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MEAN FIELD MODELS FOR SELF-GRAVITATING PARTICLES

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Abstract. This paper describes generalizations of results (presented during SNA 2004 conference) on the behavior of solutions of particular systems describing the interaction of gravitationally attracting particles that obey either Maxwell–Boltzmann or Fermi–Dirac statistics.

1. Introduction and density-pressure relations

We consider parabolic-elliptic systems of the form

$$n_t = \nabla\cdot(D_*(\nabla p + n\nabla\varphi)), \tag{1}$$

$$\Delta\varphi = n, \tag{2}$$

which appear in statistical mechanics as hydrodynamical (*mean field*) models for self-interacting particles. Here $n = n(x,t) \geq 0$ is the density function defined for $(x,t) \in \Omega\times\mathbb{R}^+$, $\Omega\subset\mathbb{R}^d$, $\varphi = \varphi(x,t)$ is the Newtonian potential generated by the particles of density n, and the pressure $p \geq 0$ is determined by the density-pressure relation with a sufficiently regular function p

$$p = p(n,\vartheta). \tag{3}$$

The parameter $\vartheta > 0$ plays the role of the temperature, and $D_* > 0$ is a diffusion coefficient which may depend on n, ϑ, φ, x, ... Such systems can be studied either in the *canonical ensemble* (i.e. the *isothermal* setting), when $\vartheta =$ const is fixed, or in the *microcanonical ensemble* with a variable temperature: $\vartheta = \vartheta(t)$. The energy balance is then described by the relation

$$E = \frac{d}{2}\int_\Omega p\,dx + \frac{1}{2}\int_\Omega n\varphi\,dx = \text{const}, \tag{4}$$

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which, for a given n, defines $\vartheta = \vartheta(t)$ in an implicit way.

We consider the system (1)–(2) with the natural no-flux boundary condition on $\partial\Omega$

$$(\nabla p + n\nabla\varphi)\cdot\overline{\nu} = 0 \tag{5}$$

($\overline{\nu}$ is the unit exterior normal vector to $\partial\Omega$), and an initial condition

$$n(x,0) = n_0(x) \geq 0. \tag{6}$$

We impose for the potential φ either a physically acceptable "free" condition

$$\varphi = E_d * n, \tag{7}$$

E_d being the fundamental solution of the Laplacian in $\mathbb{R}^d$, or the homogeneous Dirichlet boundary condition

$$\varphi_{|\partial\Omega} = 0, \tag{8}$$

which is mathematically somewhat simpler. In the case of radially symmetric solutions (8) is equivalent to (7) by adding a constant to the potential φ, cf. the discussion of this issue in [4, 5, 2]. As a consequence of (5), total mass

$$M = \int_\Omega n(x,t)\,dx \tag{9}$$

is conserved during the evolution. Moreover, sufficiently regular solutions of the evolution problem with $n_0 \geq 0$ remain positive.

In fact, the starting point of the derivation of related *mean field* models in astrophysics in [12] was an analysis of kinetic equations whose evolution in time is governed by the Maximum Entropy Production Principle. More precisely, if $0 \leq f = f(x,v,t)$ is the density of particles at the point $(x,t) \in \Omega\times\mathbb{R}^+$, $\Omega\subset\mathbb{R}^d$, moving at the velocity v, then f satisfies a kinetic equation

$$f_t + v\cdot\nabla_x f - \nabla\varphi\cdot\nabla_v f = -\nabla_v\,\mathbf{j}$$

with a general dissipation flux term $-\nabla_v\,\mathbf{j}$. Distribution functions f that maximize the functional of *local entropy* $S = \int_{\mathbb{R}^d} s(f(x,v,t))\,dv$ (under local density and pressure constraints, see (10) below) have a particular form depending on some parameters $\lambda = \lambda(x,t)$, $\vartheta = \vartheta(t)$, etc., see the examples below in Section 2. Then, the term $\nabla_v\,\mathbf{j}$ is determined (up to a positive diffusion coefficient) by the requirement that the system evolves with maximal entropy production rate at each moment t. Averaging f over the velocities $v\in\mathbb{R}^d$, and then the passage to the limit of large friction (or large times) lead to "hydrodynamic" equations in the (x,t) space. We refer for the details of that construction to [10, 12, 11] and [3].

Given the distribution f on the kinetic level, the spatio-temporal density and the pressure are

$$n(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\,dv, \quad p(x,t) = \frac{1}{d}\int_{\mathbb{R}^d} |v|^2 f(x,v,t)\,dv. \tag{10}$$

Thus, the first term in the energy (4) corresponds to the kinetic energy of particles, and the second is the potential energy of the system of self--gravitating particles. A mild assumption (24) on the form of f leads to natural density-pressure relations between n and p, see (26).

Two systems with particular density-pressure relations have been recently studied: for Brownian (or Maxwell–Boltzmann) and for Fermi–Dirac particles.

The model of self-gravitating Brownian particles, which consists of (1)–(2), (12) below, supplemented by (4), has been considered in [12] for radially symmetric solutions (n,φ), and in [13, 6] without those symmetry assumptions. Studies of the corresponding isothermal problem with $\vartheta \equiv 1$ had been conducted earlier, see e.g. [4, 1]. However, the motivations there had been a bit different – stemming from statistical mechanics of interacting charged particles in semiconductors, electrolytes, plasmas, with (2) replaced by $\Delta\varphi = -n$, and afterwards for gravitating particles and chemotaxis phenomena in biology. We refer the reader to [13, 6, 2, 5] for the Brownian particles models. The main issues are:

- gravitational collapse is possible for $d \geq 2$ in the isothermal model and for $d \geq 3$ in the nonisothermal model,
- the existence of steady states with prescribed mass and energy in $d \geq 3$ dimensions is controlled by the parameter E/M^2 which should be large enough.

The Fermi–Dirac model, see (15) below, involves nonlinear diffusion, and thus even local in time existence of solutions is much harder to establish than in the Brownian (linear diffusion) case, see [3] where a specific choice of the diffusion coefficient D_* has been considered in the isothermal case. There are also many results on radially symmetric stationary solutions in [9]. In particular:

- structure of the set of steady states with given M and ϑ is different (and less complicated) than in the Brownian case,
- the existence of steady states with given mass and energy is controlled by the parameter $\min\left\{\frac{E}{M^2}, \frac{E}{M^{1+2/d}}\right\}$ ([7, 17]),
- gravitational collapse cannot occur in $d \leq 3$ dimensions in the isothermal case ([3]),

- the gravitational collapse is possible for $d \geq 4$ for suitable initial data in the nonisothermal case ([7]).

We will study in this paper:

- examples of density-pressure relations more general than Maxwell–Boltzmann and Fermi–Dirac,
- existence of entropy functionals and entropy production rates (in Section 2),
- existence of steady states with prescribed mass and temperature, or prescribed mass and energy (in Section 3),
- nonexistence of global in time solutions of (1)–(2) with general density-pressure relations (3), $D_* = 1$, one of the conditions (7), (8), and either negative initial entropy or low energy (in Section 4), and thus, *a fortiori*, we obtain nonexistence of steady states for arbitrary D_*,
- continuation of local in time solutions of (1)–(2) with polytropic density-pressure relations (in Section 5).

Further results on general systems (1)–(2) will appear in [8]. We do not consider here the question of the local in time existence of solutions of the evolution problem. This seems to be a rather difficult question in such a general setting with (3).

Notation. In the sequel $|\,.\,|_p$ will denote the $L^p(\Omega)$ norm. The letter C will denote inessential constants which may vary from line to line.

Now we recall some important distributions and density-pressure relations in statistical mechanics, and formulate our structure assumptions on p.

1.1. Maxwell–Boltzmann distributions

They are characterized by

$$f = \lambda^{-1} \mathrm{e}^{-|v|^2/2\vartheta}, \tag{11}$$

where $\lambda = \lambda(x,t) > 0$ is a physical parameter, and $\vartheta = \vartheta(t) > 0$ is an instantaneous *temperature* uniform in x. Abusing slightly the notation, we will write $f = f(x,v,t) = f(\lambda, \vartheta, v)$. They maximize the Boltzmann entropy

$$S = \frac{-1}{\int_{\mathbb{R}^d} f\, dv} \int_{\mathbb{R}^d} f \log f\, dv.$$

The corresponding macroscopic density and pressure are

$$n = \lambda^{-1} \sigma_d 2^{d/2-1} \vartheta^{d/2} \Gamma\left(\frac{d}{2}\right), \quad p = \lambda^{-1} \sigma_d 2^{d/2-1} \vartheta^{d/2+1} \Gamma\left(\frac{d}{2}\right),$$

so that

$$p_{\mathrm{MB}}(n, \vartheta) = \vartheta n. \tag{12}$$

This is the classical Boltzmann relation leading to *linear* (Brownian) diffusion term in (1). Indeed,

$$n(x,t) = \int_{\mathbb{R}^d} \lambda^{-1} \mathrm{e}^{-|v|^2/2\vartheta}\, dv = \lambda^{-1}\sigma_d (2\vartheta)^{d/2} \int_0^\infty \mathrm{e}^{-r^2} r^{d-1}\, dr$$

and we calculate p in a similar manner.

1.2. Fermi–Dirac distributions

They have the form

$$f = \frac{\eta_0}{\lambda \mathrm{e}^{|v|^2/2\vartheta} + 1} \tag{13}$$

with a fixed $\eta_0 > 0$, and $\lambda = \lambda(x,t) > 0$, $\vartheta = \vartheta(t) > 0$. They maximize the entropy

$$S = \frac{-1}{\int_{\mathbb{R}^d} f\, dv} \int_{\mathbb{R}^d} \left(\frac{f}{\eta_0} \log \frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0}\right) \log\left(1 - \frac{f}{\eta_0}\right)\right) dv$$

whose form *a priori* prevents from the overcrowding of particles at (x,v,t): $0 \le f \le \eta_0$. Then we have (see e.g. [3, (1.1)–(1.3)])

$$n = \eta_0 2^{d/2-1} \sigma_d \vartheta^{d/2} I_{d/2-1}(\lambda), \quad p = \eta_0 2^{d/2} \frac{\sigma_d}{d} \vartheta^{d/2+1} I_{d/2}(\lambda),$$

where I_α denotes the Fermi integral of order $\alpha > -1$ defined for $\lambda > 0$

$$I_\alpha(\lambda) = \int_0^\infty \frac{y^\alpha\, dy}{\lambda \mathrm{e}^y + 1}. \tag{14}$$

Hence

$$p_{\mathrm{FD}}(n,\vartheta) = \frac{\mu}{d} \vartheta^{d/2+1} \left(I_{d/2} \circ I_{d/2-1}^{-1}\right) \left(\frac{2}{\mu} \frac{n}{\vartheta^{d/2}}\right) \tag{15}$$

for a constant $\mu > 0$, which leads to a *nonlinear* diffusion in (1). Various properties of Fermi integrals (14) are collected in [3, Sec. 2] and [7, Sec. 5].

1.3. Bose–Einstein distributions

They are of the form

$$f = \frac{\eta_0}{\lambda \mathrm{e}^{|v|^2/2\vartheta} - 1} \tag{16}$$

with $\eta_0 > 0$, $\lambda = \lambda(x,t) > 1$, $\vartheta = \vartheta(t) > 0$. Similarly as before, we obtain

$$p_{\mathrm{BE}}(n,\vartheta) = \frac{\mu}{d} \vartheta^{d/2+1} \left(J_{d/2} \circ J_{d/2-1}^{-1}\right) \left(\frac{2}{\mu} \frac{n}{\vartheta^{d/2}}\right), \tag{17}$$

where J_α denotes the integral

$$J_\alpha(\lambda) = \int_0^\infty \frac{y^\alpha\, dy}{\lambda \mathrm{e}^y - 1} \tag{18}$$

defined either for $\alpha > -1$ and $\lambda > 1$, or $\alpha > 0$ and $\lambda \geq 1$. Note that $\sup_{\lambda > 1} J_\alpha(\lambda)$ is finite, if and only if $\alpha > 0$.

Examples 1.2 and 1.3 are well known in quantum statistical mechanics of particles with spin: fermions and bosons, respectively.

1.4. Polytropic equations of state

The distributions corresponding to polytropic equations of state of a gas on the kinetic level have the form

$$f = A\left(\alpha - \varphi - \frac{|v|^2}{2}\right)^{\frac{1}{q-1}}, \tag{19}$$

where $q > 1$, $A = \left(\frac{q-1}{q\vartheta}\right)^{\frac{1}{q-1}}$, $\alpha = \frac{1-(q-1)\lambda}{q-1}\vartheta$ with $\lambda = \lambda(x,t) > 0$, $\vartheta = \vartheta(t) > 0$. They are obtained by extremizing the Rényi–Tsallis entropy

$$S = \frac{-1}{q-1}\int_{\Omega\times\mathbb{R}^d} (f^q - f)\, dx\, dv$$

at fixed mass and energy, cf. [11].

After integrating with respect to $r = |v|$ we obtain

$$\begin{aligned} n &= A2^{d/2-1}\sigma_d(\alpha-\varphi)^{\frac{1}{q-1}+\frac{d}{2}} B\left(\frac{d}{2}, 1+\frac{1}{q-1}\right), \\ p &= A2^{d/2-1}\sigma_d(\alpha-\varphi)^{\frac{1}{q-1}+\frac{d}{2}+1} B\left(\frac{d}{2}, 1+\frac{1}{q-1}\right)\frac{1}{\frac{d}{2}+\frac{1}{q-1}+1}, \end{aligned}$$

where B denotes the Euler Beta function $B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$, see [11]. Thus we arrive at the relation

$$p_{1+\gamma}(n,\vartheta) = \kappa_\gamma \vartheta^{1-\gamma\frac{d}{2}} n^{1+\gamma} \tag{20}$$

with $1/\gamma = 1/(q-1) + d/2 > d/2$, so that $q = 1 + \frac{1}{1/\gamma - d/2}$, and the polytropic constant $\kappa_\gamma = \frac{\gamma}{\gamma+1}\left(2^{d/2-1}\sigma_d B\left(\frac{d}{2}, 1+\frac{1}{\gamma}-\frac{d}{2}\right)\right)^{-\gamma}\left(1+\frac{1}{\gamma}-\frac{d}{2}\right)^{\gamma\frac{d}{2}-1}$, since $\left(\frac{q-1}{q}\right)^{\frac{1}{q-1}} = \left(1+\frac{1}{\gamma}-\frac{d}{2}\right)^{\frac{1}{\gamma}-\frac{d}{2}}$. Passing to the limit $q \nearrow \infty$, we have $\gamma = 2/d$, and p is independent of ϑ

$$p_{1+2/d}(n,\vartheta) = \kappa_{2/d} n^{1+2/d}. \tag{21}$$

The polytropic relations define evolution equations with nonlinear diffusions as, e.g., in the porous media equation.

The limit case $q \searrow 1$ corresponds to the Boltzmann density-pressure relation (12) (cf. also the limit Boltzmann entropy S).

The Fermi–Dirac model is, in certain sense, an interpolation between mean field models involving the Brownian diffusion and diffusion in gases. In the *classical limit* $\lambda \nearrow \infty$, i.e. $n/\vartheta^{d/2} \searrow 0$, the relation

$$\frac{p_{\mathrm{FD}}}{n} = \frac{2}{d}\vartheta \frac{I_{d/2}(\lambda)}{I_{d/2-1}(\lambda)} \sim \vartheta \tag{22}$$

holds. This limit corresponds to the linear Brownian diffusion as was in [6, 13]. The completely degenerate case (*white dwarf* in astrophysics), $\lambda \searrow 0$, i.e. $n/\vartheta^{d/2} \nearrow \infty$, corresponds to the relation

$$\frac{p_{\mathrm{FD}}}{n^{1+2/d}} \sim \frac{2}{d+2}\left(\frac{d}{\mu}\right)^{2/d} = \kappa = \text{const}, \tag{23}$$

i.e., to a polytropic equation of state of a gas.

In the Examples 1.1–1.4 the scaling relation

$$f(\lambda, \vartheta, \vartheta^{1/2}v) \equiv f(\lambda, 1, v) \tag{24}$$

holds for each $\vartheta > 0$. Consequently, we have (see (10))

$$p(\vartheta^{d/2}n, \vartheta) = \vartheta^{d/2+1}p(n, 1), \tag{25}$$

for all $\vartheta > 0$, and thus

$$p(n, \vartheta) = \vartheta^{d/2+1}P\left(\frac{n}{\vartheta^{d/2}}\right), \tag{26}$$

which is valid with a function P defined on $\mathbb{R}^+$ – as was for the pressure in each of the examples (12), (15), (17), (20) and (21).

2. Estimates for entropies and energies

In the isothermal setting (i.e. *canonical ensemble*) the function

$$W_{\mathrm{iso}} = \frac{1}{\vartheta}\int_\Omega \left(\vartheta n h - p + \frac{1}{2}n\varphi\right)\, dx \tag{27}$$

is an entropy for the problem (1)–(2), (5)–(6). Here the function h is defined for an arbitrary increasing function p of $n > 0$ and $\vartheta > 0$ by the relation

$$\frac{\partial h}{\partial n} = \frac{1}{\vartheta n}\frac{\partial p}{\partial n}. \tag{28}$$

A simple calculation leads to the entropy production formula

$$\frac{d}{dt}W_{\mathrm{iso}} = -\int_\Omega \vartheta n D_* \left|\nabla\left(h + \frac{\varphi}{\vartheta}\right)\right|^2\, dx \le 0. \tag{29}$$

In the nonisothermal setting (i.e. *microcanonical ensemble*) we have

Lemma 1. *If for the density-pressure relation (3) the condition (26) is satisfied with a function* $P \in C^1$*, then*

$$W = \int_\Omega \left(nh - \left(\frac{d}{2} + 1 \right) \frac{p}{\vartheta} \right) dx = W_{\text{iso}} - \frac{E}{\vartheta} \tag{30}$$

is an entropy for the problem (1)–(2), (5)–(6), one of the boundary condition (7) or (8), with the energy relation (4). Moreover, the following production of entropy formula holds

$$\frac{d}{dt} W = - \int_\Omega \vartheta n D_* \left| \nabla \left(h + \frac{\varphi}{\vartheta} \right) \right|^2 dx \le 0. \tag{31}$$

Proof. Let us compute $\frac{d}{dt}\left(\frac{1}{\vartheta}\int_\Omega p\,dx\right)$ in two ways.

First, using the energy relation (4) and denoting $\frac{d}{dt}\vartheta$ by $\dot\vartheta$, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{d}{2}\frac{1}{\vartheta}\int_\Omega p\,dx\right) &= \frac{1}{\vartheta}\frac{d}{dt}\left(\frac{d}{2}\int_\Omega p\,dx\right) - \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx = \\
&= \frac{1}{\vartheta}\frac{d}{dt}\left(E - \frac{1}{2}\int_\Omega n\varphi\,dx\right) - \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx = \\
&= -\frac{1}{\vartheta}\int_\Omega n_t\varphi\,dx - \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx.
\end{aligned}$$

Second, since by (26) the relation $\frac{\partial p}{\partial \vartheta} = \left(\frac{d}{2}+1\right)\frac{p}{\vartheta} - \frac{d}{2}\frac{n}{\vartheta}\frac{\partial p}{\partial n}$ holds, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{\vartheta}\int_\Omega p\,dx\right) &= \frac{1}{\vartheta}\int_\Omega \frac{\partial p}{\partial n} n_t\,dx + \frac{\dot\vartheta}{\vartheta}\int_\Omega \frac{\partial p}{\partial \vartheta}\,dx - \frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx = \\
&= \frac{1}{\vartheta}\int_\Omega \frac{\partial p}{\partial n} n_t\,dx - \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega n\frac{\partial p}{\partial n}\,dx + \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx.
\end{aligned}$$

Now we get

$$\begin{aligned}
\frac{d}{dt}W &= \int_\Omega n_t h\,dx + \int_\Omega n h_t\,dx + \frac{1}{\vartheta}\int_\Omega n_t\varphi\,dx + \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx + \\
&\quad -\frac{1}{\vartheta}\int_\Omega \frac{\partial p}{\partial n} n_t\,dx + \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega n\frac{\partial p}{\partial n}\,dx - \frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx.
\end{aligned}$$

Let us observe that by (26) h has the self-similar form

$$h(n,\vartheta) = H\left(\frac{n}{\vartheta^{d/2}}\right) \tag{32}$$

with a function H satisfying $H'(s) = P'(s)/s$. In other words

$$P'(s) = \frac{d}{ds}G(H(s)), \quad \text{where} \quad G' = g = H^{-1} \tag{33}$$

is the inverse function of H. So we obtain $\frac{\partial h}{\partial \vartheta} = -\frac{d}{2}\vartheta^{-2}\frac{\partial p}{\partial n}$ and $\frac{d}{dt}h = \frac{\partial h}{\partial n}\frac{d}{dt}n + \dot{\vartheta}\frac{\partial h}{\partial \vartheta}$. Therefore, we arrive at the entropy production formula (31). $\square$

Remark 1. *The entropies corresponding to the nonisothermal problems in Examples 1.1 and 1.4 read*

$$W_{\mathrm{MB}} = \int_\Omega n\left(\log n - \frac{d}{2}\log\vartheta\right)dx, \tag{34}$$

$$W_{1+\gamma} = \kappa_\gamma\left(\frac{1}{\gamma} - \frac{d}{2}\right)\vartheta^{-\gamma d/2}\int_\Omega n^{1+\gamma}\,dx, \tag{35}$$

with the limit case $\lim_{\gamma\nearrow 2/d} W_{1+\gamma} = 0$, *when* $p_{1+2/d}$ *is defined in (21) and the dependence on* ϑ *also disappears in the energy (4).*

The energy relation (4) sometimes leads to interesting *a priori* estimates.

Lemma 2. *If* $P(s) \ge \varepsilon s^{1+\gamma}$ *for some* $\varepsilon > 0$, $\gamma > 1 - 2/d$ *and large* $s > 0$, *then the total energy (4) controls the thermal energy* $\frac{d}{2}\int_\Omega p\,dx$ *and the absolute value of the potential energy* $\frac{1}{2}\left|\int_\Omega n\varphi\,dx\right|$ *from above. More precisely, for each* $0 < c_1 < d/2$ *there exists* $C = C(c_1,\Omega)$ *such that*

$$E \ge c_1\int_\Omega p\,dx + \left|\int_\Omega n\varphi\,dx\right| - CM^{1+\nu}, \tag{36}$$

where $\nu = 2\gamma/(\gamma d + 2 - d)$.

Proof. Clearly, the integral $\int_\Omega n\varphi\,dx$ is bounded from above. In fact, for either $d \ge 3$ or $d \ge 1$ and (8), we have even $\int_\Omega n\varphi\,dx \le 0$.

Using the Hölder inequality and the Sobolev imbedding theorem (and taking into account the boundary conditions for φ) we obtain for each $\varepsilon > 0$

$$\begin{aligned}\left|\int_\Omega n\varphi\,dx\right| &\le |n|_q|\varphi|_{q'} \le C|n|_q^2 \le \\ &\le C|n|_{1+\gamma}^{2r}|n|_1^{2-2r} \le \varepsilon|n|_{1+\gamma}^{1+\gamma} + CM^{1+2\gamma/(\gamma d+2-d)}\end{aligned}$$

with $1/q = r/(1+\gamma) + (1-r) = (d+2)/(2d)$. Since $\int_\Omega p\,dx$ dominates $|n|_{1+\gamma}^{1+\gamma}$, this implies (36). $\square$

Corollary 1. *Under the assumptions from Lemma 2, the relation (4) gives a uniform a priori estimate from above for the quantities* $\int_\Omega p(x,t)\,dx$ *and* $\left|\int_\Omega n(x,t)\varphi(x,t)\,dx\right|$ *in the evolution problem whenever solutions* $n(x,t)$ *with fixed* M, E *exist.*

Remark 2. *The results in Lemma 2 apply to the case of polytropic density--pressure relations and the Fermi–Dirac model* ($\gamma = 2/d$, $d \le 3$, *arbitrary* M, *see* [7, (29)]). *They are also valid (with modified proofs) for the Maxwell–Boltzmann case* ($d = 2$ *and small* M, *see* [2, 6]).

3. The steady state problem

For general density-pressure relations (3) the steady states (N, Φ) of (1)–(2) are determined from the equation

$$h(N) + \frac{\Phi}{\vartheta} = c, \tag{37}$$

where the constant c is chosen from the mass constraint $\int_\Omega N\,dx = M$, $\Delta\Phi = N$, and the function $h = h(n, \vartheta)$ is defined in (28). This is obtained multiplying the stationary version $\nabla \cdot (D_*(\nabla p + N\nabla\Phi)) = 0$ of (1) by $h(N) + \Phi/\vartheta$ and integrating over Ω, which leads to

$$\int_\Omega \vartheta N D_* \left| \nabla \left(h(N) + \frac{\Phi}{\vartheta} \right) \right|^2 dx = 0. \tag{38}$$

Whenever h is of self-similar form (32), the equation (37) can be written as

$$-\vartheta\Delta\Psi = \vartheta^{d/2} g(\Psi + c), \tag{39}$$

where $\Psi = -\Phi/\vartheta$ and $g = H^{-1}$ as in (33). Then, mass normalization becomes

$$\vartheta^{d/2-1} \int_\Omega g(\Psi + c)\,dx = \frac{M}{\vartheta}, \tag{40}$$

and the energy constraint (4) is now

$$E = \frac{d}{2} \int_\Omega p\,dx - \frac{\vartheta}{2} \int_\Omega N\Psi\,dx. \tag{41}$$

Entropies give an alternative way to derive and solve the equation (37) for steady states. Namely, the production of entropy formula leads to (37), and the variational approach is possible via an analysis of the dual functional as was in [18]. Without presenting the details (cf. [3, Sec. 4.1]), we recall that this gives for each $M > 0$ and $\vartheta > 0$, solvability of (39) supplemented with the Dirichlet condition (8) and mass constraint expressed as $\int_{\partial\Omega} \frac{\partial\Psi}{\partial\nu}\,d\sigma = -M/\vartheta$, whenever $P(s) \sim Cs^{1+\gamma}$ for large $s > 0$ with $\gamma > 1 - 2/d$. Indeed, then $H(s) \sim Cs^\gamma$, so $g(s) = H^{-1}(s) \sim Cs^{1/\gamma}$, and the sufficient condition in [18] reads $\lim_{s\to\infty} g(s)/s^{p^*} = 0$ for $p^* = d/(d-2)$, if $d \ge 3$ or $p^* < \infty$ if $d = 2$.

We get for $\vartheta =$ const and either of the boundary conditions (7), (8), the following existence results similar to those in Propositions 4.1, 4.2 in [7].

Proposition 1. *Under the assumption $P(s) \ge \varepsilon s^{1+\gamma}$ with some $\varepsilon > 0$, $\gamma > 1 - 2/d$ and all large s (as in Lemma 2), given $M > 0$ there exists at least one solution Ψ of the equation (39) satisfying the condition (7) and (40). A similar result holds true for the case of boundary condition (8).*

If $\gamma = 1 - 2/d$ those solutions exist for sufficiently small $M > 0$.

The question of the existence of multiple solutions of the equation (38), and their stability as solutions of the evolution problem (1)–(6), is rather delicate. The problem is relatively well understood in the case of the Boltzmann model. There are some numerical results in the case of radially symmetric solutions of the Fermi–Dirac model in the ball of $\mathbb{R}^3$ in [9].

Now we study stationary solutions satisfying the energy and mass constraints, i.e. in the *microcanonical ensemble*. In the problem of finding steady states for fixed E and M, (37) is supplemented by (4) and (9), so that the temperature ϑ is to be determined from these constraints. The approach via fixed point arguments is like that in [15, 16]. However, it should be noted that in the aforementioned papers $p = p_{\rm FD}$ and the dependence on ϑ was not explicitly stated (as irrelevant in the *canonical ensemble*), while it is of crucial importance in [7, 17] (still with $p = p_{\rm FD}$), and in [8] with rather general p's.

Theorem 1. *Suppose that $d \geq 2$, $0 < P'(s) < \infty$ for $s \geq 0$ and $P'(s)/s$ is a decreasing function. Then for $0 < M/\vartheta \ll 1$ there exists a solution of (39) with (40). These solutions form a branch in $L^\infty(\Omega)$ depending continuously on M and ϑ.*

Moreover, for each $M > 0$ there exists $E_0 = E_0(M) > -\infty$ such that given $E > E_0$ there exists at least one solution of (39), (40), (4) satisfying (7).

Similar result holds for the boundary condition (8).

Proof. We refer the reader for the proof (using contraction mappings arguments in the unit ball of $L^\infty(\Omega)$) to [8, Sec. 4]. The crucial estimate follows from the observation, that if $P'(0) \in (0, \infty)$, then the strictly increasing function H defined in (28), (32) satisfies the relation

$$\lim_{s \to 0} \frac{H(s)}{\log s} = \ell \in (0, \infty).$$

Since $P'(s)/s$ decreases, H is a concave function and $g = H^{-1}$ is a convex function. This is, of course, satisfied for each of the Examples 1.1–1.3, see [3], [7] for the Fermi–Dirac case. The function g defined in (33), as well as its derivative g', are strictly increasing on $(-\infty, 0)$ with

$$g(z) \sim \ell g'(z) \sim \text{const}\, \mathrm{e}^{z/\ell}, \quad z \to -\infty.$$

$\square$

A similar reasoning, but with completely different asymptotics of functions H, g, works in the case of density-pressure relations resembling the polytropic ones. Namely, we state without the proof the following

Theorem 2. *Suppose that $d \geq 2$, $0 < \gamma < 2/d$, $\lim_{s\to 0} P(s)/s^{1+\gamma} \in (0,\infty)$, and $P'(s)/s$ is a decreasing function. Then for a fixed $M > 0$ and sufficiently large ϑ there exists a solution of (39) with (40). These solutions form a branch in $L^\infty(\Omega)$ depending continuously on M and ϑ.*

4. Nonexistence of global in time solutions

Here we prove some results on the nonexistence of steady states and, more generally, nonexistence of solutions of the evolution problem defined for all $t \geq 0$. These are results on the isothermal problem with the pressure p asymptotically resembling polytropic relation (20), and in the microcanonical setting with quite general density-pressure relations, the latter are recalled from [7, Sec. 2].

Star-shaped domains $0 \in \Omega \subset \mathbb{R}^d$ are defined by the condition $\beta \geq 0$, where

$$\beta = \inf_{x\in\partial\Omega} x \cdot \overline{\nu}. \tag{42}$$

Similarly, *strictly star-shaped* domains are those with $\beta > 0$. In this section, by solutions we mean the classical ones $n \in C^2(\Omega\times(0,\infty))\cap C^1(\overline{\Omega}\times[0,\infty))$.

Let us begin with the case of polytropic p with small exponents in (20), i.e. with relatively weak diffusion for large n.

Theorem 3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \geq 3$, p is such that $p(s)/s^{2-2/d}$ is a decreasing function for all sufficiently large values of s. Then there exist initial data n_0 such that sufficiently regular, positive solutions of the problem (1)–(2), (5), (7), with a constant temperature $\vartheta > 0$, cannot be defined globally in time.*

Proof. Multiplying (1) by $|x|^2$ and integrating over Ω we get the relation

$$\begin{aligned} \frac{d}{dt}\int_\Omega n|x|^2\,dx &= -2\int_\Omega \nabla p\cdot x\,dx - 2\int_\Omega n\nabla\varphi\cdot x\,dx = \\ &= 2d\int_\Omega p\,dx - 2\int_{\partial\Omega} p\,x\cdot\overline{\nu}\,d\sigma + \\ &\quad -\frac{2}{\sigma_d}\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^d}(x-y)\cdot x\,dx\,dy \end{aligned} \tag{43}$$

for the second moment $V(t) = \int_\Omega n(x,t)|x|^2\,dx$. After symmetrization of the double integral above, we arrive at

$$\frac{d}{dt}V \;\leq\; 2d\int_\Omega p\,dx - \frac{1}{\sigma_d}\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^{d-2}}\,dx\,dy \tag{44}$$

since $(x-y)\cdot x+(y-x)\cdot y=|x-y|^2$. Next, we see that from (27) that $W_{\rm iso}$ can be used to estimate

$$\begin{aligned}\frac{d}{dt}V &\leq 2d\int_\Omega p\,dx+2(d-2)\left(\vartheta W_{\rm iso}+\int_\Omega(-\vartheta nh+p)\,dx\right)\leq\\ &\leq 2(d-2)\vartheta W_{\rm iso}(0)+2\int_\Omega((2d-2)p-(d-2)\vartheta nh)\;dx. \qquad (45)\end{aligned}$$

Now, observe that if, more generally than in the assumption, p is a C^1 function such that $p(s)/s^{1+\gamma}$ decreases for large s, $s>s_*$, then

$$\frac{p'(s)}{s^{1+\gamma}}-(1+\gamma)\frac{p(s)}{s^{2+\gamma}}\leq 0,$$

so that $(1+\gamma)\left(\frac{p(s)}{s}\right)'-\gamma\frac{p'(s)}{s}\leq 0$. This reads $\left(\frac{1}{\gamma}+1\right)\frac{p(s)}{s}\leq\vartheta h(s)$, or

$$\vartheta sh(s)-p(s)\geq\frac{1}{\gamma}p(s)$$

for large s. Finally, (45) implies

$$\frac{d}{dt}V\leq 2(d-2)\vartheta W_{\rm iso}(0)+C,$$

where $C=C(n_*)$ takes also into account the value of the integral of $(2d-2)p-(d-2)\vartheta nh$ over small values of n, $n\leq n_*$ with a fixed n_*. Evidently, for n_0 sufficiently large and well concentrated (e.g. for Gaussian n_0 with small variance), the entropy is negative: $W_{\rm iso}\ll -1$, so that we obtain for such initial data

$$\frac{d}{dt}V=\frac{d}{dt}\int_\Omega n|x|^2\,dx\leq-\varepsilon<0$$

for some $\varepsilon>0$. This leads to a contradiction with the existence of a positive solution n for $t\geq T_{\max}=\int_\Omega n_0|x|^2\,dx/\varepsilon$. □

Corollary 2. *For $d\geq 4$ there exist initial data such that solutions of the isothermal system (1)–(2) with the Fermi–Dirac density-pressure relation (15), conditions (5), (7), cease to exist after a finite time.*

Proof. Indeed, for $d\geq 4$ $\quad 1+2/d\leq 2-2/d$ holds and Theorem 3 applies. Note that for $d=4$ the initial data leading to a blow up of solutions must have a sufficiently large mass, while for $d\geq 5$ a blow up is possible for arbitrarily small mass and highly concentrated n_0. This can be inferred from the proof of Lemma 2. □

We have also a result on the nonexistence of solutions in the microcanonical setting.

Theorem 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \geq 4$, and $E/M^2 < (d/4-1)/\left(\sigma_d(d-2)(\operatorname{diam}\Omega)^{d-2}\right)$. Then sufficiently regular, positive solutions of the problem (1)–(2), (5), with $D_* = 1$, and (7), satisfying the energy relation (4) cannot be defined globally in time.*

Proof. There exist initial data (n_0, ϑ_0) leading to $E/M^2 \ll 0$. It suffices to consider an arbitrary $0 \leq n_0 \not\equiv 0$ in (6), $\vartheta_0 > 0$, and take the density Mn_0 with $M \gg 1$ large enough.

Repeating the computations (43)–(44) of the evolution of the moment $V(t) = \int_\Omega n(x,t)|x|^2\,dx$ in the beginning of the proof of Theorem 3, we arrive at the inequality

$$\begin{aligned} \frac{d}{dt}V &\leq 4E + \frac{1}{\sigma_d}\left(\frac{2}{d-2}-1\right)\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^{d-2}}\,dx\,dy \leq \\ &\leq 4E + \frac{1}{\sigma_d}\frac{4-d}{d-2}M^2\left(\operatorname{diam}\Omega\right)^{2-d}, \end{aligned}$$

where we used (4). Under the assumptions of Theorem 4 we obtain

$$\frac{d}{dt}\int_\Omega n|x|^2\,dx \leq -\varepsilon < 0$$

for some $\varepsilon > 0$. As before, this contradicts the existence of a positive solution n after $T_{\max} = \int_\Omega n_0|x|^2\,dx/\varepsilon$. $\square$

Corollary 3. *Under the assumptions of Th. 4, there is no positive steady state of the system (1)–(2), (3), (5) with the energy (4), arbitrary $D_* > 0$, in bounded star-shaped domains $\Omega \subset \mathbb{R}^d$.*

Proof. Evidently, steady states do not depend on $D_* > 0$, cf. (38), so that $\frac{d}{dt}\int_\Omega n|x|^2\,dx \equiv 0$ for each steady state. Thus, the proof of Theorem 4 applies in this situation. $\square$

5. Uniform estimates for solutions of the evolution problem for polytropes

We present in this section an argument which would permit us to continue local in time solutions of the isothermal problem with strong diffusion to the global in time ones. This reasoning is formal because, as far as we know, there are no local in time existence results for general p as in (3), except for the Boltzmann and Fermi–Dirac cases, see [4] and [3]. We will prove that $\sup_{\delta\leq t\leq T}|n(t)|_q < \infty$ for any $q < \infty$ and $0 < \delta < T < \infty$, with a bound depending also on δ and T. Since by results in Theorem 3 we know that for $\gamma \leq 1 - 2/d$ blow up of solutions, caused by $\int_\Omega n(x,t)|x|^2\,dx \to 0$ as $t \nearrow T_{\max}$, is accompanied by the unboundedness of $L^q(\Omega)$ norms as $t \nearrow T_{\max}$, we can conclude that such a phenomenon do not occur for the polytropic exponents γ, $\gamma > 1 - 2/d$, i.e. for diffusion terms $\Delta p(n)$ in (1) strong enough.

Proposition 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, p is a C^1 function such that $p(s) \geq \varepsilon s^{1+\gamma}$ for some $\varepsilon > 0$, $\gamma > 1 - 2/d$ and large s, and $p(s)/s^{1+\tilde{\gamma}}$ decreases for all sufficiently large s and some $\tilde{\gamma} > 1 - 2/d$. Suppose that the initial data n_0 are such that W_{iso} in (27) is finite. Then each local in time positive solution of the system (1)–(2), (5), with either of the condition (7), (8) satisfies*

$$\sup_{\delta \leq t \leq T} |n(t)|_q < \infty$$

for each $q > 1$, $\delta > 0$ and $T < \infty$.

Proof. Recall from the beginning of the proof of Theorem 3 that the assumption on p implies $\vartheta s h(s) - p(s) \geq \frac{1}{\tilde{\gamma}} p(s)$. This, together with the estimate recorded in Lemma 2, shows that the entropy W_{iso} controls $\int_\Omega p(x,t)\, dx$:

$$W_{\mathrm{iso}} \geq \frac{1}{2\tilde{\gamma}} \int_\Omega p \, dx - C$$

with a constant $C = C(M)$. Thus, $\int_\Omega p(x,t)\, dx \sim |n(t)|_{1+\gamma}^{1+\gamma}$ is *a priori* uniformly bounded for all $t \geq 0$, see also Corollary 1.

From now on, avoiding clumsy notation, we will write our proof for $\gamma = \tilde{\gamma}$ and $p(s) = \kappa s^{1+\gamma}$, $s \geq 0$, the modifications necessary to cover the case $p(s) \sim \kappa s^{1+\gamma}$ for $s \geq s_*$ (so that, e.g., $p(s) \geq \frac{\kappa}{2} s^{1+\gamma} - C$) being fairly standard.

Let us multiply (1) by n^{q-1} and integrate over Ω taking into account (5), and either $\frac{\partial \varphi}{\partial \nu} \leq 0$ on $\partial\Omega$ in (7) case or $\varphi = 0$ in (8) case, to estimate the boundary integrals. This leads to the differential inequality

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega n^q\, dx &\leq -(q-1)\int_\Omega n^{q+\gamma-2}|\nabla n|^2\, dx + \frac{q-1}{q}\int_\Omega n^{q+1}\, dx = \\ &= -4\frac{q-1}{(q+\gamma)^2}\int_\Omega \left|\nabla\left(n^{\frac{q+\gamma}{2}}\right)\right|^2 dx + \left(1 - \frac{1}{q}\right)|n|_{q+1}^{q+1}. \qquad (46) \end{aligned}$$

Our strategy is to estimate the term $|n|_{q+1}^{q+1} = |v|_{\frac{2(q+1)}{q+\gamma}}^{\frac{2(q+1)}{q+\gamma}}$ with $v = n^{\frac{q+\gamma}{2}}$ by $|\nabla v|_2^2$ and a function $c = c\left(|v|_1, |v|_{\frac{2(1+\gamma)}{q+\gamma}}\right)$, $|v|_{\frac{2(1+\gamma)}{q+\gamma}}^{\frac{2(1+\gamma)}{q+\gamma}} = |n|_{1+\gamma}^{1+\gamma}$. Here we will use the Sobolev and (a version of) the Poincaré inequality: $\|v\|_{H^1(\Omega)}^2 \leq C\left(|\nabla v|_2^2 + |v|_1^2\right)$. To accomplish this goal, we will apply this scheme of estimates with $q_0 = 2 + \gamma$, $q_{m+1} = 2q_m - \gamma\, (= 2^{m+1} + \gamma) \nearrow \infty$, $m = 0, 1, 2, 3 \ldots$ Note that $\frac{q_0+\gamma}{2} = 1 + \gamma$, so the recurrent procedure may be begun with the preliminary estimate of $\int_\Omega p(x,t)\,dx \sim |n(t)|_{1+\gamma}^{1+\gamma}$, and then works with $|n(t)|_{q_m}$, $q_m = \frac{q_{m+1}+\gamma}{2}$. □

Remark 3. *In the context of Fermi–Dirac isothermal systems, Proposition 2 says that solutions in either the $d \le 3$ dimensional case with arbitrary mass $M > 0$, are locally bounded in each $L^q(\Omega)$, $q < \infty$. Related results for the system with the diffusion coefficient $D_* \not\equiv 1$ are in* [3].

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