Folia Mathematica
Vol. 14, No. 1, pp. 25–31
Acta Universitatis Lodziensis

SOME REMARKS ON SMITAL'S LEMMA

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Abstract. In the paper the generalization of Smital's lemma for Lebesgue-Stieltjes measure is discussed. It is shown that for absolutely continuous Lebesgue-Stieltjes measure the Smital's lemma is always true, for singular Lebesgue-Stieltjes measure it can happen that the lemma is not true and for purely discontinuous Lebesgue-Stieltjes measure the lemma is always false.

1. Introduction

Throughout the paper R will denote the set of real numbers. Lebesgue outer (resp. inner) measure in the real line will be denoted by λ^* (resp. λ_*) whereas λ will stand for the Lebesgue measure itself. Moreover the same notation convention: τ^*, τ_* will be adopted in case of a measure τ defined on σ-algebra of subsets of R, wich is denoted by S.

We will say that a measure τ is a Borel measure if S contains all Borel subsets of R. A measure τ is called regular if:

- $\tau(F) < \infty$, for all compact sets F,
- $\tau(A) = \inf \{\tau(U) : U \text{ is open}, A \subset U\}$ for any set $A \in S$,
- $\tau(U) = \sup \{\tau(F) : F \text{ is compact }, F \subset U\}$ for any set $U \in S$.

Let $g : R \longrightarrow R$ be nondecreasing, left-continuous function. We define λ_g as the measure generated by the additive interval function:

$$\lambda_g ([a, b)) = g(b) - g(a),$$

called the Lebesgue-Stieltjes measure.

Let S_g be a σ-algebra of measurable sets with respect to the Lebesgue-Stieltjes measure λ_g. Then for any nondecreasing, left-continuous function g, σ-algebra S_g contains all Borel sets ([4, p. 71]).

In particular, when $g(x) = x$ then we obtain a Lebesgue measure λ.

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Key words and phrases: Smital's Lemma, Lebesque-Stieltjes measure.
AMS subject classifications: 28A12,28A05,54H05.

2. Regular Borel measures on R

Theorem 1 ([2, p. 329]). *Let τ be any regular Borel measure on R. Define function h by the rule:*

$$h(t) = \begin{cases} \tau(\langle 0, t)) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \\ -\tau(\langle t, 0)) & \text{if} \quad t < 0 \end{cases}$$

Then h is nondecreasing, real-valued, left-continuous function on R.

Theorem 2 ([2, p. 330]). *If g is a nondecreasing, left continuous function satysfying $g(0) = 0$, λ_g is Lebesgue-Stieltjes measure, induced by the function g, and h is the function from Theorem 1, then $g = h$.*

We say, that τ_a is absolutely continuous with respect to λ (and denote $\tau_a \ll \lambda$), if $\lambda(A) = 0$ implies $\tau_a(A) = 0$ for any $A \subset R$. We say that τ_s is singular with respect to λ, (and we denote $\tau_s \perp \lambda$) if there exists a set $A \subset R$, such that $\tau_s(A) = 0$ and $\lambda(A') = 0$.

A measure τ_d, defined on the σ-algebra 2^R we will call a purelly discontinuous measure if there exists a countable set $X = \{x_k : k \in N\} \subset R$ and a function $\varphi : X \longrightarrow [0, \infty)$ satysfying condition $\Sigma \{\varphi(x_k) : |x_k| \leq n\} < \infty$ for any $n \in N$, such that for any $A \subset R$, $\tau_d(A) = \Sigma_{k=1}^{\infty} \varphi(x_k) \cdot \chi_A(x_k)$, where χ_A denotes a characteristic function of A.

Theorem 3 ([2, p. 337]). *Let τ be any regular Borel measure on R. Then τ can be expressed in just one way of the form:*

$$\tau = \tau_a + \tau_s + \tau_d,$$

where τ_a, τ_s, τ_d are regular Borel measures on R, $\tau_a \ll \lambda$, $\tau_s \perp \lambda$, τ_d is purely discontinous and τ_s is continuous (i.e. $\tau_s(\{x\}) = 0$ for all $x \in R$).

If g, g_a, g_s, g_d are the corresponding nondecreasing functions to τ, τ_a, τ_s, τ_d, then:

$$g = g_a + g_s + g_d$$

where g_a is absolutely continuous on every compact interval, g_s is continuous and ${g_s}' = 0$ a.e., and g_d is a saltus function.

3. Generalization of Smital's lemma for Lebesque-Stieltjes measure

Lemma 1 (Smital's lemma, [1, p. 65]). *Let B, D, be such that $\lambda^*(B) > 0$ and D is dense in R. Put $A = B + D$. Then $\lambda_*(A') = 0$.*

Many mathematicians presented generalized Smital's lemma. In 1988 R. Ger, Z. Kominek and M. Sablik [3] replace the usual addition by a more general two-place function and the Lebesgue outer measure by a possibly general one. However, there exist measures which are absolutely continuous with respect to the Lebesgue measure and which are not included in their results.

In this paper we will prove that for any Lebesgue-Stieltjes measure τ_a absolutely continuous with respect to λ, the analoguous of Smital's lemma is true, whereas for singular Lebesgue-Stielties measure τ_s it can happen that the analoguous of Smital's lemma is not true and for purely discontinuous Lebesgue-Stiltjes measure τ_d the analoguous of Smital's lemma is always false.

Theorem 4. *Let τ_a be a Lebesgue-Stieltjes measure, absolutely continuous with respect to λ and different from zero. Then if B and D are such that $\tau_a^*(B) > 0$ and D is dense in R, then $\tau_{a*}(R \setminus (B + D)) = 0$.*

Proof. Let $B, D \subset R$ satisfy assumptions of Theorem 4. Since τ_a is absolutely continuous with respect to λ, $\lambda^*(B) > 0$. Put $A = B + D$. From lemma A we know that $\lambda_*\left(A'\right) = 0$. It means that if $E \subset A'$ is λ- measurable, then $\lambda(E) = 0$. We want to show that $\tau_{a*}\left(A'\right) = 0$.

Let $E \subset A'$ be τ_a - measurable. We have to prove that $\tau_a(E) = 0$. The set E, which is τ_a - measurable can be represented as: $E = E_1 \cup E_2$, where $E_1 \in F_\sigma$ and $\tau_a(E_2) = 0$, because τ_a is regular. It is obvious that the set E_1 is λ - measurable (E_1 is a Borel set) and $E_1 \subset A'$, therefore $\lambda(E_1) = 0$. Then we obtain $\tau_a(E_1) = 0$, because $\tau_a \ll \lambda$. Finally $\tau_a(E) = \tau_a(E_1) + \tau_a(E_2) = 0$ and $\tau_{a*}\left(A'\right) = 0$. □

Theorem 5. *There exists a Lebesgue-Stieltjes measure τ_s, singular with respect to λ and different from zero and there exist B and D such that ${\tau_s}^*(B) > 0$ and D is dense in R and $\tau_{s*}(R \setminus (B + D)) > 0$.*

We prove the following Lemma which will be useful in the proof of Theorem 5:

Lemma 2. *If P is a classical Cantor set, then there exists a dense set D such that for any $z \in D$ card $(P \cap (P + z)) \leq \aleph_0$.*

Proof. We show that the set D can be expressed in the form:

$$D = (-\infty, -1) \cup (1, \infty) \cup \bigcup_{n=1}^{\infty} \left(\left\{ z_k = \frac{2k-1}{3^n}, k = 1, 2, ..., \frac{3^n+1}{2} \right\} \cup \right.$$
$$\left. \cup \left\{ z_k = -\frac{2k-1}{3^n}, k = 1, 2, ..., \frac{3^n+1}{2} \right\} \right).$$

Let P be the Cantor set. We denote by $I_{n,k}$ the open intervals which are removed from interval $[0, 1]$ in the n-th step of the construction of Cantor set. After the n-th step we obtain 2^n closed intervals $J_{n,k}$ each of length $\frac{1}{3^n}$. Let $P_n = \bigcup_{k=1}^{2^n} J_{n,k}$, $n \in N$.Then $P = \bigcap_{n=1}^{\infty} P_n$.

Let ${C_n}^+$ be a set of positive z such that stright line $y = x - z$ intersects $P_n \times P_n$ in a finite number of points.

Let's consider $P_1 \times P_1 = \bigcup_{k=1}^{2^1} \bigcup_{m=1}^{2^1} J_{1,k} \times J_{1,m}$. We observe that ${C_1}^+ = \left\{\frac{1}{3}, 1\right\}$.

When we consider $P_2 \times P_2 = \bigcup_{k=1}^{2^2} \bigcup_{m=1}^{2^2} J_{2,k} \times J_{2,m}$ we observe that ${C_2}^+ = \left\{\frac{1}{9}, \frac{3}{9}, \frac{5}{9}, \frac{7}{9}, 1\right\}$ and for $P_3 \times P_3 = \bigcup_{k=1}^{2^3} \bigcup_{m=1}^{2^3} J_{3,k} \times J_{3,m}$ we have: ${C_3}^+ = \left\{\frac{1}{27}, \frac{3}{27}, \frac{5}{27}, \frac{7}{27}, \frac{9}{27}, \frac{11}{27}, \frac{13}{27}, \frac{15}{27}, \frac{17}{27}, \frac{19}{27}, \frac{21}{27}, \frac{23}{27}, \frac{25}{27}, 1\right\}$. Analoguously C_n^+

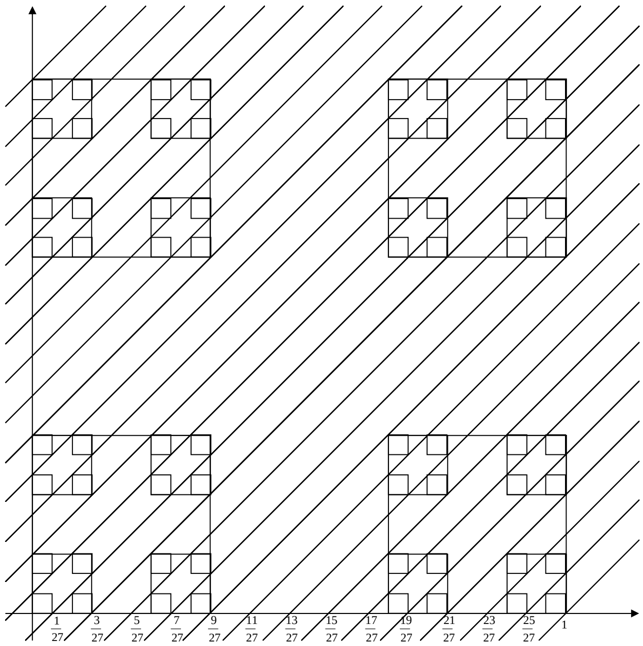

FIG. 1

can be expressed in the form:

$$C_n^+ = \left\{ z_k : z_k = \frac{2k-1}{3^n}, \qquad k = 1, 2, \ldots, \quad \frac{3^n+1}{2} \right\} \quad \text{for} \quad n \in N.$$

We will show it by induction.

We observe that if z_i, z_{i+1} are the succesive points of set ${C_n}^+$, then these points belong also to a set C_{n+1}^+.

Moreover, the points $\frac{2z_i+z_{i+1}}{3}$, $\frac{z_i+2z_{i+1}}{3}$ and the point $\frac{1}{3}z_1$ belong to set C_{n+1}^+. Indeed, if a square $J_{n,k} \times J_{n,m}$ is a component of $P_n \times P_n$ and straight line $y = x - z_i$ crosses the left upper vertex of square $J_{n,k} \times J_{n,m}$ and straight line $y = x - z_{i+1}$ crosses the right lower vertex of square $J_{n,k} \times J_{n,m}$, then the straight lines $y = x - \frac{2z_i + z_{i+1}}{3}$ and $y = x - \frac{z_i + 2z_{i+1}}{3}$ cross the set

$$(J_{n+1,2k-1} \times J_{n+1,2m-1}) \cup (J_{n+1,2k} \times J_{n+1,2m-1}) \cup (J_{n+1,2k-1} \times J_{n+1,2k}) \cup \\ \cup (J_{n+1,2k} \times J_{n+1,2m}) \subset J_{n,k} \times J_{n,m}$$

in a finite number of points (see Fig. 2).

Similarly we show that $\frac{1}{3}z_1 \in C^+_{n+1}$.

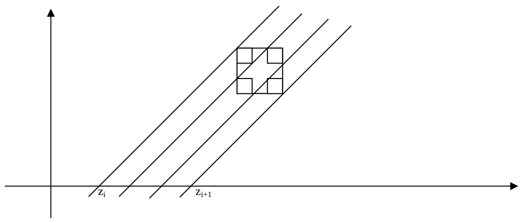

FIG. 2

Let $n \in N$. If the straight line $y = x - z$ intersects $P_n \times P_n$ in a finite number of points, then $y = x - z$ intersects $P \times P$ in finite number of points (because $P \times P \subset P_n \times P_n$).

We show that for any $z \in C^+_n$ $card\,(P \cap (P + z)) \leq \aleph_0$.

If $(x, y) \in P\times$ P then $(x, x - z) \in P \times P$. Hence $x \in P$ and $(x - z) \in P$. It means that $x \in P$ and $x \in (P + z)$, therefore $x \in (P \cap (P + z))$.

Let C^-_n be a set of negative z such that straight line $y = x + z$ intersects $P_n \times P_n$ in a finite number of points. Then we analoguously show that C^-_n can be expressed in the form:

$$C_n{}^- = \left\{ z_k : z_k = -\frac{2k-1}{3^n}, \qquad k = 1, 2, \ldots, \frac{3^n + 1}{2} \right\}$$

Now, the set D can be expressed in the form:

$$D = (-\infty, -1) \cup (1, \infty) \cup \bigcup_{n=1}^{\infty} \left(C^+_n \cup C^-_n\right)$$

. □

Now we can prove Theorem 5.

Proof. Let $g_0 : [0,1] \longmapsto [0,1]$ be a Cantor function, and

$$g(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ g_0(x) & \text{for} \quad x \in [0,1] \\ 1 & \text{for} \quad x > 1 \end{cases}$$

and λ_g be the measure generated by the function g. It is easy to check that λ_g is singular with respect to λ. Put $\tau_s = \lambda_g$. Let $B = P \cap \left[\frac{2}{3}, 1\right]$ and D be such as in lemma 1. We have $\tau_s(B) = \frac{1}{2} > 0$.

Let's consider a set $(B + D) \cap \left(P \cap \left[0, \frac{1}{3}\right]\right)$;

$$B+D = \bigcup_{z \in D} (B+z) = \bigcup_{z \in (-\infty,-1)} (B+z) \cup \bigcup_{z \in (1,\infty)} (B+z) \cup \bigcup_{z \in D \cap [-1,1]} (B+z).$$

It is easy to see that for $z < -1$ and $z > 1$ we have $(B+z) \cap \left(P \cap \left[0, \frac{1}{3}\right]\right) = \emptyset$.

Therefore

$$(B+D) \cap \left(P \cap \left[0, \frac{1}{3}\right]\right) = \bigcup_{z \in D \cap [-1,1]} (B+z) \cap \left(P \cap \left[0, \frac{1}{3}\right]\right) \subset$$

$$\subset \left(\bigcup_{z \in D \cap [-1,1]} (P+z) \right) \cap P = \bigcup_{z \in D \cap [-1,1]} ((P+z) \cap P).$$

We have

$$card\left((B+D) \cap \left(P \cap \left[0, \frac{1}{3}\right]\right)\right) \leq card\left(\bigcup_{z \in D \cap [-1,1]} (P+z) \cap P \right) \leq \aleph_0.$$

We observe that:

$$\begin{aligned} R \setminus (B+D) &\supset \left(P \cap \left[0, \frac{1}{3}\right]\right) \setminus (B+D) \\ &= \left(P \cap \left[0, \frac{1}{3}\right]\right) \setminus \left((B+D) \cap \left(P \cap \left[0, \frac{1}{3}\right]\right)\right). \end{aligned}$$

The last set is measurable since it is the difference of a measurable set and a countable set.

Moreover:

$$\tau_s\left(P \cap \left[0, \frac{1}{3}\right] \setminus (B+D)\right) = \tau_s\left(P \cap \left[0, \frac{1}{3}\right]\right) = \frac{1}{2}.$$

Hence $\tau_{s*}(R \setminus (B+D)) \geq \frac{1}{2} > 0$. □

Theorem 6. *For any Lebesgue-Stieltjes measure τ_d - purely discontinous, different from zero, there exist sets B and D, such that $\tau_d^*(B) > 0$ and D is dense in R and $\tau_{d*}(R \setminus (B + D)) > 0$.*

Proof. Since τ_d is purely discontinuous, there exists at most countable set $\{x_1, x_2, \ldots, x_n, \ldots\}$ such that $\tau_d(\{x_n\}) > 0$ for $n \in N$ and

$$\tau_d(R \setminus \{x_1, x_2, \ldots, x_n, \ldots\}) = 0.$$

Next $B = \{x_1\}$ and $D = R \setminus \{0\}$. We have $R \setminus (B + D) = \{x_1\}$, so

$$\tau_d(R \setminus (B + D)) \geq \tau_d(\{x_1\}) > 0.$$

□

Acknowledgement. I wish to thank Professor Władysław Wilczyński for his help concerning my work.

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