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ON THE SOLUTION OF A RATIONAL RECURSIVE SEQUENCE OF BIG ORDER

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Abstract. In this paper we dealed with the solution of the following nonlinear difference equations

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-5}x_{n-11}}, \quad n = 0, 1, ...,$$

where the initial values x_{-j}, $(j = 0, 1, ..., 11)$ are arbitrary non zero real numbers.

1. Introduction

Our aim in this paper is to obtain the solution of the following nonlinear difference equations

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-5}x_{n-11}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial values x_{-j}, $(j = 0, 1, ..., 11)$ are arbitrary non zero real numbers.

The study of Difference Equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear

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difference equations. For some results in this area, for example: Aloqeili [1] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [3–5] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Cinar et al.[6] studied the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [8] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

Elabbasy et al. [9] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [10] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Karatas et al. [15-16] obtained the solution of the following difference equation

$$x_{n+1} = \frac{a x_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}, \qquad x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek [19-20] solved the recursive sequences

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-1} x_{n-3} x_{n-5} x_{n-7} x_{n-9}}, \quad x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

In [22] the author obtained also the explicit form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}.$$

Other related results on rational difference equations can be found in refs. [2], [7], [11-26].

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. *A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if*

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2 (Periodicity)**.** *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

2. On the Difference Equation $x_{n+1} = \dfrac{x_{n-11}}{1 + x_{n-5}x_{n-11}}$

In this section we give a specific form of the first equation in the form

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-5}x_{n-11}}, \quad n = 0, 1, ..., \tag{3}$$

where the initial values are arbitrary positive real numbers.

Theorem 1. *Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of (3). Then for $n = 0, 1, 2, ...$*

$$x_{12n-11} = x_{-11} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}} \right), \quad x_{12n-10} = x_{-10} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{-4}x_{-10}}{1+(2i+1)x_{-4}x_{-10}} \right),$$

$$x_{12n-9} = x_{-9} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{-3}x_{-9}}{1+(2i+1)x_{-3}x_{-9}} \right), \quad x_{12n-8} = x_{-8} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{-2}x_{-8}}{1+(2i+1)x_{-2}x_{-8}} \right),$$

$$x_{12n-7} = x_{-7} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{-1}x_{-7}}{1+(2i+1)x_{-1}x_{-7}} \right), \quad x_{12n-6} = x_{-6} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{0}x_{-6}}{1+(2i+1)x_{0}x_{-6}} \right),$$

$$x_{12n-5} = x_{-5} \prod_{i=0}^{n-1} \left(\frac{1+(2i+1)x_{-5}x_{-11}}{1+(2i+2)x_{-5}x_{-11}} \right), \quad x_{12n-4} = x_{-4} \prod_{i=0}^{n-1} \left(\frac{1+(2i+1)x_{-4}x_{-10}}{1+(2i+2)x_{-4}x_{-10}} \right),$$

$$x_{12n-3} = x_{-3} \prod_{i=0}^{n-1} \left(\frac{1+(2i+1)x_{-3}x_{-9}}{1+(2i+2)x_{-3}x_{-9}} \right), \quad x_{12n-2} = x_{-2} \prod_{i=0}^{n-1} \left(\frac{1+(2i+1)x_{-2}x_{-8}}{1+(2i+2)x_{-2}x_{-8}} \right),$$

$$x_{12n-1} = x_{-1} \prod_{i=0}^{n-1} \left(\frac{1+(2i+1)x_{-1}x_{-7}}{1+(2i+2)x_{-1}x_{-7}} \right), \quad x_{12n} = x_{0} \prod_{i=0}^{n-1} \left(\frac{1+(2i+1)x_{0}x_{-6}}{1+(2i+2)x_{0}x_{-6}} \right).$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{12n-23} = x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right), \quad x_{12n-22} = x_{-10}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-4}x_{-10}}{1+(2i+1)x_{-4}x_{-10}}\right),$$

$$x_{12n-21} = x_{-9}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-3}x_{-9}}{1+(2i+1)x_{-3}x_{-9}}\right), \quad x_{12n-20} = x_{-8}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-2}x_{-8}}{1+(2i+1)x_{-2}x_{-8}}\right),$$

$$x_{12n-19} = x_{-7}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-1}x_{-7}}{1+(2i+1)x_{-1}x_{-7}}\right), \quad x_{12n-18} = x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{0}x_{-6}}{1+(2i+1)x_{0}x_{-6}}\right),$$

$$x_{12n-17} = x_{-5}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{-5}x_{-11}}{1+(2i+2)x_{-5}x_{-11}}\right), \quad x_{12n-16} = x_{-4}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{-4}x_{-10}}{1+(2i+2)x_{-4}x_{-10}}\right),$$

$$x_{12n-15} = x_{-3}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{-3}x_{-9}}{1+(2i+2)x_{-3}x_{-9}}\right), \quad x_{12n-14} = x_{-2}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{-2}x_{-8}}{1+(2i+2)x_{-2}x_{-8}}\right),$$

$$x_{12n-13} = x_{-1}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{-1}x_{-7}}{1+(2i+2)x_{-1}x_{-7}}\right), \quad x_{12n-12} = x_{0}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{0}x_{-6}}{1+(2i+2)x_{0}x_{-6}}\right).$$

Now, it follows from (3) that

$$\begin{aligned}
x_{12n-11} &= \frac{x_{12n-23}}{1+x_{12n-17}x_{12n-23}} = \\
&= \frac{x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right)}{1+x_{-5}\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_{-5}x_{-11}}{1+(2i+2)x_{-5}x_{-11}}\right)x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right)} = \\
&= \frac{x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right)}{1+x_{-11}x_{-5}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+2)x_{-5}x_{-11}}\right)} = \\
&= \frac{x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right)}{1+\left(\frac{x_{-11}x_{-5}}{1+(2n-2)x_{-5}x_{-11}}\right)}\left(\frac{1+(2n-2)x_{-5}x_{-11}}{1+(2n-2)x_{-5}x_{-11}}\right) = \\
&= \frac{x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right)(1+(2n-2)x_{-5}x_{-11})}{1+(2n-2)x_{-5}x_{-11}+x_{-11}x_{-5}} = \\
&= x_{-11}\prod_{i=0}^{n-2}\left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}}\right)\left(\frac{1+(2n-2)x_{-5}x_{-11}}{1+(2n-1)x_{-5}x_{-11}}\right).
\end{aligned}$$

Hence, we have

$$x_{12n-11} = x_{-11} \prod_{i=0}^{n-1} \left(\frac{1+2ix_{-5}x_{-11}}{1+(2i+1)x_{-5}x_{-11}} \right).$$

Similarly from (3) we see that

$$\begin{aligned} x_{12n-16} &= \frac{x_{12n-18}}{1+x_{12n-12}x_{12n-18}} = \\ &= \frac{x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}}\right)}{1+x_0\prod_{i=0}^{n-2}\left(\frac{1+(2i+1)x_0x_{-6}}{1+(2i+2)x_0x_{-6}}\right)x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}}\right)} = \\ &= \frac{x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}}\right)}{1+x_0x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+2)x_0x_{-6}}\right)} = \\ &= \frac{x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}}\right)}{1+\left(\frac{x_0x_{-6}}{1+(2n-2)x_0x_{-6}}\right)}\left(\frac{1+(2n-2)x_0x_{-6}}{1+(2n-2)x_0x_{-6}}\right) = \\ &= \frac{x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}}\right)(1+(2n-2)x_0x_{-6})}{1+(2n-2)x_0x_{-6}+x_0x_{-6}} = \\ &= x_{-6}\prod_{i=0}^{n-2}\left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}}\right)\left(\frac{1+(2n-2)x_0x_{-6}}{1+(2n-1)x_0x_{-6}}\right). \end{aligned}$$

Then, we get

$$x_{12n-16} = x_{-6} \prod_{i=0}^{n-1} \left(\frac{1+2ix_0x_{-6}}{1+(2i+1)x_0x_{-6}} \right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 2. *Equation* (3) *has one equilibrium point which is the zero.*

Proof. For the equilibrium points of Eq.(3), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^2}.$$

Then we have

$$\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}^3 = 0.$$

Thus the equilibrium point of (3) is $\overline{x} = 0$. □

Theorem 3. *Every positive solution of* (3) *is bounded.*

Proof. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of (3). It follows from (3) that

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-5}x_{n-11}} \leq x_{n-11}.$$

Then

$$x_{n+1} \leq x_{n-11} \quad \text{for all} \quad n \geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so is bounded from above by $M = \max\{x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$. □

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to (3).

Example 1. *We assume* $x_{-11} = 13,\ x_{-10} = 9,\ x_{-9} = 1.8, x_{-8} = 0.7,\ x_{-7} = 0.4,\ x_{-6} = 0.2,\ x_{-5} = 1.3,\ x_{-4} = 6, x_{-3} = 0.2,\ x_{-2} = 4,\ x_{-1} = 0.2,\ x_0 = 4$ *See Fig. 1.*

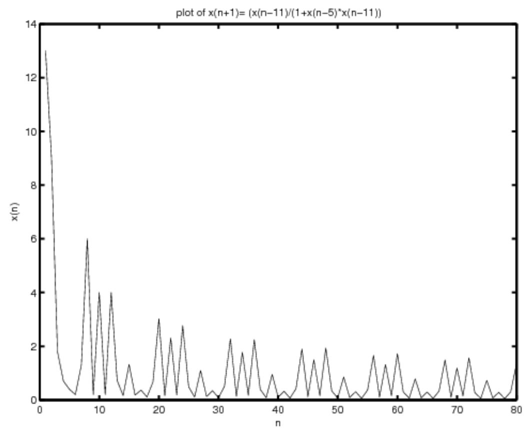

FIGURE 1.

Example 2. *See Fig. 2, since* $x_{-11} = 3,\ x_{-10} = 17,\ x_{-9} = 8,\ x_{-8} = 7,\ x_{-7} = 8,\ x_{-6} = 5,\ x_{-5} = 3,\ x_{-4} = 1.6,\ x_{-3} = 0.9,\ x_{-2} = 4,\ x_{-1} = 12,\ x_0 = 9.$

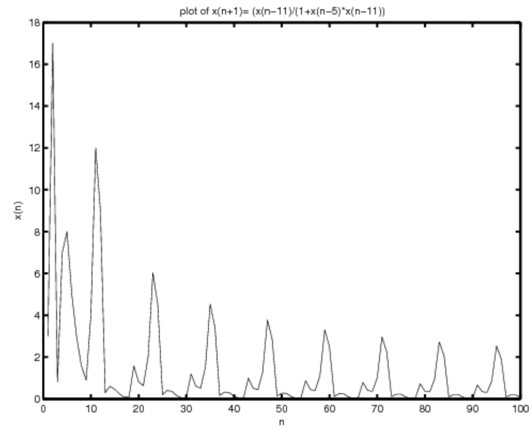

Figure 2.

3. On the Difference Equation $x_{n+1} = \frac{x_{n-11}}{-1 + x_{n-5}x_{n-11}}$

In this section we obtain the solution of the second equation in the form

$$x_{n+1} = \frac{x_{n-11}}{-1 + x_{n-5}x_{n-11}}, \quad n = 0, 1, ..., \tag{4}$$

where the initial values are arbitrary non zero real numbers with $x_{i-5}x_{i-11} \neq 1$ $(for\ i = 0, 1, 2, 3, 4, 5)$.

Theorem 4. *Let* $\{x_n\}_{n=-11}^{\infty}$ *be a solution of* (4)*. Then for* $n = 0, 1, ...$

$$x_{12n-11} = \frac{x_{-11}}{(-1 + x_{-5}x_{-11})^n}, \qquad x_{12n-10} = \frac{x_{-10}}{(-1 + x_{-4}x_{-10})^n},$$
$$x_{12n-9} = \frac{x_{-9}}{(-1 + x_{-3}x_{-9})^n}, \qquad x_{12n-8} = \frac{x_{-8}}{(-1 + x_{-2}x_{-8})^n},$$
$$x_{12n-7} = \frac{x_{-7}}{(-1 + x_{-1}x_{-7})^n}, \qquad x_{12n-6} = \frac{x_{-6}}{(-1 + x_{0}x_{-6})^n},$$
$$x_{12n-5} = x_{-5}\left(-1 + x_{-5}x_{-11}\right)^n, \qquad x_{12n-4} = x_{-4}\left(-1 + x_{-4}x_{-10}\right)^n,$$
$$x_{12n-3} = x_{-3}\left(-1 + x_{-3}x_{-9}\right)^n, \qquad x_{12n-2} = x_{-2}\left(-1 + x_{-2}x_{-8}\right)^n,$$
$$x_{12n-1} = x_{-1}\left(-1 + x_{-1}x_{-7}\right)^n, \qquad x_{12n} = x_{0}\left(-1 + x_{0}x_{-6}\right)^n.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{12n-23} = \frac{x_{-11}}{(-1+x_{-5}x_{-11})^{n-1}}, \qquad x_{12n-22} = \frac{x_{-10}}{(-1+x_{-4}x_{-10})^{n-1}},$$
$$x_{12n-21} = \frac{x_{-9}}{(-1+x_{-3}x_{-9})^{n-1}}, \qquad x_{12n-20} = \frac{x_{-8}}{(-1+x_{-2}x_{-8})^{n-1}},$$
$$x_{12n-19} = \frac{x_{-7}}{(-1+x_{-1}x_{-7})^{n-1}}, \qquad x_{12n-18} = \frac{x_{-6}}{(-1+x_{0}x_{-6})^{n-1}},$$
$$x_{12n-17} = x_{-5}\left(-1+x_{-5}x_{-11}\right)^{n-1}, x_{12n-16} = x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n-1},$$
$$x_{12n-15} = x_{-3}\left(-1+x_{-3}x_{-9}\right)^{n-1}, \ x_{12n-2} = x_{-14}\left(-1+x_{-2}x_{-8}\right)^{n-1},$$
$$x_{12n-13} = x_{-1}\left(-1+x_{-1}x_{-7}\right)^{n-1}, \ x_{12n-12} = x_{0}\left(-1+x_{0}x_{-6}\right)^{n-1}.$$

Now, it follows from (4) that

$$\begin{aligned} x_{12n-10} &= \frac{x_{12n-22}}{-1+x_{12n-16}x_{12n-22}} = \\ &= \frac{\dfrac{x_{-10}}{(-1+x_{-4}x_{-10})^{n-1}}}{-1+x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n-1}\dfrac{x_{-10}}{(-1+x_{-4}x_{-10})^{n-1}}} = \\ &= \frac{\dfrac{x_{-10}}{(-1+x_{-4}x_{-10})^{n-1}}}{-1+x_{-4}x_{-10}}. \end{aligned}$$

Hence, we have

$$x_{12n-10} = \frac{x_{-10}}{(-1+x_{-4}x_{-10})^{n}}.$$

Similarly from (4) we see that

$$\begin{aligned} x_{12n-4} &= \frac{x_{12n-16}}{-1+x_{12n-10}x_{12n-16}} = \\ &= \frac{x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n-1}}{-1+\dfrac{x_{-10}}{(-1+x_{-4}x_{-10})^{n}}x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n-1}} = \\ &= \frac{x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n-1}}{-1+\dfrac{x_{-4}x_{-10}}{(-1+x_{-4}x_{-10})}}\left(\frac{-1+x_{-4}x_{-10}}{-1+x_{-4}x_{-10}}\right) = \\ &= \frac{x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n}}{1-x_{-4}x_{-10}+x_{-4}x_{-10}}. \end{aligned}$$

Then, we get

$$x_{12n-4} = x_{-4}\left(-1+x_{-4}x_{-10}\right)^{n}.$$

Similarly, one can obtain the other relations. Thus, the proof is completed. □

Theorem 5. *Equation* (4) *has three equilibrium points which are* $0, \sqrt{2}, -\sqrt{2}$.

Proof. For the equilibrium points of Eq.(4), we can write

$$\overline{x} = \frac{\overline{x}}{-1+\overline{x}^2}.$$

Then we have

$$-\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}(\overline{x}^2 - 2) = 0.$$

Thus the equilibrium points of (4) are $0, \sqrt{2}, -\sqrt{2}$. □

Lemma 1. *It is easy to see that every solution of Eq.(4) is unbounded except in the following case.*

Theorem 6. *Equation* (4) *has a periodic solutions of period* (12) *if and only if* $x_{i-5}x_{i-11} = 2$ $(for\ i = 0, 1, 2, 3, 4, 5)$ *and will be take the form*

$$\{x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_{-11}, x_{-10}, ...\}.$$

Proof. First suppose that there exists a prime period (12) solution

$$x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_{-11}, x_{-10}, ...,$$

of (4), we see from the form of solution of (4) that

$$\begin{aligned}
x_{-11} &= \frac{x_{-11}}{(-1+x_{-5}x_{-11})^n}, & x_{-10} &= \frac{x_{-10}}{(-1+x_{-4}x_{-10})^n},\\
x_{-9} &= \frac{x_{-9}}{(-1+x_{-3}x_{-9})^n}, & x_{-8} &= \frac{x_{-8}}{(-1+x_{-2}x_{-8})^n},\\
x_{-7} &= \frac{x_{-7}}{(-1+x_{-1}x_{-7})^n}, & x_{-6} &= \frac{x_{-6}}{(-1+x_0x_{-6})^n},\\
x_{-5} &= x_{-5}\left(-1+x_{-5}x_{-11}\right)^n, & x_{-4} &= x_{-4}\left(-1+x_{-4}x_{-10}\right)^n,\\
x_{-3} &= x_{-3}\left(-1+x_{-3}x_{-9}\right)^n, & x_{-2} &= x_{-2}\left(-1+x_{-2}x_{-8}\right)^n,\\
x_{-1} &= x_{-1}\left(-1+x_{-1}x_{-7}\right)^n, & x_0 &= x_0\left(-1+x_0x_{-6}\right)^n.
\end{aligned}$$

or,

$$\begin{aligned}
(-1+x_{-5}x_{-11})^n &= 1, & (-1+x_{-4}x_{-10})^n &= 1,\\
(-1+x_{-3}x_{-9})^n &= 1, & (-1+x_{-2}x_{-8})^n &= 1,\\
(-1+x_{-1}x_{-7})^n &= 1, & (-1+x_0x_{-6})^n &= 1.
\end{aligned}$$

Then

$$x_{-5}x_{-11} = x_{-4}x_{-10} = x_{-3}x_{-9} = x_{-2}x_{-8} = x_{-1}x_{-7} = x_0x_{-6} = 2.$$

Second suppose that

$$x_{-5}x_{-11} = x_{-4}x_{-10} = x_{-3}x_{-9} = x_{-2}x_{-8} = x_{-1}x_{-7} = x_0x_{-6} = 2.$$

Then we see from (4) that

$$\begin{array}{lll} x_{12n-11} = x_{-11}, & x_{12n-10} = x_{-10}, & x_{12n-9} = x_{-9}, \\ x_{12n-8} = x_{-8}, & x_{12n-7} = x_{-7}, & x_{12n-6} = x_{-6}, \\ x_{12n-5} = x_{-5}, & x_{12n-4} = x_{-4}, & x_{12n-3} = x_{-3}, \\ x_{12n-2} = x_{-2}, & x_{12n-1} = x_{-1}, & x_{12n} = x_0. \end{array}$$

Thus we have a period (12) solution and the proof is complete. □

Numerical examples

Here we will represent different types of solutions of Eq. (4).

Example 3. *We consider* $x_{-11} = 1.3$, $x_{-10} = 1.7$, $x_{-9} = 5$, $x_{-8} = 0.7$, $x_{-7} = 1.8$, $x_{-6} = 5$, $x_{-5} = 3$, $x_{-4} = 6$, $x_{-3} = 9$, $x_{-2} = 0.4$, $x_{-1} = 1.2$, $x_0 = 0.9$. *See Fig. 3.*

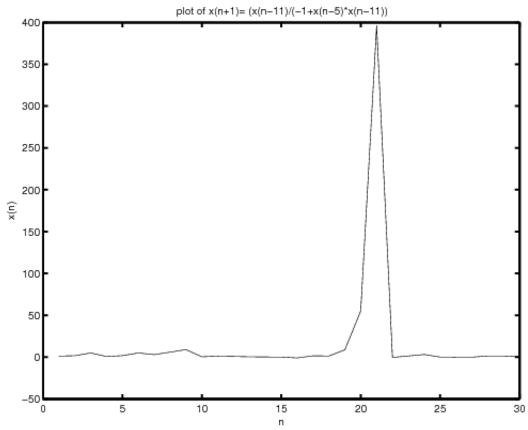

FIGURE 3.

Example 4. *See Fig. 4, since* $x_{-11} = 3$, $x_{-10} = 7$, $x_{-9} = 9$, $x_{-8} = 2$, $x_{-7} = 5$, $x_{-6} = 8$, $x_{-5} = 2/3$, $x_{-4} = 2/7$, $x_{-3} = 2/9$, $x_{-2} = 1$, $x_{-1} = 2/5$, $x_0 = 2/8$.

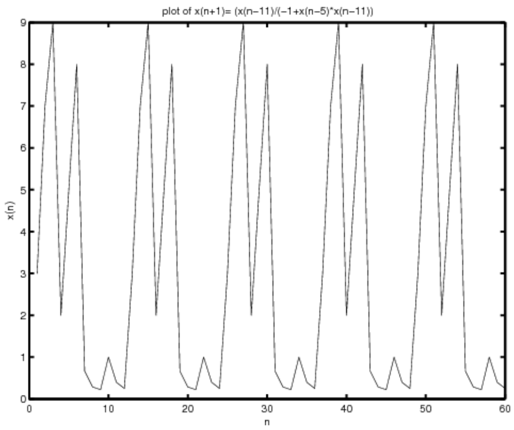

FIGURE 4.

The following cases can be proved similarly.

4. On the Difference Equation $x_{n+1}=\dfrac{x_{n-11}}{1-x_{n-5}x_{n-11}}$

In this section we get the solution of the third following equation

$$x_{n+1}=\frac{x_{n-11}}{1-x_{n-5}x_{n-11}},\quad n=0,1,...,\tag{5}$$

where the initial values are arbitrary non zero real numbers.

Theorem 7. *Let* $\{x_n\}_{n=-2k-1}^{\infty}$ *be a solution of* (5). *Then for* $n=0,1,...$

$$x_{12n-11}=x_{-11}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-5}x_{-11}}{1-(2i+1)x_{-5}x_{-11}}\right),\ x_{12n-10}=x_{-10}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-4}x_{-10}}{1-(2i+1)x_{-4}x_{-10}}\right),$$

$$x_{12n-9}=x_{-9}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-3}x_{-9}}{1-(2i+1)x_{-3}x_{-9}}\right),\quad x_{12n-8}=x_{-8}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-2}x_{-8}}{1-(2i+1)x_{-2}x_{-8}}\right),$$

$$x_{12n-7}=x_{-7}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-1}x_{-7}}{1-(2i+1)x_{-1}x_{-7}}\right),\quad x_{12n-6}=x_{-6}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{0}x_{-6}}{1-(2i+1)x_{0}x_{-6}}\right),$$

$$x_{12n-5}=x_{-5}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-5}x_{-11}}{1-(2i+2)x_{-5}x_{-11}}\right),\quad x_{12n-4}=x_{-4}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-4}x_{-10}}{1-(2i+2)x_{-4}x_{-10}}\right),$$

$$x_{12n-3}=x_{-3}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-3}x_{-9}}{1-(2i+2)x_{-3}x_{-9}}\right),\quad x_{12n-2}=x_{-2}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-2}x_{-8}}{1-(2i+2)x_{-2}x_{-8}}\right),$$

$$x_{12n-1}=x_{-1}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-1}x_{-7}}{1-(2i+2)x_{-1}x_{-7}}\right),\ x_{12n}=x_{0}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{0}x_{-6}}{1-(2i+2)x_{0}x_{-6}}\right).$$

Theorem 8. *Equation* (5) *has a unique equilibrium point which is the number zero.*

Example 5. *Assume that* $x_{-11}=3,\ x_{-10}=7,\ x_{-9}=9,\ x_{-8}=2,\ x_{-7}=5,\ x_{-6}=8,\ x_{-5}=1.3,\ x_{-4}=1.7,\ x_{-3}=0.8,\ x_{-2}=1.5,\ x_{-1}=3.5,\ x_{0}=0.7$ *see Fig. 5*

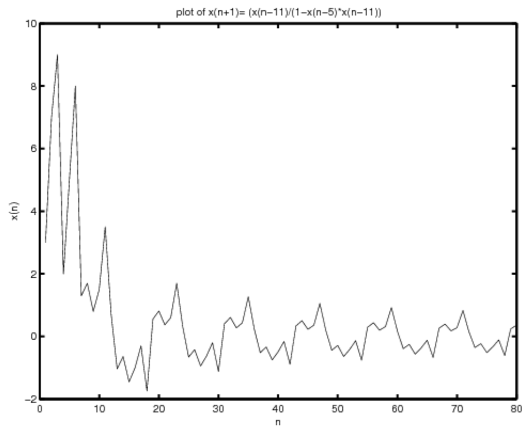

FIGURE 5.

Example 6. *See Fig. 6 since* $x_{-11}=3,\ x_{-10}=1.7,\ x_{-9}=0.9,\ x_{-8}=2,\ x_{-7}=5,\ x_{-6}=8,\ x_{-5}=0.3,\ x_{-4}=7,\ x_{-3}=5.8,\ x_{-2}=5,\ x_{-1}=2,\ x_{0}=7.$

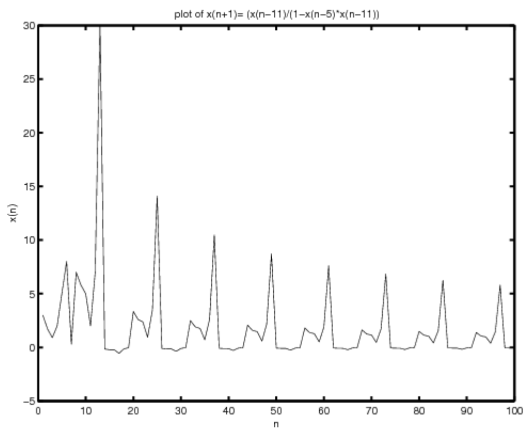

FIGURE 6.

5. On the Difference Equation $x_{n+1} = \dfrac{x_{n-11}}{-1 - x_{n-5}x_{n-11}}$

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-11}}{-1 - x_{n-5}x_{n-11}}, \quad n = 0, 1, ..., \tag{6}$$

where the initial values are arbitrary non zero real numbers with $x_{i-5}x_{i-11} \neq -1$ $(for\ i = 0, 1, 2, 3, 4, 5)$.

Theorem 9. *Let* $\{x_n\}_{n=-11}^{\infty}$ *be a solution of* (6)*. Then for* $n = 0, 1, ...$

$$x_{12n-11} = \frac{(-1)^n x_{-11}}{(1 + x_{-5}x_{-11})^n}, \qquad x_{12n-10} = \frac{(-1)^n x_{-10}}{(1 + x_{-4}x_{-10})^n},$$
$$x_{12n-9} = \frac{(-1)^n x_{-9}}{(1 + x_{-3}x_{-9})^n}, \qquad x_{12n-8} = \frac{(-1)^n x_{-8}}{(1 + x_{-2}x_{-8})^n},$$
$$x_{12n-7} = \frac{(-1)^n x_{-7}}{(1 + x_{-1}x_{-7})^n}, \qquad x_{12n-6} = \frac{(-1)^n x_{-6}}{(1 + x_{0}x_{-6})^n},$$
$$x_{12n-5} = (-1)^n x_{-5} (1 + x_{-5}x_{-11})^n, \quad x_{12n-4} = (-1)^n x_{-4} (1 + x_{-4}x_{-10})^n,$$
$$x_{12n-3} = (-1)^n x_{-3} (1 + x_{-3}x_{-9})^n, \quad x_{12n-2} = (-1)^n x_{-2} (1 + x_{-2}x_{-8})^n,$$
$$x_{12n-1} = (-1)^n x_{-1} (1 + x_{-1}x_{-7})^n, \quad x_{12n} = (-1)^n x_{0} (1 + x_{0}x_{-6})^n.$$

Theorem 10. *Equation* (6) *has a unique equilibrium point which is the number zero.*

Lemma 2. *It is easy to see that every solution of* (6) *is unbounded except in the following case.*

Theorem 11. *Equation* (6) *has a periodic solutions of period* (12) *if and only if* $x_{i-5}x_{i-11} = -2$ $(for\ i = 0, 1, 2, 3, 4, 5)$ *and will be take the form*

$$\{x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_{-11}, x_{-10}, ...\}.$$

Example 7. *Consider* $x_{-11} = 3,\ x_{-10} = 7,\ x_{-9} = 9,\ x_{-8} = 2,\ x_{-7} = 5,\ x_{-6} = 8,\ x_{-5} = 1.8,\ x_{-4} = 2.7,\ x_{-3} = 0.8,\ x_{-2} = 11,\ x_{-1} = 0.2,\ x_0 = 13$ *see Fig. 7*

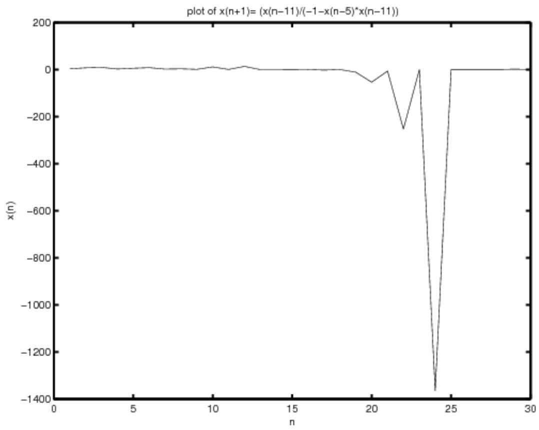

FIGURE 7.

Example 8. *Fig. 8. shows the solutions when* $x_{-11} = 3,\ x_{-10} = 7,\ x_{-9} = 9,\ x_{-8} = 2,\ x_{-7} = 5,\ x_{-6} = 8,\ x_{-5} = -2/3,\ x_{-4} = -2/7,\ x_{-3} = -2/9,\ x_{-2} = -1,\ x_{-1} = -2/5,\ x_0 = -2/8.$

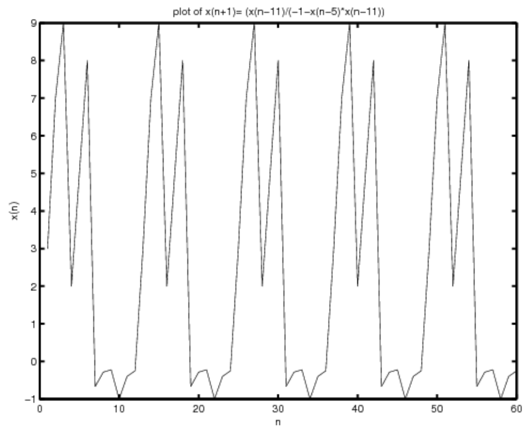

FIGURE 8.

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