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UPPER AND LOWER LIMITS OF SEQUENCES OF MEASURABLE SETS AND OF SETS WITH THE BAIRE PROPERTY

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Abstract. The note presents the study of the behaviour of upper and lower limits of sequence $\{A_n\}_{n\in N}$ of S-measurable subsets of the unit interval $[0,1]$ when all sets A_n are subject to small translations. S is here the class of all Lebesgue measurable sets or the class of sets with the Baire property. The last two theorems describe a new (density point) approach to the convergence of Rademacher's type sequences of sets and functions and generalise the result of K. P. Rath from [2]

Among interesting properties of the Lebesgue measure one can find the following one (compare [1], p. 901): for each measurable set $A \subset [0,1]$ we have $\lim_{x\to 0}\mu(A\triangle(A+x)) = 0$, where $A+x = \{a+x : a \in A\}$. In this note we shall study the behaviour of upper and lower limits of sequence $\{A_n\}_{n\in N}$ of measurable subsets of the unit interval, when all sets A_n are subject to small translations (it may happen that $A_n + \epsilon_n$ is no longer included in $[0,1]$).

Theorem 1. *For each sequence $\{A_n\}_{n\in N}$ of measurable subsets of $[0,1]$ such that $\mu(\limsup_n A_n) = 0$ there exists a sequence $\{\epsilon_n\}_{n\in N}$ of numbers different from zero such that $\lim_{n\to\infty}\epsilon_n = 0$ and $\mu(\limsup_n(A_n+\epsilon_n)) = 0$.*

Proof. By virtue of the quoted property of Lebesgue measure for each $n \in N$ there exists $\epsilon_n \in (0,\frac{1}{n})$ such that $\mu(A_n\triangle(A_n+\epsilon_n)) < \frac{1}{2^n}$.

Observe that for each $n \in N$ we have the following inclusions and equations:

$$\bigcup_{m=n}^{\infty}(A_m+\epsilon_m) \subset \bigcup_{m=n}^{\infty}(A_m\cup(A_m+\epsilon_m)) = \bigcup_{m=n}^{\infty}(A_m\cup((A_m+\epsilon_m)-A_m)) =$$

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$$= \bigcup_{m=n}^{\infty} A_m \cup \bigcup_{m=n}^{\infty} ((A_m + \epsilon_m) - A_m) = \bigcup_{m=n}^{\infty} A_m \cup \bigcup_{m=n}^{\infty} ((A_m + \epsilon_m) \triangle A_m).$$

Hence $\mu(\bigcup_{m=n}^{\infty}(A_m + \epsilon_m)) \leq \mu(\bigcup_{m=n}^{\infty} A_m) + \frac{1}{2^{n-1}}$. From the assumption it follows that $\lim_{n\to\infty} \mu(\bigcup_{m=n}^{\infty} A_m) = 0$, so $\lim_{n\to\infty} \mu(\bigcup_{m=n}^{\infty}(A_m+\epsilon_m)) = 0$ and finally $\mu(\limsup_n (A_n + \epsilon_n)) = 0$. □

Remark 1. *Observe that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then for arbitrary sequence $\{\epsilon_n\}_{n\in N}$ of real numbers $\mu(\limsup_n (A_n+\epsilon_n)) = 0$. The situation is different if $\sum_{n=1}^{\infty} \mu(A_n) = +\infty$ and $\mu(\limsup_n A_n) = 0$. It can happen either that $\mu(\limsup_n (A_n + \epsilon_n)) = 0$ for each sequence $\{\epsilon_n\}_{n\in N}$ convergent to zero, or only for some sequences.*

We shall illustrate it with the following two examples.

Example 1. *Put $A_n = [0, \frac{1}{n}]$ for $n \in N$, and let $\{\epsilon_n\}_{n\in N}$ be an arbitrary sequence convergent to zero. Let $h \in (0, 1)$. We have $-h < \epsilon_n < \frac{1}{n} + \epsilon_n < h$ for n large enough, which yields $\limsup_n (A_n + \epsilon_n) \subset (-h, h)$. From arbitrariness of h it follows that $\mu(\limsup_n (A_n + \epsilon_n)) = 0$.*

Example 2. *Now, we shall construct two sequences $\{A_n\}_{n\in N}$ and $\{\epsilon_n\}_{n\in N}$ such that $A_n \subset [0, 1]$, $\lim_{n\to\infty} \epsilon_n = 0$, $\mu(\limsup_n (A_n)) = 0$ and $\mu(\limsup_n (A_n+\epsilon_n)) = 1$. The sequence $\{\epsilon_n\}_{n\in N}$ will be here strictly decreasing.*

We start with $A_1 = (0, \frac{1}{2})$. Next are $A_2 = A_3 = A_4 = (0, \frac{1}{8}) \cup (\frac{1}{2}, \frac{1}{2} + \frac{1}{8})$. They are followed by $A_5 = A_6 = \ldots = A_{11} = (0, \frac{1}{64}) \cup (\frac{1}{8}, \frac{1}{8} + \frac{1}{64}) \cup \ldots \cup (\frac{7}{8}, \frac{7}{8} + \frac{1}{64})$, and so on. Appropiate ϵ_n's are $\epsilon_1 = \frac{1}{2}$, $\epsilon_2 = \frac{3}{8}$, $\epsilon_3 = \frac{2}{8}$, $\epsilon_4 = \frac{1}{8}$, $\epsilon_5 = \frac{7}{64}$, $\epsilon_6 = \frac{6}{64}$, ..., $\epsilon_{11} = \frac{1}{64}$ and so on.

In general we define by induction a sequence $\{a_k\}_{k\in N\cup\{0\}}$ in the following way: $a_0 = 0$, $a_{k+1} = a_k + k$ for $k \in N \cup \{0\}$. Let

$$E_k = \bigcup_{i=0}^{2^{a_k}-1} \left(\frac{i}{2^{a_k}}, \frac{i}{2^{a_k}} + \frac{1}{2^{a_{k+1}}} \right)$$

and for n of the form

$$n = \sum_{i=1}^{k} (2^{i-1} - 1) + j, \text{ where } j = 1, \ldots, 2^k - 1$$

put

$$A_n = E_k \quad \text{and} \quad \epsilon_n = \frac{2^k - j}{2^{a_{k+1}}}.$$

The sequence $\{\epsilon_n\}_{n\in N}$ is strictly decreasing. Observe that

$$\mu(E_k) = 2^{a_k} \cdot \frac{1}{2^{a_{k+1}}} = 2^{-(a_{k+1}-a_k)} = 2^{-k},$$

which implies that $\lim_{n\to\infty} \mu(\bigcup_{m=n}^{\infty} A_m) = 0$, *and consequently we have* $\mu(\limsup_n(A_n)) = 0$.

On the other hand we have

$$\mu(\bigcup_{n\in M} (A_n + \epsilon_n)) = 1 - 2^{-k},$$

where

$$M = \left\{ n \in N : \sum_{i=1}^{k} (2^{i-1} - 1)) + 1 \leq n \leq \sum_{i=1}^{k+1} (2^{i-1} - 1) \right\}$$

and therefore $\mu(\bigcup_{m=n}^{\infty}(A_m + \epsilon_m)) = 1$ *for every* n. *We thus get*

$$\mu(\limsup_n (A_n + \epsilon_n)) = 1.$$

The lower limit of sequence of measurable sets behaves similarly with respect to "small" translations.

Theorem 2. *For each sequence* $\{A_n\}_{n\in N}$ *of measurable subsets of* $[0, 1]$ *such that* $\mu(\liminf_n A_n) = 0$ *there exists a sequence* $\{\epsilon_n\}_{n\in N}$ *of numbers different from zero such that* $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\mu(\liminf_n(A_n + \epsilon_n)) = 0$.

Proof. First observe that for every two families of sets $\{A_t\}_{t\in T}$ and $\{B_t\}_{t\in T}$ we have

$$\left(\bigcap_{t\in T} A_t\right) \triangle \left(\bigcap_{t\in T} B_t\right) \subset \bigcup_{t\in T} (A_t \triangle B_t).$$

Similarly as in proof of Theorem 1 for each $n \in N$ there exists $\epsilon_n \in (0, 1)$ such that $\mu(A_n \triangle (A_n + \epsilon_n)) < \frac{1}{2^n}$. From the assumption concerning the lower limit it follows that there exists an increasing sequence $\{n_k\}_{k\in N\cup\{0\}}$ of natural numbers such that

$$\mu\left(\bigcap_{m=n_{k-1}+1}^{n_k} A_m\right) < \frac{1}{k} \text{ for } k \in N.$$

Since by virtue of the above observation

$$\bigcap_{m=n_{k-1}+1}^{n_k} (A_m + \epsilon_m) \subset \left(\bigcap_{m=n_{k-1}+1}^{n_k} A_m\right) \cup \bigcup_{m=n_{k-1}+1}^{n_k} (A_m \triangle (A_m + \epsilon_m)),$$

then we have

$$\mu\left(\bigcap_{m=n_{k-1}+1}^{n_k} (A_m + \epsilon_m)\right) < \frac{1}{k} + \frac{1}{2^{n_{k-1}}}.$$

This clearly implies $\mu(\bigcap_{m=n}^{\infty}(A_m+\epsilon_m)) = 0$ for each $n \in N$ and consequently $\mu(\liminf_n(A_n+\epsilon_n)) = 0$. □

Remark 2. *Observe that if* $\liminf_n \mu(A_n) = 0$*, then for arbitrary sequence* $\{\epsilon_n\}_{n\in N}$ *of real numbers* $\mu(\liminf_n(A_n+\epsilon_n)) = 0$*. The situation again is different if* $\liminf \mu(A_n) > 0$ *and* $\mu(\liminf_n A_n) = 0$*. Here also can happen either that* $\mu(\liminf_n(A_n+\epsilon_n)) = 0$ *for each sequence* $\{\epsilon_n\}_{n\in N}$ *convergent to zero, or only for some sequences.*

Example 3. *We shall construct here the sequence* $\{A_n\}_{n\in N}$ *of measurable subsets of* $[0,1]$ *such that* $\mu(A_n) = \frac{1}{2}$ *for each* $n \in N$ *and* $\mu(\liminf_n(A_n + \epsilon_n)) = 0$ *for each sequence* $\{\epsilon_n\}_{n\in N}$ *such that* $\liminf_n |\epsilon_n| = 0$.

Let $\{A_n\}_{n\in N}$ *be a sequence of sets*

$$A_n = \bigcup_{i=0}^{2^{n-1}-1}\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right)$$

and $\{\epsilon_n\}_{n\in N}$ *a sequence of positive numbers with* $\liminf_n |\epsilon_n| = 0$*. We denote* $B_n = A_n + \epsilon_n$.

We shall show that for arbitrary $n \in N$, $\mu\left(\bigcap_{m=n}^{\infty}(A_m+\epsilon_m)\right) = 0$*. This will imply that*

$$\begin{aligned}\mu\left(\liminf_m (A_m+\epsilon_m)\right) &= \mu\left(\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}(A_m+\epsilon_m)\right) \leq \\ &\leq \sum_{n=1}^{\infty}\mu\left(\bigcap_{m=n}^{\infty}(A_m+\epsilon_m)\right) = \sum_{n=1}^{\infty}(0) = 0.\end{aligned}$$

It is enough to define for arbitrary $n \in N$*, a subsequence* $\{m_k\}_{k\in N}$*, such that* $m_1 \geq n$ *and* $\mu\left(\bigcap_{k=1}^{\infty}(A_{m_k}+\epsilon_{m_k})\right) = 0$.

Fix $n \in N$*. We shall define* $\{m_k\}_{k\in N}$ *by induction. Put* $m_1 = n$*. Assume* $m_1, m_2, ..., m_{k-1}$ *are already defined, such that* $m_1 < m_2 < ... < m_{k-1}$ *and for arbitrary* $j \in \{1,2,...,k-2\}$ *we have inequality*

$$\mu\left(\bigcap_{i=1}^{j+1}(A_{m_i}+\epsilon_{m_i})\right) \leq \frac{2}{3}\mu\left(\bigcap_{i=1}^{j}(A_{m_i}+\epsilon_{m_i})\right),$$

We shall choose m_k *to have*

$$(*) \qquad \mu\left(\bigcap_{i=1}^{k}(A_{m_i}+\epsilon_{m_i})\right) \leq \frac{2}{3}\mu\left(\bigcap_{i=1}^{k-1}(A_{m_i}+\epsilon_{m_i})\right).$$

Observe, that the set $\bigcap_{i=1}^{k-1}(A_{m_i}+\epsilon_{m_i})$ *is a sum of finite number of closed intervals and finite number of points. Let* d_{k-1} *be the length of the shortest interval. We choose* m_k *big enough to have*

$$\frac{1}{2^{m_k}} < \frac{1}{3}d_{k-1}.$$

Then for arbitrary closed, non-degenerate component I *of the set*

$$\bigcap_{i=1}^{k-1}(A_{m_i}+\epsilon_{m_i}),$$

we have

$$\mu\,(I\cap A_{m_k}) \le \frac{2}{3}\mu\,(I),$$

which ensures $(*)$.

Example 4. *We shall construct here two sequences* $\{B_n\}_{n\in N}$ *and* $\{\epsilon_n\}_{n\in N}$ *such that* $B_n\subset[0,1]$, $\mu(\liminf_n B_n)=0$, $\lim_{n\to\infty}\epsilon_n=0$ *and* $\mu(\liminf_n(B_n+\epsilon_n))=1$. *The sequence* $\{\epsilon_n\}_{n\in N}$ *will be here strictly decreasing. We shall write* $\widehat{A}=\{1-x : x\in A\}$.

Consider the sets $C_n=A_n+\epsilon_n$, $n\in N$ *where* A_n, ϵ_n, $n\in N$ *are as defined in Example 2. Put* $B_n=[0,1]-\widehat{C_n}$. *Let us notice that* $B_n+\epsilon_n=[0,1]-\widehat{A}_n$. *From what was established in Example 2. we have* $\mu(\liminf_n B_n)=0$ *as* $\mu(\limsup_n(A_n+\epsilon_n))=1$ *and* $\mu(\liminf_n(B_n+\epsilon_n))=1$ *as* $\mu(\limsup_n A_n)=0$.

Now we shall discuss similar properties of upper and lower limits of sequences with the Baire property.

Let S be a σ-algebra of sets with the Baire property and I the σ-ideal of sets of the first category. If $A\in S$ we say that A is S-measurable.

Theorem 3. *For each sequence* $\{A_n\}_{n\in N}$, *of* S*-measurable (Baire measurable) subsets of* $[0,1]$ *such that* $\liminf_n A_n\in I$ *we have also* $\liminf_n(A_n+\epsilon_n)\in I$ *for every sequence* $\{\epsilon_n\}_{n\in N}$ *of numbers different from zero such that* $\lim_{n\to\infty}\epsilon_n=0$.

Proof. We may assume that all sets A_n, $n\in N$ are open here. From the assumption we have $\bigcap_{n\ge m}A_n\in I$ for every $m\in N$. Since $\bigcap_{n\ge m}A_n$ is a $\mathcal{G}_\delta$ set it has to be nowhere dense. We shall show that also $\bigcap_{n\ge m}(A_n+\epsilon_n)$ is nowhere dense for every $m\in N$. Fix $m_0\in N$ and let $(a,b)\subset[0,1]$. We shall show that there exists interval $(c,d)\subset(a,b)$ such that $(c,d)\cap\bigcap_{n\ge m_0}(A_n+\epsilon_n)=\phi$.

Let $(a_1,b_1)=(a+\frac{1}{3}(b-a),b-\frac{1}{3}(b-a))$, and let $m_1\ge m_0$ be a natural number such that $|\epsilon_n|<\frac{1}{3}(b-a)$ for every $n\ge m_1$. So $(a_1+\epsilon_n,b_1+\epsilon_n)\subset$

$(a,b), n \geq m_1$. Since $\bigcap_{n\geq m} A_n$ is nowhere dense, there exists $n_1 \geq m_1$ such that A_{n_1} is not dense in (a_1,b_1). Hence there exists interval $(c_1,d_1) \subset (a_1,b_1)$ such that $(c_1,d_1) \cap A_{n_1} = \phi$. Consequently $(c_1+\epsilon_{n_1}, d_1+\epsilon_{n_1}) \cap (A_{n_1}+\epsilon_{n_1}) = \phi$ and therefore $(c_1+\epsilon_{n_1}, d_1+\epsilon_{n_1}) \cap \bigcap_{n\geq m_0}(A_n+\epsilon_n) = \phi$. We define now $(c,d) = (c_1+\epsilon_{n_1}, d_1+\epsilon_{n_1}) \subset (a_1+\epsilon_{n_1}, b_1+\epsilon_{n_1}) \subset (a,b)$ and have $(c,d) \cap \bigcap_{n\geq m_0}(A_n+\epsilon_n) = \phi$.

Therefore $\bigcap_{n\geq m_0}(A_n+\epsilon_n)$ is nowhere dense set and so is $\bigcap_{n\geq m}(A_n+\epsilon_n)$ for every $m \in N$, hence the theorem is proved. □

Theorem 4. *For each sequence $\{A_n\}_{n\in N}$, of S-measurable subsets of $[0,1]$ such that $\limsup_n A_n \in I$, we have also $\limsup_n(A_n+\epsilon_n) \in I$ for every sequence $\{\epsilon_n\}_{n\in N}$ of numbers different from zero such that $\lim_{n\to\infty} \epsilon_n = 0$.*

Proof. Again we may assume that all sets A_n, $n \in N$ are open here. Observe that $\bigcap_{m=1}^{\infty} \bigcup_{n\geq m} A_n$ as $\mathcal{G}_\delta$ set of first category is easily seen to be nowhere dense. We shall show that $\limsup_n(A_n+\epsilon_n) \in I$ is also nowhere dense, hence of first category. Let $(a,b) \subset [0,1]$. We shall show that there exists an interval $(c,d) \subset (a,b)$ such that $\limsup_n(A_n+\epsilon_n) \cap (c,d) = \phi$. There exists $m_0 \in N$ such that $\bigcup_{n\geq m_0} A_n$ is not dense in (a,b). Hence there exists interval $(c_0,d_0) \subset (a,b)$ such that $(c_0,d_0) \cap \bigcup_{n\geq m_0} A_n = \phi$. Let now $m_1 \geq m_0$ be a natural number such that $|\epsilon_n| < \frac{1}{3}(d_0-c_0)$. Define $(c,d) = \left(c_0+\frac{1}{3}(d_0-c_0), d_0-\frac{1}{3}(d_0-c_0)\right) \subset (a,b)$. We have $(c,d) \cap (A_n+\epsilon_n) = \phi$, for $n \geq m_1$, which implies $(c,d) \cap \bigcup_{n\geq m_0}(A_n+\epsilon_n) = \phi$. From arbitrariness of (a,b), $\limsup_n(A_n+\epsilon_n)$ is nowhere dense, hence the theorem is proved. □

K. P. Rath in [2] has proved the following theorem:

Theorem 5. *Let $\{\epsilon_n\}_{n\in N}$ be a sequence of positive numbers converging to 0 and $\{f_n\}_{n\in N}$ a sequence of real valued measurable functions on the unit interval $[0,1]$ given by*

$$f_n(t) = (-1)^k \text{ if } k\epsilon_n \leq t < (k+1)\epsilon_n,\ k = 0,1,2,\ldots$$

Let $\{f^{n_k}\}_{k\in N}$ be any subsequence of $\{f^n\}_{n\in N}$. Then the set of points at which $\{f^{n_k}\}_{k\in N}$ converges has Lebesgue measure zero.

We shall prove the theorem describing much wider class of sequences of measurable functions divergent in measure. We shall write $A^c = [0,1] - A$.

Theorem 6. *Suppose that a sequence $\{A_n\}_{n\in N}$ of measurable subsets of $[0,1]$ fulfils the following condition: there exists a positive number $\epsilon > 0$ such that for each open interval $(a,b) \subset [0,1]$ and for each natural number m there exists a natural number $n > m$ for which*

$$\frac{m(A_n \cap (a,b))}{m((a,b))} > \epsilon.$$

Then $m\left(\limsup_n A_n\right) = 1$.

Proof. Suppose that $m(\liminf_n A_n^c \cap [0,1]) > 0$. Since $\liminf_n A_n^c \cap [0,1] = \bigcup_{m=1}^{\infty} \bigcap_{n\geq m} A_n^c$, then there exists $m \in N$ such that $m\left(\bigcap_{n\geq m} A_n^c\right) > 0$. Let $x_0 \in (0,1)$ be a point of density of $\bigcap_{n\geq m} A_n^c$. There exists an interval $(a,b) \subset [0,1]$ such that $x_0 \in (a,b)$ and

$$\frac{m\left(\bigcap_{n\geq m} A_n^c \cap (a,b)\right)}{b-a} > 1 - \frac{\epsilon}{2}.$$

Hence

$$\frac{m\left(A_n^c \cap (a,b)\right)}{b-a} > 1 - \frac{\epsilon}{2}$$

for each $n > m$, so

$$\frac{m\left(A_n \cap (a,b)\right)}{b-a} < \frac{\epsilon}{2}$$

for each $n > m$ -a contradiction. □

As a simple consequence of the above we obtain:

Theorem 7. *Suppose that a sequence* $\{A_n\}_{n\in N}$ *of measurable subsets of* $[0,1]$ *fulfils the following condition: there exist two positive numbers* ϵ_1, ϵ_2 *such that for each open interval* $(a,b) \subset [0,1]$ *and for each natural number* m *there exist* $n,p > m$ *such that*

$$\frac{m\left(A_n \cap (a,b)\right)}{m\left((a,b)\right)} > \epsilon_1 \text{ and } \frac{m\left(A_p \cap (a,b)\right)}{m\left((a,b)\right)} < 1 - \epsilon_2.$$

Then $m\left(\limsup_n A_n\right) = 1$ *and* $m(\liminf_n A_n) = 0$.

Corollary 1. *Under the assumptions of Theorem 6 the sequence* $\{\chi_{A_n}\}_{n\in N}$ *of measurable functions is divergent in measure.*

Remark 3. *Observe that if* $\{f^n\}_{n\in N}$ *is a sequence from theorem of Rath, then for each subsequence* $\{f^{n_k}\}_{k\in N}$ *the condition from our Theorem 6 is fulfilled for* $A_k = \{x \in [0,1] : f^{n_k}(x) = 1\}$.

Remark 4. *Observe that the order of quantifiers in condition from Theorem 6 is important. Really, let* $\delta < 1$ *be arbitrary small real number. We shall define a sequence* $\{A_n\}_{n\in N}$ *of measurable subsets of* $[0,1]$ *fulfilling the condition: for each open interval* $(a,b) \subset [0,1]$ *there exists a positive number* $\epsilon > 0$ *such that for each natural number* m *there exists a natural number* $n > m$ *for which*

$$\frac{m\left(A_n \cap (a,b)\right)}{m\left((a,b)\right)} > \epsilon.$$

but $m\left(\limsup_n A_n\right) < \delta$.

Let C^n be a Cantor set in $[0,1]$ of measure $\frac{1}{2^n}$. Put $C^n_{(a,b)} = (b-a)C^n+a$. Take $n_0 \in N$, such that $\frac{1}{2^n} < \frac{\delta}{2}$. We define the sequence $\{A_n\}_{n\in N}$ by induction

$$A_1 = C^{n_0}.$$

Suppose that all $A_1, A_2, ..., A_{n-1}$ are already defined, we put

$$A_n = A_{n-1} \cup \bigcup_{\substack{(a,b) \text{ is component} \\ \text{of } A_{n-1}}} C^{n_0+n-1}_{(a,b)}$$

We have $m\left(\limsup_n A_n\right) < \delta$, and the set $\limsup_n A_n$ is of first category.

References

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