Folia Mathematica
Vol. 14, No. 1, pp. 17–23
Acta Universitatis Lodziensis

NONLINEAR DIFFERENCE EQUATIONS IN BANACH SPACES

ANNA KISIOŁEK[‡], IRENEUSZ KUBIACZYK[‡‡]

Abstract. In this paper we consider the third order difference equations in Banach spaces:

$$\Delta(r_n \Delta^2 x_n) + p_n f(n+2, x_{n+2}) = h_n, \quad p_n \in R, r_n \in R, h_n \in X$$

and the second order difference equations of the form:

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n f(n-k, x_{n-k}) = h_n.$$

We show, that this equations has a solution asymptotically equal to c.

We suppose that f has values in Banach space and satisfies some conditions with respect to the measure of noncompactness and measure of weak noncompactness.

1. Introduction

Let $l_1(C)$ be the space of complex valued sequences (c_n) such that:

$$\|(c_n)\|_1 := \sum_{i=1}^{\infty} |c_n| < \infty.$$

Let $(X, \|\cdot\|)$ be a real Banach space and $l_\infty(X)$ denote the space of bounded sequences $x = (x_n)$ in X with the norm:

$$\|x\|_\infty = \|(x_n)\|_\infty = \sup_n \|x_n\|.$$

With this norm $l_\infty(X)$ is a Banach space.

In this note, the results in [5] are extend, function f is a continuous mapping in Banach space, which is condensing with respect to the measure of noncompactness. We give sufficient conditions for the existence of asymptotically constants solutions.

[‡] *University of Technology, Institute of Mathematics*, 5 Maria Sklodowska-Curie Sq., 60-965 Poznań, Poland. E-mail: akisiolek@fabryka.net.pl.

[‡‡] *Adam Mickiewicz University, Collegium Mathematicum*, Umultowska 87 St., 61-614 Poznań, Poland. E-mail: kuba@amu.edu.pl.

Key words and phrases: difference equations, Banach spaces, asymptotic behaviour, measure of noncompactness, existence theorem.

AMS subject classifications: 39A10.

The second order difference equations:

$$\Delta(r_{n-1}\Delta x_{n-1} + a_n f(x_{n-k}) = b_n$$

was considered by M. Migda [5]. She gave sufficient conditions on the a_n and on the function f so that, this equations has solutions asymptotically constants.

2. Main results

Let f be the function from $N^+ \times X$ to X and (p_n), (r_n) are sequences of real numbers, (h_n) is a sequence in Banach space.

Consider the third order nonlinear difference equations:

$$(*) \qquad \Delta(r_n\Delta^2 x_n) + p_n f(n+2, x_{n+2}) = h_n, \quad n \in N,$$

where Δ denotes the difference operator and is defined as usual i.e.:

$$\Delta x_n = x_{n+1} - x_n, \quad \text{for} \ \ n \in N.$$

By a solution of equation $(*)$ we mean a sequence (x_n), which satisfies $(*)$ for all $n > N_0$ for some $N_0 \in N$.

Our result will be proved by the following fixed point theorem.

Theorem 1. [1] *Let D be a nonempty, closed, convex and bounded subset of Banach space. Let $F\, D \to D$ be a continuous mapping, which is condensing with respect to the measure of noncompactness α:*

$$(**) \qquad \alpha(F(V)) \le L\alpha(V), \quad L < 1,$$

for any subset V of D, where α is the Kuratowski's measure of noncompactness. Then F has fixed point.

Theorem 2. *Let $V \subset C(N^+, X)$ be a family of functions. Then:*

$$\alpha(V) = \alpha(V(N^+)) = \sup\{\alpha(V(i)) : i \in N^+\},$$

where $\alpha(V)$ denotes the measure of noncompactness in $C(N^+, X)$.

Theorem 3. *Let $f\, N^+ \times X \to X$ be the bounded and uniformly continuous function and:*

$$(1) \qquad \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}|p_i| < \infty$$

$$(2) \qquad \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}\|h_i\| < \infty$$

Assume that:

$$\alpha(f(n, V)) \le a\alpha(V),$$

for any $n > N$ and for any bounded subset V of X, and for some $a > 0$ such that:

$$L = a \sum_{j=m}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |p_i| < 1.$$

Then for $c \in X$ there exists a solution $x = (x_n)$ of equation $(*)$ such that:

$$x_n \to c.$$

Proof. Let $c \in X$. Then there exists a constant $M > 1$ such that $\|f(i,t)\| \leq M$ for $t \in X$, $i \in N$.

Assume that:

$$k_i = |p_i| + \|h_i\| \quad \text{for} \quad i = 1, 2, \ldots$$

By (1) and (2) we obtain that $\sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} k_i$ is convergent.

Assume that:

$$b = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} k_i \quad \text{for} \quad n \in N.$$

Now define the operator $T\ D \to D$, where:

$$D = \{y = (y_1, y_2, y_3, \ldots),\ \|y_i - c\| \leq Mb,\ i = 1, 2, 3, \ldots\}.$$

For $x \in D$, let $Tx = (Tx)_n$ be given by:

$$(Tx)_n = \begin{cases} c, n \leq m, \\ c - \sum_{j=n}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} (h_i - p_i f(i+2, x_{i+2})), n > m. \end{cases}$$

For $n > m$ we have:

$$\left\| c - \sum_{j=n}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} (h_i - p_i f(i+2, x_{i+2})) - c \right\| =$$

$$= \left\| \sum_{j=n}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} (h_i - p_i f(i+2, x_{i+2})) \right\| \leq$$

$$\leq \sum_{j=n}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} \|h_i - p_i f(i+2, x_{i+2})\| \leq$$

$$\leq \sum_{j=n}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} (\|h_i\| + |p_i|\, \|f(i+2, x_{i+2})\|) \leq$$

$$\leq \sum_{j=n}^{\infty} \sum_{k=j}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} (\|h_i\| + M|p_i|) \leq Mb.$$

So the operator $T D \to D$ and the continuity of f, implies that T is continuous.

Now, we will prove that T satisfies condition $(**)$ of Theorem 1.

Let $V \subset D$, where $V = \{v v = (v_1, v_2, \ldots)\}$ and $T(i, V) = \{T(i, v) v \in V\}$.

Let $V_k = \{v_k\, v \in V, v = (v_1, v_2, \ldots, v_k, \ldots)\}$.

For $n \leq m$ we obtain:

$$\alpha(T(i, V)) = \alpha(c) = 0.$$

For $n > m$ we have:

$$\begin{aligned}
\alpha(T(n,V)) &= \sup_n \alpha\left(c - \sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}(h_i - p_i f(n, V_n))\right) \leq \\
&\leq \sup_n \left[\alpha(c) + \alpha\left(\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}(h_i - p_i f(n, V_n))\right)\right] \leq \\
&\leq \sup_n \alpha(c) + \sup_n \alpha\left(\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}(h_i - p_i f(n, V_n))\right) \leq \\
&\leq \sup_n \alpha\left[\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}h_i + \sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}p_i f(n, V_n)\right] \leq \\
&\leq \sup_n \alpha\left[\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}p_i f(n, V_n)\right] \\
&\quad + \sup_n \alpha\left[\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}h_i\right] \leq \\
&\leq \sup_n \left(\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}\alpha(p_i f(n, V_n))\right) \leq \\
&\leq \sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}|p_i|\alpha(f(n, V)).
\end{aligned}$$

Using the inequality:

$$\alpha(f(n, V)) \leq a\alpha(V)$$

we obtain, that:

$$\alpha(T(n,V)) \le a\alpha(V)\sum_{j=n}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}|p_i| \le L\alpha(V).$$

By Theorem 1, T has a fixed point, and by the definition of T the solution $x=(x_n)$ satisfies the condition:

$$x_n \to c.$$

□

Example 1. *We adduce a sequence, which fulfil condition (1) and (2). Let* $p_i = q^i,\ 0 < q < 1$ *then*

$$\sum_{i=k}^{\infty} q^i = \frac{q^k}{1-q}.$$

Assume $r_k = \frac{1}{1-q}$, *then we obtain* $\frac{1}{r_k}\sum_{i=k}^{\infty}|p_i| = q^k$ *and further* $\sum_{k=n}^{\infty} q^k = \frac{q^n}{1-q}$ *and*

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q} = \frac{q}{(1-q)^2}.$$

Consequently, this sequence is convergent, so the condition (1) is satisfy. We can choose $r_k = \frac{q_1^k}{1-q}$, *where* $q < q_1 < 1$ *and then the condition (1) will be fulfil too. Hence, the condition (2) will be satisfy when we can take* h_i *such that*

$$\|h_i\| = q^i.$$

So, if $0 \le p_i \le q^i$ *conditions (1) and (2) are satisfies. Now, we consider the following second order difference equation:*

$$(***) \qquad \Delta(r_{n-1}\Delta x_{n-1}) + p_n f(n-l, x_{n-l}) = h_n,$$

where f *is a function from* $N^+ \times X$ *to* X, (p_n), (r_{n-1}) *are sequences of real numbers,* (h_n) *is a sequences in Banach space.*

By a solution of equation $(***)$ *we mean a sequences* (x_n) *which is defined for* $n \ge -l$ *and which satisfies equation* $(***)$ *for* $n > N_0$ *for some* $N_0 \in N$

Theorem 4. *Let* $f\, N^+ \times X \to X$ *be the bounded and uniformly continuous function. Suppose that:*

$$(3) \qquad \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{1}{r_j}|p_n| < \infty,$$

(4) $$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{1}{r_j}\|h_n\| < \infty.$$

Assume that for some $a > 0$:

$$L = a\sum_{j=m}^{\infty}\sum_{k=0}^{n-1}\frac{1}{r_k}|p_j| < 1.$$

Moreover for any bounded $V \subset X$, and for any $n > N$:

$$\alpha(f(n, V)) \leq a\alpha(V).$$

Let $c \in X$. Then there exists a generalized solution (x_n) of equation $(***)$ such that

$$\lim_{n\to\infty} x_n = c.$$

We can prove this theorem, work similarly like in the proof of Theorem 3.

Remark 1. *We can generalizes Theorem 3 for measure of the weak non-compactness β (see* [4]*).*

Let $f\ N^+ \times X \to X$ be the bounded and weakly–weakly continuous function and:

$$\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{1}{r_j}\sum_{i=k}^{\infty}|p_i| < \infty,$$

$$\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{1}{r_j}\sum_{i=k}^{\infty}\|h_i\| < \infty.$$

Assume that:

$$\beta(f(n, V)) \leq a\beta(V),$$

for any bounded subset V of X, and for some $a > 0$ such that:

$$L = a\sum_{j=m}^{\infty}\sum_{k=j}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}|p_i| < 1.$$

Then for any $c \in X$ there exists a solution $x = (x_n)$ of equation $(*)$ such that:

$$x_n \to c.$$

References

[1] G. Darbo, *Punti uniti in transformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), pp. 84–92.

[2] A. Drozdowicz, J. Popenda, *Asymptotic behaviour of the solutions of difference equation of second order*, J. Comp. Appl. Math. **47** (1993), pp. 141–149.

[3] R. E. Edwards, *Functional Analysis*, Holt. Rinehart and Winston Inc., New York 1965.

[4] I. Kubiaczyk, *On a fixed point theorem for weakly sequentially continuous mapping*, Discussiones Math. – Differential Inclusions **15** (1995), pp. 15–20.

[5] M. Migda, *Asymptotic behaviour of solutions of nonlinear delay difference equations*, Fasciculi Math. **31** (2001), pp. 57–62.

[6] M. Migda, J. Migda, *Asymptotic properties of the solutions of the second order difference equation*, Archivum Math., **34** (1998), pp. 467–476.

[7] M. Migda, E. Schmeidel, A. Drozdowicz *Nonoscillation Results For Some Third Order Nonlinear Difference Equations*, Folia Facultatis Scientarium Naturalium Universitatis Masarykianae Brunesis Mathematica **13** (2003), pp. 185–192.

[8] J. Popenda, E. Schmeidel, *Nonoscillatory solutions of third order difference eqiation*, Portugaliae Mathematica **49**, 2 (1992), pp. 233–239.

[9] B. Smith, W.E. Taylor, *Asymptotic behaviour of the solutions of the third order difference equation*, Portugaliae Mathematica **44**, 2 (1987), pp. 113–117.

[10] B.G. Zhang, *Oscillation and asymptotic behaviour of second order difference equation*, J. Math. Anal. Appl. **173** (1993), pp. 58–68.