Folia Mathematica
Vol. 14, No. 1, pp. 33–38
Acta Universitatis Lodziensis

DESCRIPTIVE CHARACTER OF SETS OF ψ-DENSITY POINTS

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Abstract. Let $X = [0,1]$ and $A \subset X^2$ is a Borel set. In the paper the following problem is discussed: consider the set $D_\psi(A)$ of all $(x,y) \in X^2$ such that A_x is measurable and y is ψ-density point of A_x. Is $D_\psi(A)$ a Borel set? Can we estimate the Borel class of this set if the Borel class of A is assumed? Is this set analytic (coanalytic) while A is analytic (coanalytic)?

Let $\mathbb{N}$ be the sets of all natural numbers (including 0). $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ will stand for the sets of all rational, real and positive real numbers. By $\mathcal{L}_k$ we will mean the class of Lebesgue measurable sets in $\mathbb{R}^k$, $k = 1,2$. The Lebesgue measure on the real line and on the real plane will be denoted by m and m_2, respectively. Let $\mathcal{C}$ be the family of all nondecreasing, continuous functions $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to 0^+} \psi = 0$. Put $X = [0,1]$.

We say that x is a ψ-density point of a measurable set $A \subset \mathbb{R}$ if

$$\lim_{h\to 0^+} \frac{m(A' \cap [x-h, x+h])}{2h\psi(2h)} = 0,$$

where A' denotes a complement of A. For any measurable set $A \subset \mathbb{R}$, we put

$$\Phi_\psi(A) = \{x \in \mathbb{R}\colon x \text{ is a } \psi\text{-density point of } A\}.$$

From [4] we obtain that the set $\Phi_\psi(A)$ is $F_{\sigma\delta}$ and the family $\mathcal{T}_\psi = \{A \in \mathcal{L}_1\colon A \subset \Phi_\psi(A)\}$ is a topology stronger than the natural topology $\mathcal{T}_o$ and weaker than the density topology $\mathcal{T}_d$. For $A \subset X^2$, $x \in X$ and a function $\psi \in \mathcal{C}$ we put

$$A_x = \{y \in X\colon (x,y) \in A\}$$

and

$$D_\psi(A) = \{(x,y) \in X^2\colon A_x \in \mathcal{L}_1 \wedge y \in \Phi_\psi(A_x)\}.$$

In the analogous way we can define the set $D(A)$ for the density topology. The behaviour of operator $D(\cdot)$ with respect to classes of Borel, analytic and

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Key words and phrases: analytic set, Borel set, density point, ψ - density point.
AMS subject classifications: 04A15, 28A05, 54H05.

coanalytic subsets was studied in [1] and [2]. We are going to obtain similar results for $D_\psi(\cdot)$. From [5] it follows that the symmetric difference $A \triangle D(A)$ is a set of plane measure zero for each $A \in \mathcal{L}_2$. We do not obtain similar result for the set of the form $A \triangle D_\psi(A)$ due to the following theorem

Theorem 1 ([4, Theorem 0.2])**.** *For any function $\psi \in \mathcal{C}$ and any $\alpha \in (0,1)$ there exists a perfect set $P \subset [0,1]$ such that $m(P) = \alpha$ and $\Phi_\psi(P) = \emptyset$.*

Proposition 1. *There exists a measurable set $E \subset X^2$ of positive plane measure such that $m_2(E \triangle D_\psi(E)) > 0$.*

Proof. Let $\alpha \in (0,1)$ be an arbitrary real number and $\psi \in \mathcal{C}$. Let P be a perfect set from Theorem 1 such that $m(P) = \alpha > 0$ and $\Phi_\psi(P) = \emptyset$. We put $E = X \times P$. Hence $m_2(E) > 0$ and $D_\psi(E) = \emptyset$, since $E_x = P$ so $E_x \in \mathcal{L}_1$ for $x \in X$ and $\Phi_\psi(P) = \emptyset$. So $m_2(E \triangle D_\psi(E)) > 0$. □

The following example shows that even if $A \subset X^2$ is open, the set $D_\psi(A)$ does not need to be open.

Example 1. *There exist an open set $A \subset \mathbb{R}^2$ and a function $\psi \in \mathcal{C}$ such that $D_\psi(A)$ is not open.*

Let $A = \mathbb{R} \times B$, where

$$B = \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^n} - \frac{1}{2^n (n+2)!}, \frac{1}{2^n} \right] \cup \{0\} \right).$$

Obviously A is open in the natural topology on the plane. Let the function ψ be defined as follows:

$$\psi(x) = \begin{cases} \frac{1}{(n+2)!} & \text{for} \quad x \in \left[\frac{1}{2^n} - \frac{1}{2^n (n+2)!}, \frac{1}{2^n} \right], n \in \mathbb{N} \setminus \{0\}, \\ \text{linear} & \text{for} \quad x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{2^n (n+2)!} \right), n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

From [4, Theorem 0.1] it follows that 0 is a ψ-density point of the open set B. Hence $(x,0) \in D_\psi(A)$ for any $x \in \mathbb{R}$, and $(x,0)$ is not an interior point of $D_\psi(A)$, so this set is not open in the natural topology on the plane.

Proposition 2. *Let $A \subset X^2$ and let A_x be measurable for all $x \in X$. Then*

$$D_\psi(A) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcap_{h \in (0, \frac{1}{k+1}) \cap \mathbb{Q}} T(n,h), \tag{1}$$

where

$$T(n,h) = \left\{ (x,y) \in X^2 \colon m(A'_x \cap [y-h, y+h]) \le \frac{2h\psi(2h)}{n+1} \right\}.$$

Proof. The inclusion $D_\psi(A) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{h\in(0,\frac{1}{k+1})\cap\mathbb{Q}} T(n,h)$ is obvious. Indeed, let $(x,y) \in D_\psi(A)$. Hence A_x is measurable and y is a ψ-density point of A_x. From the definition of ψ-density points we have

$$\lim_{h\to 0^+} \frac{m(A'_x \cap [y-h, y+h])}{2h\psi(2h)} = 0.$$

Therefore for any $n \in \mathbb{N}$ there exists a number $k \in \mathbb{N}$ such that for any rational $h \in (0, \frac{1}{k+1})$ we have

$$\frac{m(A'_x \cap [y-h, y+h]}{2h\psi(2h)} \le \frac{1}{n+1}.$$

This by the definition of $T(n,h)$ ends the proof of this inclusion.

We will show the contrary inclusion. We consider the function

$$g(h) = \frac{m(A'_x \cap [y-h, y+h])}{2h\psi(2h)}$$

for $h > 0$. It is continuous for $h > 0$. Hence from inequality

$$g(h) \le \frac{1}{n+1}$$

fulfilled for $h \in (0, \frac{1}{k+1}) \cap \mathbb{Q}$ we have the same inequality for $h \in (0, \frac{1}{k+1})$. □

Following [3] (page 68) we introduce the Borel hierarchy of sets, consisting of the open, closed, F_σ, G_δ, etc., sets. Let Y be a metrizable space, so any closed subset of Y is a G_δ set. Let ω_1 be the first uncountable ordinal. For any $1 \le \alpha < \omega_1$ we define the classes Σ^0_α and Π^0_α of subsets of Y as follows:

$\Sigma^0_1 = \{U \subset Y : U \text{ is open }\}$, $\Pi^0_\alpha = \sim \Sigma^0_\alpha$,

$\Sigma^0_\alpha = \{\bigcup_{n\in\mathbb{N}} A_n : A_n \in \Pi^0_{\alpha_n},\ \alpha_n < \alpha,\ n \in \mathbb{N}\}$, if $\alpha > 1$.

Hence $\Sigma^0_2 = F_\sigma$, $\Pi^0_2 = G_\delta$, $\Sigma^0_3 = G_{\delta\sigma}$, $\Pi^0_3 = F_{\sigma\delta}$, etc.

In the next considerations we will assume that $A \subset X^2$.

Proposition 3 ([1]). *If $A \in \Sigma^0_\alpha$, $0 < \alpha < \omega_1$, then for $r \in \mathbb{R}$*

$$\{x \in X : m(A_x) > r\} \in \Sigma^0_\alpha.$$

Remark 1. *If $A \in \Sigma^0_\alpha$, $0 < \alpha < \omega_1$, then for $r \in \mathbb{R}$*

$$\{x \in X : m(A'_x) < r\} \in \Sigma^0_\alpha.$$

Proof. Obviously $m(A'_x) < r$ if and only if $m(A_x) > m(X) - r$, so

$$\{x \in X : m(A'_x) < r\} = \{x \in X : m(A_x) > m(X) - r\}$$

and the remark follows from Proposition 3. □

The following theorem estimates the Borel class of the set $D_\psi(A)$ when the Borel class of the set A is assumed.

Theorem 2. *If $A \in \Sigma^0_\alpha$, then $D_\psi(A) \in \Pi^0_{\alpha+3}$.*

Proof. If $A \in \Sigma^0_\alpha$ then A is a Borel set. Hence A_x is also Borel for any $x \in X$, so it is measurable. From Proposition 2 we obtain that

$$D_\psi(A) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{h\in(0,\frac{1}{k+1})\cap\mathbb{Q}} T(n,h),$$

where

$$T(n,h) = \left\{(x,y) \in X^2 \colon m(A'_x \cap [y-h, y+h]) \leq \frac{2h\psi(2h)}{n+1}\right\}.$$

Let $p \in \mathbb{N}$. The function $y \mapsto m(A'_x \cap [y-h, y+h])$ is continuous, so for $y \in X$ there exists a rational number s such that $|y-s| < \frac{1}{p+1}$ and

$$m(A'_x \cap [s-h, s+h]) < \frac{2h\psi(2h)}{n+1} + \frac{1}{p+1}.$$

Hence

$$\begin{aligned} T(n,h) &= \bigcap_{p\in\mathbb{N}} \bigcup_{s\in\mathbb{Q}} \Bigg(\left\{ x \in X : m(A'_x \cap [s-h, s+h]) < \frac{2h\psi(2h)}{n+1} + \frac{1}{p+1} \right\} \\ &\times \left\{ y \in X \colon |y-s| < \frac{1}{p+1} \right\} \Bigg). \end{aligned}$$

Let us notice that

$$\begin{aligned} &\{x \in X \colon m(A'_x \cap [s-h, s+h]) < r\} = \\ =\ &\{x \in X \colon m\left(A_x \cap [s-h, s+h]\right) > 2h - r\} = \\ =\ &\{x \in X \colon m((A \cap (X \times [s-h, s+h]))_x) > 2h - r\}. \end{aligned}$$

From the above and Proposition 3 we obtain that

$$\{x \in X \colon m(A'_x \cap [s-h, s+h]) < r\} \in \Sigma^0_\alpha.$$

Therefore $T(n,h)$ belongs to the class $\Pi^0_{\alpha+1}$. Thus by (1) we obtain that $D_\psi(A) \in \Pi^0_{\alpha+3}$. □

In the next considerations we will answer the question: is $D_\psi(A)$ analytic (coanalytic) if A is so?

Proposition 4 ([1, Prop. 2.1])**.** *If $A \subset X^2$ is analytic, then for any $h > 0$ and $r \in \mathbb{R}$ the set*

$$\{(x,y) \in X^2 \colon m(A_x \cap [y-h, y+h]) > r\}$$

is analytic.

Remark 2. *If $A \subset X^2$ is analytic, then for any $h > 0$ and $r \in \mathbb{R}$ the set*

$$\{(x,y) \in X^2 \colon m(A'_x \cap [y-h, y+h]) \le r\}$$

is analytic.

Proof. We have the following equalites

$$\begin{aligned} & \{(x,y) \in X^2 \colon m(A'_x \cap [y-h, y+h]) \le r\} = \\ = \ & \{(x,y) \in X^2 \colon m(A_x \cap [y-h, y+h]) \ge 2h - r\} = \\ = \ & \bigcap_{n=1}^{\infty} \{(x,y) \in X^2 \colon m(A_x \cap [y-h, y+h]) > 2h - r - \frac{1}{n}\}. \end{aligned}$$

The class of analytic sets is closed under countable intersections (see [3, Prop. 14.4]). This and Proposition 4 finishes the proof. □

Theorem 3. *If $A \subset X^2$ is analytic (coanalytic), then $D_\psi(A)$ is analytic (coanalytic).*

Proof. Let us notice that if A is analytic (coanalytic), then A_x is analytic (coanalytic). So it is measurable ([3], Theorem 29.7) for any $x \in X$.

Let us assume that A is analytic. From Proposition 2 we have

$$D_\psi(A) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{h\in(0,\frac{1}{k+1})\cap\mathbb{Q}} T(n,h),$$

where

$$T(n,h) = \left\{(x,y) \in X^2 \colon m(A'_x \cap [y-h, y+h]) \le \frac{2h\psi(2h)}{n+1}\right\}.$$

From Remark 2 the set $T(n,h)$ is analytic for arbitrary $n \in \mathbb{N}$ and $h > 0$. Therefore $D_\psi(A)$ is analytic as countable union and countable product of analytic sets (see [3, Prop.14.4]).

Let us assume now that A is coanalytic. Then $X^2 \setminus A$ is analytic and A_x is measurable for any $x \in X$, so from Proposition 2

$$D_\psi(A) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{h\in(0,\frac{1}{k+1})\cap\mathbb{Q}} T(n,h),$$

where

$$T(n,h) = \left\{(x,y) \in X^2 \colon m((X^2 \setminus A)_x \cap [y-h, y+h]) \le \frac{2h\psi(2h)}{n+1}\right\}.$$

Obviously

$$T(n,h) = X^2 \setminus \left\{(x,y) \in X^2 \colon m((X^2 \setminus A)_x \cap [y-h, y+h]) > \frac{2h\psi(2h)}{n+1}\right\}.$$

The set $X^2 \setminus A$ is analytic, so by Proposition 4 the set

$$\left\{(x, y) \in X^2 \colon m((X^2 \setminus A)_x \cap [y-h, y+h]) > \frac{2h\psi(2h)}{n+1}\right\}$$

is also analytic. Consequently, the set $T(n, h)$ is coanalytic and so is $D_\psi(A)$. □

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