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CHARACTERIZATIONS OF SOLVABILITY OF NONLINEAR NONCONVEX VECTORIAL OPTIMIZATION PROBLEMS IN BANACH SPACES BY η-APPROXIMATION APPROACH

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Abstract. In this paper, we propose an η-approximation method for characterizing solvability of a nonlinear nonconvex constrained vectorial optimization problem defined in arbitrary Banach spaces. The equivalence between the original vector optimization problem and its associated η-approximated vectorial optimization problem, constructed in this method, is established under invexity assumption. Finally, the characterization of a (weakly) efficient solution in the original vectorial optimization problem by the help of an η-saddle point of the modified η-Lagrange function in its η-approximated vectorial optimization problem is also presented.

1. Introduction

In the past several decades there has been a lot of emphasis in the study of vector optimization. This class of problems has many applications in mathematical economy and engineering. Several optimality conditions and approaches to solve a nonlinear constrained vector optimization may be found in the literature (e.g. [1], [2], [5], [8], [10], [12], [13], [17]). However, there are various methods to obtain optimality conditions for vector optimization problems. One of them consists of a direct consideration of the vector optimization problem. However, many ideas and tools used in scalar optimization seem to be applicable to vector optimization (see, for example, [17]). For example, the separation theorems (often used in the form of theorems of the alternative) are the source of necessary optimality conditions for both scalar and vector optimization (see, for example, [14], [19], [22]). The second method is based on a replacement of the original vector optimization problem by scalar optimization problems (see, for example, [9], [12]). If the original multiobjective programming problem is essentially (in a certain

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sense) finite dimensional, it may prove sufficiently simple to get optimality conditions for vector optimization by this method. But sometimes it is required only that the number of objectives is finite. We add here that a great deal of study dealt with relating vector optimization problems to scalar optimization and there by developing the Karush-Kuhn-Tucker type optimality conditions (see, for example, [13], [20]). If the vector optimization problem is infinite dimensional, some scalarization may be applied (see, for example, [16]).

But in most of these studies, an assumption of convexity on the problem was made. However, in the last few years, various generalizations of the classic concept of convexity have been derived from it. One of such an useful generalization of convexity to extent of optimality conditions applicability is an invexity notion introduced by Hanson [6]. After the work of Hanson, other types of differentiable functions have appeared with the intent of generalizing invex functions from different points of view, also in the vectorial case (see, for example, [3], [5], [10], [21], and others).

In the recent years, considerable attention has been given to devising new methods by the help which solvability of the original multiobjective programming problem is characterized by solvability of some associated vector optimization problem. Two of such approaches are the so-called modified objective function method [1] and, as its extension, the so-called η-approximation method [2]. Antczak used them for characterizing solvability differentiable nonconvex multiobjective programming problems defined in finite dimensional spaces.

This paper deals with characterizations of solvability of vector optimization problems in arbitrary Banach spaces in terms of solvability of η-approximated vector optimization problems also defined in Banach spaces. Thus, the results obtained recently by Antczak in [2] for finite-dimensional case are extended to the case of vectorial optimization problems defined in arbitrary Banach spaces. Therefore, the aim of the present paper is to show how one can obtain optimality conditions for a nonlinear nonconvex constrained vectorial optimization problem in arbitrary Banach spaces by constructing for it an equivalent vector minimization problem. This associated η-approximated optimization problem is constructed that involves η-approximated functions (at an arbitrary but fixed feasible point $\overline{x}$) constructing the original vectorial optimization problem.

This construction depends heavily on results proved in this paper which connects the (weak) efficient points of the original multiobjective programming problems to the (weak) efficient points of the η-approximated vector minimization problems. To prove this equivalence, all functions constituting the original vector optimization problem are assumed to be invex (with respect to the same function η) and, moreover, some constraint is imposed on the function η. Thus, we obtain the associated η-approximated vectorial optimization problem with the same (weak) Pareto optimality solution and the optimality value equal to the optimality value in the original problem.

Furthermore, we apply the introduced η-approximation method to develop saddle points criteria for differentiable vectorial optimization problems in Banach spaces involving invex functions with respect to the same function η. We define an η-Lagrange function and η-saddle points criteria for the constructed η-approximated vectorial optimization problem. Thus, a characterization of solvability of the original multiobjective programming problems in Banach spaces by η-saddle points of the η-Lagrange function in its η-approximated vectorial optimization problems under invexity assumption is also presented.

2. Preliminaries

In what follows we will use the following notations in the present paper. Let Y be a Banach space, $K \subset Y$ a pointed closed, convex cone (i.e. $K \cap (-K) = \{0\}$) with nonempty interior. Let $a, b \in Y$, we define three cone orders with respect to K as follows:

$$a \leqq_K b \Longleftrightarrow b - a \in K,$$
$$a \leq_K b \Longleftrightarrow b - a \in K \backslash \{0\},$$
$$a <_K b \Longleftrightarrow b - a \in intK.$$

Throughout this section we shall assume that S is a nonempty open subset of a Banach space X and $f : S \to Y$ is a Fréchet differentiable function at $u \in S$. Furthermore, we denote by $Df(u)$ the Fréchet derivative of f at the point u.

Definition 1. *The function f is said to be invex at $u \in S$ on S with respect to η if there exists $\eta : S \times S \to X$ such that, for all $x \in S$,*

$$f(x) - f(u) \geqq_K Df(u)\eta(x, u)\,, \tag{1}$$

where $Df(u)\eta(x, u)$ is the value of the function $Df(u)$ applied in the vector $\eta(x, u) \in X$. If the inequality (1) holds for any $u \in S$ then f is invex on S with respect to η.

Definition 2. *The function f is said to be strictly invex at $u \in S$ on S with respect to η if there exists $\eta : S \times S \to X$ such that, for all $x \in S$, $x \neq u$,*

$$f(x) - f(u) >_K Df(u)\eta(x, u)\,. \tag{2}$$

If the inequality (2) holds for any $u \in S$ then f is strictly invex on S with respect to η.

The invexity notion for Fréchet differentiable functions in Banach spaces was considered, for example, by Batista dos Santos et al. [4], Kazmi [11]. We remark that, if we take R^n and R, respectively, in place of X and Y, then f is called invex in the sense of the Hanson's definition (see [6]).

In [6], Hanson also defined classes of generalized invex functions, that is, pseudo-invex and quasi-invex function (with respect to η). Now, we give definitions of these classes of generalized invex functions defined between Banach spaces.

Definition 3. *The function f is said to be pseudo-invex at $u \in S$ on S with respect to η if there exists $\eta : S \times S \to X$ such that, for all $x \in S$,*

$$Df(u)\eta(x,u) \geqq_K 0 \Longrightarrow f(x) - f(u) \geqq_K 0. \tag{3}$$

If the inequality (3) holds for any $u \in S$ then f is pseudo-invex on S with respect to η.

Definition 4. *The function f is said to be strictly pseudo-invex at $u \in S$ on S with respect to η if there exists $\eta : S \times S \to X$ such that, for all $x \in S$,*

$$Df(u)\eta(x,u) \geqq_K 0 \Longrightarrow f(x) - f(u) >_K 0. \tag{4}$$

If the inequality (4) holds for any $u \in S$ then f is strictly pseudo-invex on S with respect to η.

Definition 5. *The function f is said to be quasi-invex at $u \in S$ on S with respect to η if there exists $\eta : S \times S \to X$ such that, for all $x \in S$,*

$$f(x) - f(u) \leqq_K 0 \Longrightarrow Df(u)\eta(x,u) \leqq_K 0. \tag{5}$$

If the inequality (4) holds for any $u \in S$ then f is quasi-invex on S with respect to η.

In this paper, we consider the nonlinear constrained minimization problem

$$\begin{array}{rl} & K - \min f(x) \\ \text{subject to} & -g(x) \in C, \end{array} \qquad \text{(VP)}$$

where X, Y and Z are Banach spaces, $f : S \to Y$ and $g : S \to Z$ are (Fréchet) differentiable functions on a nonempty open set $S \subset X$. Further, we assume that Y and Z are ordered partially by the closed, convex, pointed cone, with nonempty interior $K \subset Y$, and by a closed convex cone $C \subset Z$ with $C \neq Z$, respectively.
Let

$$D := \{x \in S : -g(x) \in C\}$$

denote the set of all feasible solutions in (VP).

We denote by Y^* the topological dual of Y, and $\langle \cdot, \cdot \rangle$ the duality pairing between Y^* and Y (that is, $\langle y^*, y \rangle = y^*(y)$, $\forall y^* \in Y^*$, $\forall y \in Y$). Given $K \subset Y$ a convex cone, we define the dual cone of K,

$$K^* := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in K\}.$$

We use the following Lagrange function or the Lagrangian for problem (VP)

$$L(x, \lambda, \xi) := \langle \lambda, f(x) \rangle + \langle \xi, g(x) \rangle.$$

where $\lambda \in Y^*$ and $\xi \in Z^*$.

For such an optimization problem the following definitions will be accepted (see, for example, [8], [13], [15])

Definition 6. *The feasible point $\overline{x} \in D$ is said to be an efficient solution (Pareto optimal solution) if there does not exist a feasible solution x such that*

$$f(\overline{x}) - f(x) \in K \backslash \{0\},$$

or equivalently, the feasible point $\overline{x}$ is efficient if does not exist a feasible solution x such that

$$f(x) \leq_K f(\overline{x}).$$

Definition 7. *The feasible point $\overline{x} \in D$ is said to be a weakly efficient solution (weak Pareto optimal solution) if there does not exist a feasible solution x such that*

$$f(\overline{x}) - f(x) \in intK,$$

or equivalently, the feasible point $\overline{x}$ is efficient if does not exist a feasible solution x such that

$$f(x) <_K f(\overline{x})$$

It is easy to verify that every efficient solution is a weakly efficient solution.

The problem (VP) will be said to satisfy the generalized Slater constraint qualification at $\overline{x}$ if $-g(\overline{x}) \in intC$ and g is invex at $\overline{x}$ on D.

The following the Karush-Kuhn-Tucker necessary optimality conditions for Frèchet differentiable vectorial optimization problems will be required (see, for example, [13], [7]):

Theorem 1. *Let $\overline{x}$ be a (weak) Pareto solution in (VP) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then, there exist $\lambda^* \in Y^*$, $\xi^* \in Z^*$ such that the following Karush-Kuhn-Tucker necessary optimality conditions are fulfilled:*

$$\lambda^* Df(\overline{x}) + \xi^* Dg(\overline{x}) = 0, \tag{6}$$

$$\langle \xi^*, g(\overline{x}) \rangle = 0, \tag{7}$$

$$\lambda^* \in K^* \backslash \{0\}, \;\; \xi^* \in C^*. \tag{8}$$

3. An η-approximated optimization problem and optimality conditions for vectorial optimization problems defined in Banach spaces.

In the recent years, considerable attention has been given to devising new methods by the help which solvability of the original multiobjective programming problem is characterized by solvability of some associated vector optimization problem, for example, the η-approximation method introduced by Antczak in [2] for differentiable multiobjective programming problems defined in finite dimensional spaces. We now extend a definition of a so-called η-approximation vectorial optimization problem, constructed in this approach, to the case of vector optimization problems defined in infinite dimensional spaces.

Let $\overline{x}$ be the given feasible solution in the original multiobjective programming problem (VP) defined in Banach spaces. We consider the following η-approximated vectorial optimization problem ($\mathrm{VP}_\eta(\overline{x})$) given by

$$\begin{array}{rl} K-\min & \left(f(\overline{x})+Df(\overline{x})\eta\left(x,\overline{x}\right)\right) \\ \text{subject to} & g(\overline{x})+Dg(\overline{x})\eta\left(x,\overline{x}\right)\in -C, \end{array} \qquad (\mathrm{VP}_\eta(\overline{x}))$$

where f, g, K, C are defined as in the original vectorial optimization problem (VP), and η is a vectorial function defined by $\eta : S \times S \to X$ satisfying $\eta\left(x,\overline{x}\right) \neq 0$ whenever $x \neq \overline{x}$.

Let

$$D\left(\overline{x}\right) := \{x \in S : g(\overline{x}) + Dg(\overline{x})\eta\left(x,\overline{x}\right) \in -C\}$$

denote the set of all feasible solutions in ($\mathrm{VP}_\eta(\overline{x})$).

Now, we establish the equivalence between the vectorial optimization problems (VP) and ($\mathrm{VP}_\eta(\overline{x})$) defined in Banach spaces, that is, we prove that if $\overline{x}$ is a (weakly) efficient solution in the original vectorial optimization problem (VP) then it is also (weakly) efficient in its associated η-approximated vectorial optimization problem ($\mathrm{VP}_\eta(\overline{x})$), and conversely, if $\overline{x}$ is a (weakly) efficient solution in the η-approximated vectorial optimization problem ($\mathrm{VP}_\eta(\overline{x})$), then it is also (weakly) efficient in the original vectorial optimization problem (VP). It turns out that to establish this equivalence, we don't need to assume that the sets of all feasible solutions in the vectorial optimization problems (VP) and ($\mathrm{VP}_\eta(\overline{x})$), respectively, are the same.

Theorem 2. *Let $\overline{x}$ be a (weakly) efficient solution in the original vectorial optimization problem (VP) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. If $\eta\left(\overline{x},\overline{x}\right) = 0$, then $\overline{x}$ is also (weakly) efficient in its associated η-approximated vectorial optimization problem ($VP_\eta(\overline{x})$).*

Proof. By assumption, $\overline{x}$ is a (weakly) optimal solution in (VP). Then there exist $\lambda^* \in K^* \backslash \{0\}$, $\xi^* \in C^*$ such that the Karush-Kuhn-Tucker conditions (6)-(8) are satisfied.
We proceed by contradiction. Let $\overline{x}$ be not weakly efficient in $(\mathrm{VP}_\eta(\overline{x}))$. This implies that there exists $\widetilde{x}$ feasible for $(\mathrm{VP}_\eta(\overline{x}))$, i.e. $\widetilde{x} \in S$, $g(\overline{x}) + Dg(\overline{x})\eta(\widetilde{x}, \overline{x}) \in -C$, such that

$$f(\overline{x}) + Df(\overline{x})\eta(\widetilde{x}, \overline{x}) - (f(\overline{x}) + Df(\overline{x})\eta(\overline{x}, \overline{x})) \in -intK, \tag{9}$$

and, by $\eta(\overline{x}, \overline{x}) = 0$,

$$Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \in -intK. \tag{10}$$

Since $\lambda^* \in K^* \backslash \{0\}$ then (10) gives

$$\langle \lambda^*, Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle < 0. \tag{11}$$

By the feasibility of $\widetilde{x}$ in $(\mathrm{VP}_\eta(\overline{x}))$ together with $\xi^* \in C^*$ we obtain

$$\langle \xi^*, g(\overline{x}) + Dg(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle \leqq 0.$$

Thus, the Karush-Kuhn-Tucker necessary optimality condition (7) implies

$$\langle \xi^*, Dg(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle \leqq 0. \tag{12}$$

By (11) and (12), we get the following inequality

$$\langle \lambda^*, Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle + \langle \xi^*, Dg(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle < 0,$$

which is a contradiction to the Karush-Kuhn-Tucker optimality condition (6) and, therefore, it contradicts weak Pareto optimality of $\overline{x}$ in the original vectorial optimization problem (VP). Hence, $\overline{x}$ is weakly efficient in $(\mathrm{VP}_\eta(\overline{x}))$.

Proof for efficiency is analogous, but $intK$ should be replaced by $K \backslash \{0\}$. □

Remark 1. *Note that we have established Theorem 2 without any assumption to which a class of functions belong the functions involving in problems (VP) and ($VP_\eta(\overline{x})$), respectively. However, we assume that the generalized Slater constraint qualification is fulfilled at a (weak) Pareto optimal solution $\overline{x}$ in problem (VP). It turns out that this assumption is essential to establish Theorem 2 and it will be not omitted (see* [2], *Example 17).*

Now, under invexity assumption imposed on the objective function f and the constraint function g, we prove that a (weakly) efficient solution $\overline{x}$ in the η-approximized vectorial optimization problem $(\mathrm{VP}_\eta(\overline{x}))$ is also (weakly) efficient in the original vectorial optimization problem (VP).

Theorem 3. *Let $\overline{x}$ be a (weakly) efficient solution in the η-approximized vectorial optimization problem (VP$_\eta(\overline{x})$). Moreover, we assume that f and g are invex at $\overline{x}$ on D with respect to the same function η satisfying $\eta(\overline{x},\overline{x}) = 0$. Then $\overline{x}$ is also (weakly) efficient in the original vectorial optimization problem (VP).*

Proof. Suppose that $\overline{x}$ is not weakly efficient in the original vectorial optimization problem (VP). Then, by Definition 7, there exists $\widetilde{x}$ feasible for (VP), that is, $g(\widetilde{x}) \in -C$, such that

$$f(\widetilde{x}) - f(\overline{x}) \in -intK. \tag{13}$$

By assumption, f and g are invex at $\overline{x}$ on D with respect to the same function η. Then, by Definition 1, it follows that, respectively,

$$f(\widetilde{x}) - f(\overline{x}) - Df(\overline{x})\eta(\widetilde{x},\overline{x}) \in K, \tag{14}$$

$$g(\widetilde{x}) - g(\overline{x}) - Dg(\overline{x})\eta(\widetilde{x},\overline{x}) \in C. \tag{15}$$

Since $g(\widetilde{x}) \in -C$ then (15) gives

$$-g(\overline{x}) - Dg(\overline{x})\eta(\widetilde{x},\overline{x}) \in C + C = C.$$

This means that $\widetilde{x}$ is feasible for the η-approximized vectorial optimization problem (VP$_\eta(\overline{x})$), that is, $\widetilde{x} \in D(\overline{x})$. By (13) and (14), we get

$$-Df(\overline{x})\eta(\widetilde{x},\overline{x}) \in K + intK = intK. \tag{16}$$

By assumption, $\eta(\overline{x},\overline{x}) = 0$. Then, (16) gives

$$f(\overline{x}) + Df(\overline{x})\eta(\widetilde{x},\overline{x}) <_K f(\overline{x}) + Df(\overline{x})\eta(\overline{x},\overline{x}).$$

This means that $\overline{x}$ is not weakly efficient solution in the η-approximized vectorial optimization problem (VP$_\eta(\overline{x})$). Thus, the conclusion of this theorem is proved.

For the case of an efficient solution the proof of theorem is the same; just use $K + (K\backslash\{0\}) = K\backslash\{0\}$ in (16). □

In view of Theorem 2 and Theorem 3, if we assume that both the objective and the constraint functions involving in the original vectorial optimization problem (VP) are invex at $\overline{x}$ on the set of all feasible solutions D with respect to the same function η, and, moreover, some suitable constraint qualification (for example, the generalized Slater constraint qualification) is satisfied at $\overline{x}$, then problems (VP) and (VP$_\eta(\overline{x})$) defined in Banach spaces are equivalent in the sense discussed above. Further, the optimal value in the η-approximated vectorial optimization problem (VP$_\eta(\overline{x})$) is the same as the optimal value in the original vectorial optimization problem (VP).

As follows from the proof of Theorem 3, the invexity assumption can be weakened to the generalized invexity.

Theorem 4. *Let $\overline{x}$ be a feasible point for ($VP_\eta(\overline{x})$). Further, we assume that f is (pseudo-invex) strictly pseudo-invex with respect to η at $\overline{x}$ on D, g is quasi-invex with respect to η at $\overline{x}$ on D, and $\eta(\overline{x},\overline{x}) = 0$. If $\overline{x}$ is a (weakly) efficient solution in ($VP_\eta(\overline{x})$) then $\overline{x}$ is also a (weakly) efficient solution in (VP).*

Some examples of vectorial optimization problems in finite dimensional spaces, for which solvability is characterized by the η-approximation approach, can be found in [2].

4. η-saddle point criteria for vectorial optimization problems defined in Banach spaces.

In this section, we use the η-approximation method to obtain new saddle point criteria for Fréchet differentiable nonconvex vectorial optimization problems defined in Banach spaces.

Now, we introduce a definition of an η-Lagrange function for the constructed η-approximated vectorial optimization problem ($\mathrm{VP}_\eta(\overline{x})$) defined in Banach spaces.

Definition 8. *An η-approximated Lagrange function is said to be the Lagrange function for the vectorial optimization problem ($VP_\eta(\overline{x})$)*

$$\begin{aligned} L_\eta(x,\lambda,\xi) \quad &: \quad = \langle \lambda, f(\overline{x}) + Df(\overline{x})\eta(x,\overline{x})\rangle + \langle \xi, g(\overline{x}) + Dg(\overline{x})\eta(x,\overline{x})\rangle \\ &: \quad = L(\overline{x},\lambda,\xi) + DL(\overline{x},\lambda,\xi)\,\eta(x,\overline{x}) \end{aligned}$$

Remark 2. *Note that if $\eta(\overline{x},\overline{x}) = 0$ then, by Definition 8, it follows that*

$$L_\eta(\overline{x},\lambda,\xi) = L(\overline{x},\lambda,\xi)\,.$$

For the Lagrange function, some kinds of saddle points have been introduced (see, for example, [1], [22]). Now, in the natural way, we introduce a definition of a so-called η-saddle point for the η-Lagrange function in the η-approximated vector optimization problem ($\mathrm{VP}_\eta(\overline{x})$) defined in Banach spaces.

Definition 9. *A point $(\overline{x},\lambda^*,\xi^*) \in D\times K^*\backslash\{0\}\times C^*$ is said to be a (Pareto) η-saddle point for the η-approximated Lagrange function if,*

i): $L_\eta(\overline{x},\lambda^*,\xi) \leq L_\eta(\overline{x},\lambda^*,\xi^*)\ \forall\xi\in C^*$,
ii): $L_\eta(\overline{x},\lambda^*,\xi^*) \leq L_\eta(x,\lambda^*,\xi^*)\ \forall x\in D$.

Theorem 5. *Let f be (invex) strictly invex at $\overline{x}$ on D with respect to η satisfying $\eta(\overline{x},\overline{x}) = 0$. Moreover, we assume that the generalized Slater constraint qualification is satisfied at $\overline{x}$ for (VP). If $(\overline{x},\lambda^*,\xi^*)$ is an η-saddle point for L_η then $\overline{x}$ is a (weakly) efficient solution in (VP).*

Proof. Let $(\overline{x}, \lambda^*, \xi^*) \in D \times K^* \backslash \{0\} \times C^*$ be an η-saddle point for L_η. Then by Definition 9 i) and Remark 2 we have, for any $\xi \in C^*$,

$$\langle \lambda^*, f(\overline{x}) + Df(\overline{x})\eta(\overline{x}, \overline{x}) \rangle + \langle \xi, g(\overline{x}) + Dg(\overline{x})\eta(\overline{x}, \overline{x}) \rangle - \\ - \langle \lambda^*, f(\overline{x}) + Df(\overline{x})\eta(\overline{x}, \overline{x}) \rangle - \langle \xi^*, g(\overline{x}) + Dg(\overline{x})\eta(\overline{x}, \overline{x}) \rangle \leq 0.$$

and, so by $\eta(\overline{x}, \overline{x}) = 0$, it follows that

$$\langle \xi, g(\overline{x}) \rangle \leq \langle \xi^*, g(\overline{x}) \rangle . \tag{17}$$

In (17), let $\xi = 0$. Thus,

$$\langle \xi^*, g(\overline{x}) \rangle \geq 0. \tag{18}$$

We proceed by contradiction. Let us suppose that $\overline{x}$ is not a weakly efficient solution in (VP). Then there exists $\widetilde{x} \in D$ such that

$$f(\overline{x}) - f(\widetilde{x}) \in intK. \tag{19}$$

Since $\overline{x} \in D(\overline{x})$ then

$$g(\overline{x}) + Dg(\overline{x})\eta(\overline{x}, \overline{x}) \in -C,$$

and by $\xi^* \in C^*$ we have

$$\langle \xi^*, g(\overline{x}) + Dg(\overline{x})\eta(\overline{x}, \overline{x}) \rangle \leq 0.$$

Thus, $\eta(\overline{x}, \overline{x}) = 0$ gives,

$$\langle \xi^*, g(\overline{x}) \rangle \leq 0. \tag{20}$$

Hence, by (18) and (20),

$$\langle \xi^*, g(\overline{x}) \rangle = 0. \tag{21}$$

Thus, using the feasibility of $\widetilde{x}$ in $(\mathrm{VP}_\eta(\overline{x}))$ together with (21), we get

$$\langle \xi^*, Dg(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle \leq 0. \tag{22}$$

By assumption, f is invex with respect to η on D. Then, by Definition 1,

$$f(\widetilde{x}) - f(\overline{x}) - Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \in K.$$

Hence, by (19), it follows that

$$-Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \in K + intK = intK, \tag{23}$$

and so, by $\lambda^* \in K^* \backslash \{0\}$,

$$\langle \lambda^*, Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle < 0. \tag{24}$$

Comparing (22) and (24), we obtain

$$\langle \lambda^*, Df(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle + \langle \xi^*, Dg(\overline{x})\eta(\widetilde{x}, \overline{x}) \rangle < 0,$$

and so $\eta(\overline{x},\overline{x})=0$ gives,

$$\begin{aligned}\langle\lambda^*, Df(\overline{x})\eta(\widetilde{x},\overline{x})\rangle \quad &+ \quad \langle\xi^*, Dg(\overline{x})\eta(\widetilde{x},\overline{x})\rangle < \\ &< \quad \langle\lambda^*, Df(\overline{x})\eta(\overline{x},\overline{x})\rangle + \langle\xi^*, Dg(\overline{x})\eta(\overline{x},\overline{x})\rangle\end{aligned}$$

Now, using the definition of L_η, we get

$$\begin{aligned}L_\eta(\widetilde{x},\lambda^*,\xi^*) - L_\eta(\overline{x},\lambda^*,\xi^*) &= \langle\lambda^*, Df(\overline{x})\eta(\widetilde{x},\overline{x})\rangle + \langle\xi^*, Dg(\overline{x})\eta(\widetilde{x},\overline{x})\rangle \\ &- \langle\lambda^*, Df(\overline{x})\eta(\overline{x},\overline{x})\rangle + \langle\xi^*, Dg(\overline{x})\eta(\overline{x},\overline{x})\rangle < 0.\end{aligned}$$

This contradicts Definition 9 ii), provided $\overline{x}$ is a weak Pareto solution in (VP).

The proof of efficiency is similar, but $intK$ should be replaced by $K\backslash\{0\}$.

□

Now, we prove a converse condition, that is, a sufficient condition for a point $(\overline{x},\lambda^*,\xi^*) \in D \times K^*\backslash\{0\} \times C^*$ to be an η-saddle point for the η-Lagrange function in problem ($\mathrm{VP}_\eta(\overline{x})$) defined in Banach spaces.

Theorem 6. *Let $\overline{x}$ be a (weakly) efficient solution in (VP) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Further, we assume that η satisfies $\eta(\overline{x},\overline{x})=0$. Then there exist $\lambda^* \in K^*\backslash\{0\}$, $\xi^* \in C^*$, such that $(\overline{x},\lambda^*,\xi^*)$ is an η-saddle point for the η-Lagrange function in the η-approximated vectorial optimization problem ($VP_\eta(\overline{x})$).*

Proof. By assumption, $\overline{x}$ is a weakly efficient solution for (VP). Thus, by Theorem 1, it follows that Karush-Kuhn-Tucker necessary optimality conditions (6)-(8) are satisfied. Then, by the Karush-Kuhn-Tucker condition (7), it follows that the inequality

$$\langle\xi^*, g(\overline{x})\rangle \geq \langle\xi, g(\overline{x})\rangle$$

holds for all $\xi \in C^*$. Then, the assumption $\eta(\overline{x},\overline{x})=0$ implies

$$\begin{aligned}&\langle\lambda^*, f(\overline{x}) + Df(\overline{x})\eta(\overline{x},\overline{x})\rangle + \langle\xi, g(\overline{x}) + Dg(\overline{x})\eta(\overline{x},\overline{x})\rangle \leq \\ &\qquad \leq \langle\lambda^*, f(\overline{x}) + Df(\overline{x})\eta(\overline{x},\overline{x})\rangle + \langle\xi^*, g(\overline{x}) + Dg(\overline{x})\eta(\overline{x},\overline{x})\rangle\end{aligned}$$

From Definition 8 follows that the inequality $L_\eta(\overline{x},\lambda^*,\xi^*) - L_\eta(\overline{x},\lambda^*,\xi) \in K$ is satisfied for all $\xi \in C^*$. This means that the inequality i) from Definition 9 is established.

We now prove the second inequality in Definition 9. We proceed by contradiction. Suppose that there exists $\widetilde{x} \in D$ such that

$$L_\eta(\overline{x},\lambda^*,\xi^*) > L_\eta(\widetilde{x},\lambda^*,\xi^*)$$

Then, by Definition 9,

$$\begin{aligned}&\langle\lambda^*, f(\overline{x}) + Df(\overline{x})\eta(\overline{x},\overline{x})\rangle + \langle\xi^*, g(\overline{x}) + Dg(\overline{x})\eta(\overline{x},\overline{x})\rangle > \\ &\qquad > \langle\lambda^*, f(\overline{x}) + Df(\overline{x})\eta(\widetilde{x},\overline{x})\rangle + \langle\xi^*, g(\overline{x}) + Dg(\overline{x})\eta(\widetilde{x},\overline{x})\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\langle \lambda^*, Df(\overline{x})\eta(\overline{x},\overline{x})\rangle &+ \langle \xi^*, Dg(\overline{x})\eta(\overline{x},\overline{x})\rangle > \\ &> \langle \lambda^*, Df(\overline{x})\eta(\widetilde{x},\overline{x})\rangle + \langle \xi^*, Dg(\overline{x})\eta(\widetilde{x},\overline{x})\rangle .\end{aligned}$$

By assumption, $\eta(\overline{x},\overline{x}) = 0$. Thus,

$$\langle \lambda^* Df(\overline{x}) + \xi^* Dg(\overline{x}), \eta(\widetilde{x},\overline{x})\rangle < 0.$$

This is a contradiction to the Karush-Kuhn-Tucker condition (6). Thus, by Definition 9, we conclude that $(\overline{x}, \lambda^*, \xi^*)$ is an η-saddle point for the η-Lagrange function in the η-approximated vector optimization problem $(\mathrm{VP}_\eta(\overline{x}))$. □

In view of Theorem 5 and Theorem 6, we see that, if we assume that f is (invex) strictly invex with respect to η and g is also invex at $\overline{x}$ on D with respect to the same function η satisfying $\eta(\overline{x},\overline{x}) = 0$, and, moreover, the generalized Slater constraint qualification is satisfied at $\overline{x}$, then the η-approximation approach guarantees the equivalence between a (weakly) efficient solution $\overline{x}$ in (VP) and an η-saddle point of the η-Lagrange function in its associated η-approximated vectorial optimization problem $(\mathrm{VP}_\eta(\overline{x}))$ in the sense discussed above.

5. Conclusions

In this paper, the study of (weakly) efficient solutions of differentiable multiobjective problems in Banach spaces are approached from two aspects: one, trying to relate them with the solutions to the approximated vector optimization problems being, in general, less complicated and another, trying to locate conditions which are easier to deal with computationally and which guarantee (weak) efficiency. This work is an extension of the results obtained in [2] for arbitrary Banach spaces context with dominance structure given by cones. There are characterizations of solvability of differentiable vector optimization problems (VP) in infinite spaces in terms of solvability of η-approximated vector optimization problems $(\mathrm{VP}_\eta(\overline{x}))$. One of them is the equivalence between (weak) efficient points of multiobjective programming problems and (weak) efficient points of their modified vector minimization problems. In the second one, (weak) efficient points of multiobjective programming problems have been characterized by η-saddle points of the η-Lagrange function in their η-approximated vector optimization problems. Furthermore, the η-approximated vector optimization problems $(\mathrm{VP}_\eta(\overline{x}))$ and the proposed characterizations exploit the concept of invex functions defined in Banach spaces.

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