \mathbb{N}$, $(2n-1) \nmid r$. Then the following equality holds:*

$$\sigma_n(r) := \sum_{k=0}^{n-1} \exp\left(i \, \frac{2\,k\,r}{2n-1}\,\pi\right) = \tag{31}$$
$$= \begin{cases} \frac{1}{2} - \frac{i}{2}\,\tan\left(\dfrac{r\,\pi}{2(2n-1)}\right), & \text{whenever } r \in 2\mathbb{N}, \\ \frac{1}{2} + \frac{i}{2}\,\cot\left(\dfrac{r\,\pi}{2(2n-1)}\right), & \text{whenever } r \in 2\mathbb{N}-1, \end{cases}$$

where $2\mathbb{N}$ $(2\mathbb{N}-1)$ denotes the set of even (odd) positive integers.

Proof. We perform the following transformations:

$$\sigma_n(r) = \left(1 - \exp\left(i\,\frac{2\,n\,r\,\pi}{2n-1}\right)\right)\left(1 - \exp\left(i\,\frac{2\,r\,\pi}{2n-1}\right)\right)^{-1} =$$
$$= \left(1 - \exp\left(i\,r\,\pi\right)\exp\left(\frac{i\,r\,\pi}{2n-1}\right)\right)\left(1 - \exp\left(i\,\frac{2\,r\,\pi}{2n-1}\right)\right)^{-1} =$$
$$= \left(1 - (-1)^r\,\exp\left(\frac{i\,r\,\pi}{2n-1}\right)\right)\left(1 - \exp\left(i\,\frac{2\,r\,\pi}{2n-1}\right)\right)^{-1} =$$
$$= \left(1 - (-1)^{r-1}\,\exp\left(\frac{i\,r\,\pi}{2n-1}\right)\right)^{-1} =$$
$$= \begin{cases} \frac{1}{2}\,\exp\left(-i\,\dfrac{r\,\pi}{2(2n-1)}\right)\,\cos^{-1}\left(\dfrac{r\,\pi}{2(2n-1)}\right), & \text{for } r \in 2\mathbb{N}, \\ \frac{i}{2}\,\exp\left(-i\,\dfrac{r\,\pi}{2(2n-1)}\right)\,\sin^{-1}\left(\dfrac{r\,\pi}{2(2n-1)}\right), & \text{for } r \in 2\mathbb{N}-1, \end{cases}$$

which implies the desired identity. □

Corollary 6. *Let $n, r \in \mathbb{N}$. Then:*

$$(32) \quad 1 + 2\sum_{k=1}^{n-1} \cos\left(\frac{2\,k\,r}{2n-1}\,\pi\right) = \begin{cases} 0, & \text{whenever } (2n-1) \not| \, r, \\ 2n-1, & \text{whenever } (2n-1) \mid r, \end{cases}$$

and

$$(33) \quad 2\sum_{k=1}^{n-1} \sin\left(\frac{2kr}{2n-1}\pi\right) =$$

$$= \begin{cases} 0, & \text{whenever}(2n-1) \mid r, \\ -\tan\left(\dfrac{r\pi}{2(2n-1)}\right), & \text{whenever } (2n-1) \not| \, r \wedge \; r \in 2\mathbb{N}, \\ \cot\left(\dfrac{r\pi}{2(2n-1)}\right), & \text{whenever } (2n-1) \not| \, r \;\wedge\; r \in 2\mathbb{N}-1. \end{cases}$$

In the next Lemma, identity (1) shall be generalized. The Lemma is derived on the grounds of (6), (7) and Corollary 6.

Lemma 10. *Let us set (for $n, r \in \mathbb{N}$):*

$$(34) \qquad \Phi^+_{r,n}(x) := C^+_r(x, 0) + 2\sum_{k=1}^{n-1} C^+_r\left(x, \frac{2k\pi}{2n-1}\right)$$

and

$$(35) \qquad \Phi^-_{r,n}(x) := \sum_{k=1}^{n-1} C^-_r\left(x, \frac{2k\pi}{2n-1}\right).$$

Then we have for $r \in 2\mathbb{N}-1$:

$$\Phi^+_{r,n}(x) = \begin{cases} 0, & r < 2n-1, \\ (2n-1)\dbinom{r}{\frac{r-2n+1}{2}} 2^{2-r} \cos\big((2n-1)x\big), & 2n-1 \le r < 3(2n-1), \\ (2n-1)\left(\dbinom{r}{\frac{r-2n+1}{2}} 2^{2-r} \times \right. & \\ \quad \times \cos\big((2n-1)x\big) + & \\ \quad \left. + \dbinom{r}{\frac{r-6n+3}{2}} 2^{2-r} \cos\big(3(2n-1)x\big)\right), & 3(2n-1) \le r < 5(2n-1), \end{cases}$$

and for $r \in 2\mathbb{N}$*:*

$$\Phi^{+}_{r,n}(x) = \begin{cases} (2n-1)\binom{r}{r/2}2^{1-r}, & r < 2(2n-1), \\ (2n-1)\Big(\binom{r}{r/2}2^{1-r}+ \\ \quad +\binom{r}{\frac{r-2(2n-1)}{2}}2^{2-r}\cos\big(2(2n-1)x\big)\Big), & 2(2n-1) \le r < 4(2n-1), \\ (2n-1)\,\Big(\binom{r}{r/2}2^{1-r}+ \\ \quad +\binom{r}{\frac{r-4n+2}{2}}2^{2-r}\cos\big(2(2n-1)x\big)+ \\ \quad +\binom{r}{\frac{r-8n+4}{2}}2^{2-r}\cos\big(4(2n-1)x\big)\Big), & 4(2n-1) \le r < 6(2n-1). \end{cases}$$

We have for $r \in 2\mathbb{N}-1$ *and* $r \le 2\,n-1$*:*

(36) $$\Phi^{-}_{r,n}(x) = 2^{1-r}\sum_{k=0}^{(r-1)/2}\binom{r}{k}\cot\left(\frac{(r-2\,k)\pi}{2\,(2\,n-1)}\right)\sin\big((r-2\,k)\,x\big)$$

and for $r \in 2\mathbb{N}$ *and* $r \le 2(2\,n-1)$*:*

(37) $$\Phi^{-}_{r,n}(x) = -2^{1-r}\sum_{k=0}^{r/2}\binom{r}{k}\tan\left(\frac{(r-2\,k)\pi}{2\,(2\,n-1)}\right)\sin\big((r-2\,k)\,x\big).$$

Lemma 11. *The following identity hold:*

$$\sum_{k=1}^{n-1}(-1)^k\exp\left(i\frac{(2\,k-1)\,r\,\pi}{2\,n-1}\right) = \frac{1-(-1)^{n-1}\,e^{i\,\frac{2\,(n-1)\,r\,\pi}{2\,n-1}}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}}(-1)\,e^{i\,\frac{r\,\pi}{2\,n-1}} =$$

$$= \frac{1-(-1)^{n+r-1}\,e^{-i\,\frac{r\,\pi}{2\,n-1}}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}}(-1)\,e^{i\,\frac{r\,\pi}{2\,n-1}} = -\frac{e^{i\,\frac{r\,\pi}{2\,n-1}}-(-1)^{n+r-1}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}} =$$

(38)
$$= \begin{cases} -\dfrac{e^{i\frac{r\,\pi}{2\,n-1}}-1}{e^{i\frac{2r\pi}{2\,n-1}}+1} = \dfrac{-i\sin\left(\frac{r\pi}{2\,(2n-1)}\right)}{\cos\left(\frac{r\,\pi}{2\,n-1}\right)}\,e^{-i\,\frac{r\,\pi}{2\,(2\,n-1)}} = \\ = \frac{1}{2}\left(1-\sec\left(\frac{r\,\pi}{2\,n-1}\right)\right)-\frac{i}{2}\tan\left(\frac{r\,\pi}{2\,n-1}\right), & \text{for } (n+r) \in 2\mathbb{N}-1, \\ -\dfrac{e^{i\,\frac{r\,\pi}{2\,n-1}}+1}{e^{i\,\frac{2\,r\,\pi}{2\,n-1}}+1} = -\dfrac{\cos\left(\frac{r\,\pi}{2\,(2\,n-1)}\right)}{\cos\left(\frac{r\,\pi}{2\,n-1}\right)}\,e^{-i\,\frac{r\,\pi}{2\,(2\,n-1)}} = \\ = -\frac{1}{2}\left(1+\sec\left(\frac{r\,\pi}{2\,n-1}\right)\right)+\frac{i}{2}\tan\left(\frac{r\,\pi}{2\,n-1}\right), & \text{for } (n+r) \in 2\mathbb{N}-1. \end{cases}$$

Corollary 7. *We have:*

$$(39)\qquad (-1)^{n+r} + 2\sum_{k=1}^{n-1}(-1)^k \cos\left(\frac{(2\,k-1)\,r\,\pi}{2\,n-1}\right) = -\sec\left(\frac{r\,\pi}{2\,n-1}\right)$$

and

$$(40)\qquad 2\sum_{k=0}^{n-1}(-1)^k \sin\left(\frac{(2\,k-1)\,r\,\pi}{2\,n-1}\right) = (-1)^{n+r}\tan\left(\frac{r\,\pi}{2\,n-1}\right).$$

Corollary 8. *Using identities* (6), (7) *and Corollary* 7 *we find:*

$$(41)\quad \Xi^+_{r,n}(x) := \sum_{k=1}^{n-1}(-1)^k C^+_r\left(x, \frac{(2\,k-1)\,\pi}{2n-1}\right) =$$
$$= -2^{1-r}\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r}{k}\left((-1)^{n+r} + \sec\left(\frac{(r-2\,k)\pi}{2\,n-1}\right)\right)\cos\big((r-2\,k)\,x\big)$$

and

$$(42)\quad \Xi^-_{r,n}(x) := \sum_{k=1}^{n-1}(-1)^k\, C^-_r\left(x, \frac{(2\,k-1)\,\pi}{2\,n-1}\right) =$$
$$= (-1)^{n+r}\,2^{1-r}\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r}{k}\tan\left(\frac{(r-2\,k)\,\pi}{2\,n-1}\right)\sin\big((r-2\,k)\,x\big).$$

Lemma 12. *We have:*

$$(43)\quad \sum_{k=0}^{n-1}(-1)^k \exp\left(i\,\frac{2\,k\,r\,\pi}{2\,n-1}\right) = \frac{1-(-1)^n\,e^{i\,\frac{2\,n\,r\,\pi}{2\,n-1}}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}} = \frac{1-(-1)^{n+r}\,e^{i\,\frac{r\,\pi}{2\,n-1}}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}} =$$
$$= \begin{cases} \dfrac{1+e^{i\,\frac{r\,\pi}{2\,n-1}}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}} = \dfrac{\cos\left(\frac{r\,\pi}{2\,(2\,n-1)}\right)}{\cos\left(\frac{r\,\pi}{2\,n-1}\right)}\,e^{-i\,\frac{r\,\pi}{2\,(2\,n-1)}} = \\ = \frac{1}{2}\left(1+\sec\left(\frac{r\,\pi}{2\,n-1}\right)\right) - \frac{i}{2}\tan\left(\frac{r\,\pi}{2\,n-1}\right), & \text{for } (n+r)\in 2\mathbb{N}-1, \\ \\ \dfrac{1-e^{i\,\frac{r\,\pi}{2\,n-1}}}{1+e^{i\,\frac{2\,r\,\pi}{2\,n-1}}} = \dfrac{-i\,\sin\left(\frac{r\,\pi}{2\,(2\,n-1)}\right)}{\cos\left(\frac{r\,\pi}{2\,n-1}\right)}\,e^{-i\,\frac{r\,\pi}{2\,(2\,n-1)}} = \\ = \frac{1}{2}\left(1-\sec\left(\frac{r\,\pi}{2\,n-1}\right)\right) - \frac{i}{2}\tan\left(\frac{r\,\pi}{2\,n-1}\right), & \text{for } (n+r)\in 2\mathbb{N}. \end{cases}$$

Corollary 9. *We have:*

$$(44)\qquad (-1)^{n+r-1}\sec\left(\frac{r\,\pi}{2\,n-1}\right) = 1 + 2\sum_{k=1}^{n-1}(-1)^k\cos\left(\frac{2\,k\,r\,\pi}{2\,n-1}\right)$$

and

$$2\sum_{k=0}^{n-1}(-1)^k\,\sin\left(\frac{2\,k\,r\,\pi}{2\,n-1}\right)=-\tan\left(\frac{r\,\pi}{2\,n-1}\right). \tag{45}$$

Corollary 10. *Using identities* (6), (7) *and Corollary* 9 *we obtain:*

$$\begin{aligned}\Psi^+_{r,n}(x) &:= C^+_r\big(x,0\big)+2\sum_{k=1}^{n-1}(-1)^k\,C^+_r\left(x,\frac{2\,k\,\pi}{2\,n-1}\right)=\\ &=2^{2-r}\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r}{k}\left(1+2\sum_{l=1}^{n-1}(-1)^l\cos\left(\frac{2l(r-2\,k)\pi}{2\,n-1}\right)\right)\cos\big((r-2\,k)\,x\big)+\\ &\quad+(-1)^n\,\tfrac12\,\left(\lfloor\tfrac r2\rfloor-\lfloor\tfrac{r-1}{2}\rfloor\right)\binom{r}{\lfloor\frac r2\rfloor}=\\ &=(-1)^{n+r-1}\,2^{2-r}\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r}{k}\sec\left(\frac{(r-2\,k)\,\pi}{2\,n-1}\right)\cos\big((r-2\,k)\,x\big)+\\ &\quad+(-1)^n\,\tfrac12\left(\lfloor\tfrac r2\rfloor-\lfloor\tfrac{r-1}{2}\rfloor\right)\binom{r}{\lfloor\frac r2\rfloor}\end{aligned} \tag{46}$$

and

$$\begin{aligned}\Phi^-_{r,n}(x) &:= \sum_{k=1}^{n-1}(-1)^k\,C^-_r\left(x,\frac{2\,k\,\pi}{2\,n-1}\right)=\\ &=-2^{1-r}\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r}{k}\tan\left(\frac{(r-2\,k)\,\pi}{2\,n-1}\right)\sin\big((r-2\,k)\,x\big).\end{aligned} \tag{47}$$

Lemma 13. *We have:*

$$\sum_{k=1}^{n}\exp\left(i\,\frac{(2\,k-1)\,r\,\pi}{2\,n}\right)=e^{i\,\frac{r\,\pi}{2\,n}}\,\frac{e^{i\,r\,\pi}-1}{e^{i\,\frac{r\,\pi}{n}}-1}=$$

$$=\begin{cases}0, & \textit{whenever } r \textit{ is even}\\ & \textit{and } n\nmid r\vee(n|r\wedge\frac rn\in 2\mathbb{N}-1),\\ \dfrac{-2\,e^{i\,\frac{r\,\pi}{2\,n}}}{e^{i\,\frac{r\,\pi}{n}}-1}=i\,\csc\left(\frac{r\,\pi}{2\,n}\right), & \textit{whenever } r \textit{ is odd.}\end{cases} \tag{48}$$

Corollary 11. *We have:*

$$\sum_{k=1}^{n}\cos\left(\frac{(2\,k-1)\,r\,\pi}{2\,n}\right)=\begin{cases}n & n|r\wedge\frac{r}{2n}\in 2\mathbb{N},\\ -n & n|r\wedge\frac{r}{2n}\in 2\mathbb{N}-1,\\ 0 & \textit{otherwise.}\end{cases} \tag{49}$$

and

$$\sum_{k=1}^{n} \sin\left(\frac{(2\,k-1)\,r\,\pi}{2\,n}\right) = \begin{cases} 0 & \textit{whenever } r \textit{ is even,} \\ \csc\left(\frac{r\,\pi}{2\,n}\right) & \textit{whenever } r \textit{ is odd.} \end{cases} \tag{50}$$

Corollary 12. *By* (6), (7) *and Corollary* 11 *we get:*

$$\Delta^{+}_{r,n}(x) := \sum_{k=1}^{n} C^{+}_{r}\left(x, \frac{(2\,k-1)\,\pi}{2\,n}\right) = 0, \qquad \textit{whenever } r \textit{ is odd,} \tag{51}$$

and

$$\Delta^{-}_{r,n}(x) := \sum_{k=1}^{n} C^{-}_{r}\left(x, \frac{(2\,k-1)\,\pi}{2\,n}\right) = \tag{52}$$

$$= \begin{cases} 0 & \textit{whenever } r \textit{ is even,} \\ 2^{2-r} \sum\limits_{k=0}^{\lfloor r/2 \rfloor} \binom{r}{k} \csc\left(\frac{(r-2\,k)\,\pi}{2\,n}\right) \sin\big((r-2\,k)\,x\big) & \textit{whenever } r \textit{ is odd.} \end{cases}$$

Lemma 14. *Let $n \in \mathbb{N}$, $r \in \mathbb{Z}$ and $n \nmid r$. Then we have:*

$$\sum_{k=1}^{n-1} (-1)^k \exp\left(i\,\frac{(2\,k-1)\,r\,\pi}{2\,n}\right) = \frac{1-(-1)^{n-1}\,e^{i\,\frac{(n-1)\,r\,\pi}{n}}}{1+e^{i\,\frac{r\,\pi}{n}}}(-e^{i\,\frac{r\,\pi}{2\,n}}) = \tag{53}$$

$$= \frac{-1-(-1)^{n+r}\,e^{-i\,\frac{r\,\pi}{n}}}{2\,\cos(\frac{r\,\pi}{2\,n})} =$$

$$= \begin{cases} \dfrac{-(1+e^{-i\,\frac{r\,\pi}{n}})}{2\,\cos(\frac{r\,\pi}{2\,n})} = -e^{-i\,\frac{r\,\pi}{2\,n}}, & \textit{whenever } (n+r) \textit{ is even,} \\ \dfrac{-1+e^{-i\,\frac{r\,\pi}{n}}}{2\,\cos(\frac{r\,\pi}{2\,n})} = \dfrac{-i\,\sin(\frac{r\,\pi}{2\,n})}{\cos(\frac{r\,\pi}{2\,n})}\,e^{-i\,\frac{r\,\pi}{2\,n}} = & \\ \quad = \cos(\frac{r\,\pi}{2\,n}) - \sec(\frac{r\,\pi}{2\,n}) - i\,\sin(\frac{r\,\pi}{2\,n}), & \textit{whenever } (n+r) \textit{ is odd.} \end{cases}$$

Corollary 13. *We have:*

$$\sum_{k=1}^{n-1} (-1)^k \cos\left(\frac{(2\,k-1)\,r\,\pi}{2\,n}\right) = \tag{54}$$

$$= \begin{cases} -\cos(\frac{r\,\pi}{2\,n}), & \textit{whenever } (n+r) \textit{ is even,} \\ \cos(\frac{r\,\pi}{2\,n}) - \sec(\frac{r\,\pi}{2\,n}), & \textit{whenever } (n+r) \textit{ is odd,} \end{cases}$$

and

$$\sum_{k=1}^{n-1} (-1)^k \sin\left(\frac{(2\,k-1)\,r\,\pi}{2\,n}\right) = (-1)^{n+r} \sin\left(\frac{r\,\pi}{2\,n}\right). \tag{55}$$

Corollary 14. *We have:*

$$\Theta^{+}_{r,n}(x) := \sum_{k=1}^{n-1} (-1)^k\, C^{+}_{r}\left(x, \frac{(2\,k-1)\,\pi}{2\,n}\right) =$$

$$= \begin{cases} -2^{1-r} \sum\limits_{k=0}^{\lfloor r/2 \rfloor} \binom{r}{k} \cos\left(\frac{(r-2\,k)\,\pi}{2\,n}\right) \cos\left((r-2\,k)\,x\right), & \text{for } r \in 2\mathbb{N}, \\ 2^{2-r} \sum\limits_{k=0}^{\lfloor r/2 \rfloor} \binom{r}{k} \Big[\csc\left(\frac{(r-2\,k)\,\pi}{2\,n}\right) - & \\ \qquad - \sec\left(\frac{(r-2\,k)\,\pi}{2\,n}\right) \Big] \cos\big((r-2\,k)\,x\big), & \text{for } r \in 2\mathbb{N}-1. \end{cases}$$

and

$$\Theta^{-}_{r,n}(x) := \sum_{k=1}^{n-1} (-1)^k\, C^{-}_{r}\left(x, \frac{(2\,k-1)\,\pi}{2\,n}\right) = \tag{56}$$

$$= (-1)^{n+r} \sum_{k=0}^{\lfloor r/2 \rfloor} \binom{r}{k} \sin\left(\frac{r\,\pi}{2\,n}\right) \sin\big((r-2\,k)\,x\big).$$

Remark 2. *All the trigonometric identities of* (31)–(56)*, on the bases of identities* (8)–(11)*, also may be given for* S^{+}_{n} *and* S^{-}_{n} *functions.*

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