Folia Mathematica
Vol. 13, No. 1, pp. 31–39
Acta Universitatis Lodziensis

THE BIRKHOFF AND VARIATIONAL MCSHANE INTEGRALS OF VECTOR VALUED FUNCTIONS

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Abstract. In the paper we prove some properties of the Birkhoff integral of Banach space valued functions. We also investigate connections between this integral and the variational McShane integral considered by K. Musiał and L. Di Piazza in [13].

1. Some properties of the Birkhoff integral

Integration of vector valued functions is strongly motivated by general problems of modern analysis including differential equations theory for Banach space valued mappings. In many situations it is enough to use the well-known Bochner integral involved with the classical Lebesgue theory. Except for a general and elegant Pettis integral, which in some cases seems to be hardly applicable, at least two other less general types of integrals are known, they are due to G. Birkhoff and E. J. McShane (this latter, for vector valued function was defined by R. A. Gordon [7] and D. H. Fremlin [5]).

The Birkhoff integral was introduced in [1] where also many basic properties were proved. Some of them are obvious because of connections with a wider notion of the Pettis integral (see [11], [12], [16]). Recently, V. M. Kadets and others [9], [10] investigated the unconditional Riemann–Lebesgue integral. However, their definition turned out to be equivalent with that given by G. Birkhoff what was discovered independently in [2] and [14]. Here we prove some new properties of the Birkhoff integral. They will be useful in Section 2. We do not use the original definition of the Birkhoff integral but its equivalent form given by D. H. Fremlin [6].

Through the paper, (Ω, Σ, μ) denotes a measure space, where μ is a σ-finite measure on a σ-field Σ of subsets of Ω. Let X stand for a Banach

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Key words and phrases: Birkhoff integral, variational McShane integral, vector valued functions.

AMS subject classifications: 28B05.

space. Let $A \subset X$. If A is bounded we denote $\|A\| = \sup_{x\in A}\|x\|$. If $\mu(A) < \infty$ and $f\colon \Omega \to X$ we write $f(A)\,\mu(A) = \{f(t)\,\mu(A) : t \in A\}$.

By *a (countable) partition* of Ω we mean a countable family Π of pairwise disjoint, nonempty, measurable sets E_i of finite measure such that $\bigcup_{i\in\mathbb{N}} E_i = \Omega$. By *a finite partition* of Ω we mean a finite family Π of pairwise disjoint, nonempty, measurable sets E_i, $i \le n$, such that $\bigcup_{i\le n} E_i = \Omega$.

A set function Var $\phi : \Sigma \to X$ is called *of bounded variation on* Ω, if Var $\phi = \sup_\Gamma \sum_{k=1}^{n} \|\mathrm{Var}\ \phi(D_k)\| < \infty$, where the supremum is taken over all finite partitions $\Gamma = \{D_k\}_{k\le n}$ of Ω ([4]). A set function Var $\phi : \Sigma \to X$ is called *of σ-finite variation*, if there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of measurable sets covering Ω, where the variation of Var ϕ is finite on each E_n.

A function $f\colon \Omega \to X$ is called *strongly measurable (μ-measurable)*, if there exists a sequence of countably valued functions (i.e. assuming at most a countable set of values in X, where each value appears on a measurable set) converging almost everywhere to f ([8, Def. 3.5.4.(2)]).

Definition 1. *A function $f\colon \Omega \to X$ is called Pettis integrable, if $x^* f \in L_1(\Omega)$ for all $x^* \in X^*$ and for any $E \in \Sigma$ there exists an $x \in X$ such that*

$$x^* x = \int_E x^* f d\mu,$$

whenever $x^* \in X^*$. The point x is then called the Pettis integral of f on E.

Definition 2. (Fremlin [6]). *A function $f : \Omega \to X$ is called Birkhoff integrable (in short, B- -integrable), if there exists a point $x \in X$ such that for each $\varepsilon > 0$ there exists a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ such that for every choice of points $t_i \in E_i$, $i \in \mathbb{N}$, we have $\|\sum_{i=1}^{\infty} f(t_i)\mu(E_i) - x\| \le \varepsilon$ and the series $\sum_{i=1}^{\infty} f(t_i)\,\mu(E_i)$ is unconditionally convergent. Then we say that the B-integrability conditions for f and ε are satisfied. The point x is called the B-integral of f on Ω. The analogous statement with the absolutely convergent series yields the notion of absB-integral of f.*

Remark 1. *Obviously, if $f\colon \Omega \to X$ is absB-integrable, then f is also B- -integrable.*

First we prove some lemmas.

Lemma 1. *Let $f\colon \Omega \to X$ be B-integrable and $\Pi = \{E_i\}_{i\in\mathbb{N}}$ satisfies the B-integrability conditions for f and a given $\varepsilon > 0$. Then each image $f(E_i)$, $i \in \mathbb{N}$, is bounded whenever $\mu(E_i) > 0$.*

Proof. Consider an arbitrary set $E_{i_0} \in \Pi$ such that $\mu(E_{i_0}) > 0$ and pick any $t_{i_0}, t'_{i_0} \in E_{i_0}$. Fix points $t_i \in E_i$ *for* $i \ne i_0$. Then,

$$\left\|f(t_{i_0})\,\mu(E_{i_0}) - f\left(t'_{i_0}\right)\mu(E_{i_0})\right\| =$$

$$=\left\|\left(\sum_{i\neq i_0} f(t_i)\,\mu(E_i)+f(t_{i_0})\,\mu(E_{i_0})\right)-\left(\sum_{i\neq i_0} f(t_i)\,\mu(E_i)+f\left(t'_{i_0}\right)\mu(E_{i_0})\right)\right\|\leq$$

$$\leq\left\|\sum_{i\neq i_0} f(t_i)\,\mu(E_i)+f(t_{i_0})\,\mu(E_{i_0})-\int_\Omega f d\mu\right\|+$$

$$+\left\|\sum_{i\neq i_0} f(t_i)\,\mu(E_i)+f\left(t'_{i_0}\right)\mu(E_{i_0})-\int_\Omega f d\mu\right\|\leq 2\varepsilon.$$

Thus $\left\|f(t_{i_0})-f\left(t'_{i_0}\right)\right\|\leq 2\varepsilon/\mu(E_{i_0})$ and consequently $\|f(t_{i_0})\|\leq\left\|f\left(t'_{i_0}\right)\right\|+2\varepsilon/\mu(E_{i_0})$. By the arbitrariness of $t_{i_0}\in E_{i_0}$, we have $\|f(E_{i_0})\|\leq\left\|f\left(t'_{i_0}\right)\right\|+2\varepsilon/\mu(E_{i_0})$ as desired. □

Lemma 2. *Let (Ω,Σ,μ) be a σ-finite measure space, let X be a Banach space and assume that $f:\Omega\to X$ is B-integrable. Let $\Pi=\{E_i\}_{i\in\mathbb{N}}$ be a partition satisfying the B-integrability conditions for f on Ω and a given $\varepsilon>0$. Then the set $A=\{\|f(E_i)\,\mu(E_i)\|:i\in\mathbb{N}\}$ is bounded.*

Proof. By Lemma 1 each set $f(E_i)\,\mu(E_i)$, $i\in\mathbb{N}$ is bounded. For each $i\in\mathbb{N}$ there exists a $t_i\in E_i$ such that $\|f(t_i)\,\mu(E_i)\|>\|f(E_i)\,\mu(E_i)\|-\varepsilon$. Suppose that A is unbounded. Then the sequence $(f(t_i)\,\mu(E_i))_{i\in\mathbb{N}}$ is unbounded. Hence the series $\sum_{i=1}^\infty f(t_i)\,\mu(E_i)$ is not convergent what gives a contradiction. □

Lemma 3. *Let (Ω,Σ,μ) be a σ-finite measure space, let X be a Banach space and assume that $f:\Omega\to X$ is B-integrable. Let $A\in\Sigma$ be a fixed set of finite measure such that $f(A)$ is bounded and let $\Gamma=\{D_k\}_{k\leq n}$ be a finite partition of A. Then we have*

$$\sum_{k=1}^n\left\|\int_{D_k} f d\mu\right\|\leq\|f(A)\|\,\mu(A).$$

Proof. Let $\varepsilon>0$. Observe that f is B-integrable on each $D_k,\ k\leq n$ [1, Th. 14]. For $k=1,...,n$ let $\{D_{jk}\}_{j\in\mathbb{N}}$ be a partition satisfying the B-integrability conditions for f on D_k and ε/n, with an arbitrary choice $t_{jk}\in D_{jk},\ j\in\mathbb{N}$. Then we have

$$\left\|\int_{D_k} f d\mu-\sum_{j=1}^\infty f(t_{jk})\mu(D_{jk})\right\|\leq\frac{\varepsilon}{n},$$

and

$$\left\|\int_{D_k} f d\mu\right\| \leq \left\|\sum_{j=1}^{\infty} f\left(t_{jk}\right)\mu\left(D_{jk}\right)\right\| + \frac{\varepsilon}{n}.$$

Thus,

$$\sum_{k=1}^{n}\left\|\int_{D_k} f d\mu\right\| \leq \sum_{k=1}^{n}\left(\left\|\sum_{j=1}^{\infty} f\left(t_{jk}\right)\mu\left(D_{jk}\right)\right\| + \frac{\varepsilon}{n}\right) =$$

$$= \sum_{k=1}^{n}\left\|\sum_{j=1}^{\infty} f\left(t_{jk}\right)\mu\left(D_{jk}\right)\right\| + \sum_{k=1}^{n}\frac{\varepsilon}{n} \leq \sum_{k=1}^{n}\sum_{j=1}^{\infty}\sup_{t\in A}\left\|f\left(t\right)\right\|\mu\left(D_{jk}\right) + \varepsilon =$$

$$= \sup_{t\in A}\left\|f\left(t\right)\right\|\sum_{k=1}^{n}\sum_{j=1}^{\infty}\mu\left(D_{jk}\right) + \varepsilon = \left\|f\left(A\right)\right\|\mu\left(A\right) + \varepsilon.$$

By the arbitrariness of ε we have the assertion. □

Assume that $f\colon \Omega \to X$ is B-integrable on Ω. By [1, Th. 14] f is B--integrable on each set $E \in \Sigma$. Thus the B-integral of f can be considered as a set function on Σ. Birkhoff also proved that for every B-integrable function $f\colon \Omega \to X$, the B-integral as a set function is a *countably (completely) additive measure* [1, Th. 14], i.e. for arbitrary pairwise disjoint subsets E_n, $n \in \mathbb{N}$, of Σ we have $\int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu = \sum_{n=1}^{\infty}\int_{E_n} f d\mu$, where the series is unconditionally convergent. Note that, since every B-integrable function $f : \Omega \to X$ is Pettis integrable [1, Th. 21], this fact can be also obtained from an analogous property for the Pettis integral [8, Th. 3.7.2].

V. I. Rybakov [15] proved that the Pettis integral is of σ-finite variation (see also [11]). One of our aims is to present a direct proof of a little bit stronger property for the Birkhoff integral (Theorem 1). Moreover, we show that the abs$B$-integral is of finite variation (Theorem 2). This extends the property obtained in [9, Prop. 4] to the case of σ-finite measure.

Theorem 1. *Let (Ω, Σ, μ) be a σ-finite measure space, let X be a Banach space and assume that $f : \Omega \to X$ is B-integrable. Then the B-integral of f as a set function is of σ-finite variation. Moreover, the integral variations on sets from a partition satisfying the B-integrability conditions for f and a given $\varepsilon > 0$, are uniformly bounded.*

Proof. Let $\varepsilon > 0$. Pick a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ satisfying the B-integrability conditions for f and ε. Let $M > 0$ be an upper bound of the set

$$\left\{\left\|f\left(E_i\right)\mu\left(E_i\right)\right\| : i \in \mathbb{N}\right\}$$

(see Lemma 2). Let $\Gamma = \{D_k\}_{k\leq n}$ be a finite partition of Ω. Fix $i \in \mathbb{N}$ and observe that $\{E_i \cap D_k\}_{k\leq n}$ is a partition of E_i (we may eliminate empty sets). Now we use Lemma 3 to estimate the B-integral variation on $A = E_i$:

$$Var\left\{(B)\int_{E_i} f d\mu\right\} = \sup_{\Gamma}\sum_{k=1}^{n}\left\|\int_{E_i\cap D_k} f d\mu\right\| \leq \|f(E_i)\|\,\mu(E_i) \leq M < \infty.$$

Since M does not depend on i, we obtain that $Var\left\{(B)\int_{E_i} f d\mu\right\} \leq M$ for all $i \in \mathbb{N}$. So we have proved that f is of σ-finite variation, and the set of B-integral variations of f on E_i (for $i \in \mathbb{N}$) is bounded. ☐

Theorem 2. *Let (Ω, Σ, μ) be a σ-finite measure space, let X be a Banach space and assume that $f : \Omega \to X$ is absB-integrable. Then the absB-integral of f as a set function is of finite variation.*

Proof. For $\varepsilon = 1$ pick a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ satisfying the B-integrability conditions for f. Let $M > 0$ be an upper bound of the set

$$\{\|f(E_i)\,\mu(E_i)\| : i \in \mathbb{N}\}$$

(see Lemma 2). Let $\Gamma = \{D_k\}_{k\leq n}$ be a finite partition of Ω. Then (using the fact that the B-integral is countably additive [1, Th. 14]) we have

$$Var\left\{(B)\int_{\Omega} f d\mu\right\} = \sup_{\Gamma}\sum_{k=1}^{n}\left\|\int_{D_k} f d\mu\right\| = \sup_{\Gamma}\sum_{k=1}^{n}\left\|\int_{\left(\bigcup_{i\in\mathbb{N}} E_i\right)\cap D_k} f d\mu\right\| =$$

$$= \sup_{\Gamma}\sum_{k=1}^{n}\left\|\sum_{i=1}^{\infty}\int_{E_i\cap D_k} f d\mu\right\| \leq \sup_{\Gamma}\sum_{k=1}^{n}\sum_{i=1}^{\infty}\left\|\int_{E_i\cap D_k} f d\mu\right\|.$$

Note that $\{E_i \cap D_k\}_{k\leq n}$ is a partition of E_i (we may assume that it contains only nonempty sets). Using Lemma 3 with $A = E_i$, we obtain

$$Var\left\{(B)\int_{\Omega} f d\mu\right\} \leq \sup_{\Gamma}\sum_{i=1}^{\infty}\sum_{k=1}^{n}\left\|\int_{E_i\cap D_k} f d\mu\right\| \leq$$

$$\leq \sup_{\Gamma}\sum_{i=1}^{\infty}\|f(E_i)\|\,\mu(E_i) = \sum_{i=1}^{\infty}\|f(E_i)\|\,\mu(E_i) < \infty.$$

The last inequality follows from the fact that the series $\sum_{i=1}^{\infty} f(t_i)\,\mu(E_i)$, with $t_i \in E_i$, $i \in \mathbb{N}$, is absolutely convergent. Indeed, for every $i \in \mathbb{N}$, if $\mu(E_i) > 0$, pick a point $t_i \in E_i$ such that $\|f(E_i)\| \leq \|f(t_i)\| + 1/(2^i\mu(E_i))$. Consequently

$$\sum_{i=1}^{\infty}\sup_{t\in E_i}\|f(t)\|\,\mu(E_i) \leq \sum_{i=1}^{\infty}\left(\|f(t_i)\| + \frac{1}{2^i\mu(E_i)}\right)\mu(E_i) =$$

$$= \sum_{i=1}^{\infty} \|f(t_i)\| \mu(E_i) + 1 < \infty.$$

□

It is known that there exist absB-integrable functions that are not strongly measurable (cf. [10, Ex. 1.5]). If $f\colon \Omega \to X$ is strongly measurable and Pettis integrable then $Var\left\{\int_E f d\mu\right\} = \int_E \|f\| d\mu$ ([11, Th. 4.1 and Rem. 4.1]). Thus using Theorem 2 and [1, Th. 21] we have

Corollary 1. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a Banach space. If $f : \Omega \to X$ is strongly measurable and absB-integrable then it is Bochner integrable.*

This generalizes the result of Kadets and others [10, Th. 1.8].

2. A comparison of the Birkhoff and variational McShane integrals

In [6] it is shown that a vector valued Birkhoff integrable function on a σ-finite outer regular quasi Radon measure space is McShane integrable. In [13] the authors considered variationally McShane integrable functions (being a subclass of McShane integrable functions) and gave their characterization. We show that every variationally McShane integrable function is Birkhoff integrable but not conversely. On the other hand, every strongly measurable absolutely Birkhoff integrable function is variationally McShane integrable. Let us recall some definitions.

A quadruple $(\Omega, \tau, \Sigma, \mu)$ is called *a topological measure space*, if (Ω, Σ, μ) is a measure space and τ is a topology on Ω such that $\tau \subset \Sigma$. A measure μ is called *moderated*, if there exists a sequence $(G_n)_{n=1}^{\infty}$ of open sets of finite μ-measure covering Ω.

Definition 3. [13] *A topological measure space $(\Omega, \tau, \Sigma, \mu)$ is called a σ--finite outer regular quasi Radon measure space, when the following conditions are satisfied:*

- *(Ω, Σ, μ) is a σ-finite, complete and outer regular measure space,*
- *τ is a topology such that $\tau \subset \Sigma$,*
- *$\mu(E) = \sup\{\mu(F) : F \subseteq E,\ F$ is closed$\}$ for every $E \in \Sigma$,*
- *μ is τ-additive, i.e., if $\mathcal{G} \subseteq \tau$ is non-empty, upwards directed by inclusion, then $\mu\left(\bigcup_{G\in\mathcal{G}} G\right) = \sup\{\mu(G) : G \in \mathcal{G}\}$.*

Let $(\Omega, \tau, \Sigma, \mu)$ be a σ-finite outer regular quasi Radon measure space.

By *a generalized McShane partition of* Ω we mean a sequence $\Pi = \{(E_i, t_i)\}_{i\in\mathbb{N}}$ such that $\{E_i\}_{i\in\mathbb{N}}$ is a family of pairwise disjoint sets of positive finite measure, $t_i \in \Omega$ for each i and $\mu(\Omega \setminus \bigcup_i E_i) = 0$. A function $\Delta : \Omega \to \tau$ such that $t \in \Delta(t)$ for each $t \in \Omega$ is called *a gauge.* Given a gauge Δ we say that a generalized McShane partition $\Pi = \{(E_i, t_i)\}_{i\in\mathbb{N}}$ in Ω is Δ-*fine*, if $E_i \subset \Delta(t_i)$ for each i.

The fact of existence for any gauge Δ of Δ-fine generalized McShane partition was proved by D. H. Fremlin in [5, 1B(d)].

Definition 4. [13] *A function $f : \Omega \to X$ is called variationally McShane integrable (in short, VM-integrable), if it is Pettis integrable and for each $\varepsilon > 0$ there exists a gauge $\Delta : \Omega \to \tau$ such that $\sum_{i=1}^{\infty} \left\| f(t_i)\mu(E_i) - \int_{E_i} f d\mu \right\| \leq \varepsilon$, for each Δ-fine generalized McShane partition $\Pi = \{(E_i, t_i)\}_{i\in\mathbb{N}}$ of Ω. The value of variational McShane integral on Ω is the same as the value of Pettis integral.*

Corollary 2. *Let $(\Omega, \tau, \Sigma, \mu)$ be a σ-finite outer regular quasi Radon measure space. Let X be a Banach space and assume that $f : \Omega \to X$ is strongly measurable and absB-integrable. Then f is VM-integrable. Both integrals coincide and equal the Pettis integral.*

Proof. By Corollary 1 the function f is Bochner integrable. So by [13, Lem. 1] we obtain the assertion. □

Fact 1. [13, Th. 1] *Let $(\Omega, \tau, \Sigma, \mu)$ be a σ-finite outer regular quasi Radon measure space. Let X be a Banach space. A function $f : \Omega \to X$ is variationally McShane integrable, if and only if it is strongly measurable and Pettis integrable, and the variation of Pettis integral as a set function is a moderated measure.*

Now we give an example witnessing that a strongly measurable, B-integrable function need not be VM-integrable. We use the construction from [13, Ex. 2].

Example 1. In [13] the authors gave an example of a quasi Radon measure space $([0,1], \tau, \mathcal{P}([0,1]), \mu)$ for which there exists a Pettis integrable function $f : [0,1] \to X$ (with an infinite dimensional Banach space X), strongly measurable (as Pettis integrable, countably valued [8, Th. 3.5.3]), which is not variationally McShane integrable. We will show that the same function f is Birkhoff integrable. Let us recall the example. We take τ as the natural topology on $[0,1]$. The support of μ will be equal to the set of all rationals in $[0,1]$ enumerated in a sequence $\{r_n : n \in \mathbb{N}\}$. Pick in X an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ which is not absolutely convergent (see [3]). The idea which leads to the construction of

μ and f is as follows. Inductively (we omit details) we define a permutation ψ of $\mathbb{N}$ in such a way that $\mu(\{r_n\}) = 2^{-\psi(n)}$ for each $n \in \mathbb{N}$, and $\sum_{r_n \in I} \|x_{\psi(n)}\| = +\infty$ for every open subinterval I of $[0,1]$ with rational endpoints. We put $f = \sum_{n=1}^{\infty} 2^{\psi(n)} x_{\psi(n)} \chi_{\{r_n\}}$.

Now, let us show that the B-integral $\int_{[0,1]} f d\mu$ equals $\sum_{n=1}^{\infty} x_n$. Let $\Pi = \{E_i\}_{i \in \mathbb{N} \cup \{0\}}$ be a partition such that $E_0 = \left\{[0,1] \setminus \bigcup_{n \in \mathbb{N}} \{r_n\}\right\}$ and $E_i = \{r_i\}$ for $i \in \mathbb{N}$. Since $f(t) = 0$ for $t \in E_0$, we have

$$\left\|\sum_{n=1}^{\infty} f(t_n)\mu(E_n) - \sum_{n=1}^{\infty} x_n\right\| = \left\|\sum_{n=1}^{\infty} 2^{\psi(n)} x_{\psi(n)} \mu(\{r_n\}) - \sum_{n=1}^{\infty} x_n\right\| =$$

$$= \left\|\sum_{n=1}^{\infty} x_{\psi(n)} - \sum_{n=1}^{\infty} x_n\right\| = 0.$$

It proves that f is B-integrable and the B-integral equals $\sum_{n=1}^{\infty} x_n$.

Moreover, from the construction it follows that, for every interval I of $[0,1]$ with rational endpoints, the variation of the Birkhoff integral of f on I is $+\infty$. So this variation is not a moderated measure on any nonempty interval. Hence by Fact 1 and [1, Th. 21] the function f is VM-integrable on no nonempty interval.

Since integrals of VM-integrable functions need not be of finite variation, these functions need not be abs$B$-integrable. It shows that the converse of Corollary 2 is not true. Anyway, every VM-integrable function is B--integrable, which will be proved in Theorem 3.

Fact 2. [12, Cor. 5.11] *Let (Ω, Σ, μ) be a measure space, where μ is finite, let X be a Banach space and assume that $f : \Omega \to X$ is strongly measurable. Then Pettis integrability is equivalent to Birkhoff integrability.*

Fact 3. *Let (Ω, Σ, μ) be a complete, σ-finite measure space, let X be a Banach space and assume that $f : \Omega \to X$. Then f is Birkhoff integrable, if and only if f is Pettis integrable and there is a countable partition $\Pi = \{A_i\}_{i \in \mathbb{N}}$ of Ω such that $f|_{A_i}$ is Birkhoff integrable for every i.*

Proof. It is enough to observe that the proof of [2, Lemma 3.2] given for probability measure spaces works also for σ-finite measure spaces. □

Theorem 3. *Let $(\Omega, \tau, \Sigma, \mu)$ be a σ-finite outer regular quasi Radon measure space. Let X be a Banach space and assume that $f : \Omega \to X$ is VM--integrable. Then f is B-integrable.*

Proof. Since f is VM-integrable on Ω, then it is VM-integrable (thus Pettis integrable and strongly measurable) on every set A_i from an arbitrary countable partition of Ω ([13, Prop. 1(b)]). Thus by Fact 2, f is B-integrable on A_i, $i \in \mathbb{N}$. This and Fact 3 imply that f is B-integrable on Ω. □

Combining the above theorem and Corollary 2, we obtain:

Corollary 3. *Let $(\Omega, \tau, \Sigma, \mu)$ be a σ-finite outer regular quasi Radon measure space. Let X be a Banach space and $f : \Omega \to X$. The strong measurability with absB-integrability of f implies its VM-integrability, while the latter gives just the B-integrability of f.*

Acknowledgement. I would like to thank the Referees for their valuable comments and remarks.

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