UPPER AND LOWER LIMITS OF SEQUENCES OF MEASURABLE SETS AND OF SETS WITH THE BAIRES PROPERTY

WŁADYSŁAW WILCZYŃSKI, WOJCIECH WOJDOWSKI

Abstract. The note presents the study of the behaviour of upper and lower limits of sequence \( \{A_n\}_{n \in \mathbb{N}} \) of \( S \)-measurable subsets of the unit interval \([0, 1]\) when all sets \( A_n \) are subject to small translations. \( S \) is here the class of all Lebesgue measurable sets or the class of sets with the Baire property. The last two theorems describe a new (density point) approach to the convergence of Rademacher's type sequences of sets and functions and generalise the result of K. P. Rath from [2].

Among interesting properties of the Lebesgue measure one can find the following one (compare [1], p. 901): for each measurable set \( A \subset [0, 1] \) we have \( \lim_{x \to 0} \mu(A \triangle (A + x)) = 0 \), where \( A + x = \{a + x : a \in A\} \). In this note we shall study the behaviour of upper and lower limits of sequence \( \{A_n\}_{n \in \mathbb{N}} \) of measurable subsets of the unit interval, when all sets \( A_n \) are subject to small translations (it may happen that \( A_n + \epsilon_n \) is no longer included in \([0, 1]\)).

Theorem 1. For each sequence \( \{A_n\}_{n \in \mathbb{N}} \) of measurable subsets of \([0, 1]\) such that \( \mu(\limsup_n A_n) = 0 \) there exists a sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) of numbers different from zero such that \( \lim_{n \to \infty} \epsilon_n = 0 \) and \( \mu(\limsup_n (A_n + \epsilon_n)) = 0 \).

Proof. By virtue of the quoted property of Lebesgue measure for each \( n \in \mathbb{N} \) there exists \( \epsilon_n \in (0, \frac{1}{n}) \) such that \( \mu(A_n \triangle (A_n + \epsilon_n)) < \frac{1}{2^n} \).

Observe that for each \( n \in \mathbb{N} \) we have the following inclusions and equations:

\[
\bigcup_{m=n}^{\infty} (A_m + \epsilon_m) \subset \bigcup_{m=n}^{\infty} (A_m \cup (A_m + \epsilon_m)) = \bigcup_{m=n}^{\infty} (A_m \cup ((A_m + \epsilon_m) - A_m)) =
\]

\[= \bigcup_{m=n}^{\infty} (A_m + \epsilon_m) \subset \bigcup_{m=n}^{\infty} (A_m \cup (A_m + \epsilon_m)) = \bigcup_{m=n}^{\infty} (A_m \cup ((A_m + \epsilon_m) - A_m)) = \]

Key words and phrases: upper limit, lower limit, convergence of sequences of measurable functions.
AMS subject classifications: Primary 28A20, 54A20; Secondary 29A99, 54A99.
Example 2.  
\[ \epsilon \text{arbitrariness of } h \text{ and for } n \in \mathbb{N} \text{ it follows that } \lim_{n \to \infty} \mu(\bigcup_{m=n}^{\infty} (A_m + \epsilon_m) \Delta A_m). \]
Hence \( \mu(\bigcup_{m=n}^{\infty} (A_m + \epsilon_m)) \leq \mu(\bigcup_{m=n}^{\infty} A_m) + \frac{1}{2^n h}. \) From the assumption it follows that \( \lim_{n \to \infty} \mu(\bigcup_{m=n}^{\infty} A_m) = 0, \) so \( \lim_{n \to \infty} \mu(\bigcup_{m=n}^{\infty} (A_m + \epsilon_m)) = 0 \) and finally \( \mu(\limsup_n (A_n + \epsilon_n)) = 0. \)

Remark 1. Observe that if \( \sum_{n=1}^{\infty} \mu(A_n) < \infty, \) then for arbitrary sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) of real numbers \( \mu(\limsup_n (A_n + \epsilon_n)) = 0. \) The situation is different if \( \sum_{n=1}^{\infty} \mu(A_n) = +\infty \) and \( \mu(\limsup_n A_n) = 0. \) It can happen either that \( \mu(\limsup_n (A_n + \epsilon_n)) = 0 \) for each sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) convergent to zero, or only for some sequences.

We shall illustrate it with the following two examples.

Example 1. Put \( A_n = [0, \frac{1}{n}] \) for \( n \in \mathbb{N}, \) and let \( \{\epsilon_n\}_{n \in \mathbb{N}} \) be an arbitrary sequence convergent to zero. Let \( h \in (0, 1). \) We have \( -h < \epsilon_n < \frac{1}{n} + \epsilon_n < h \) for \( n \) large enough, which yields \( \limsup_n (A_n + \epsilon_n) \subset (-h, h). \) From arbitrariness of \( h \) it follows that \( \mu(\limsup_n (A_n + \epsilon_n)) = 0. \)

Example 2. Now, we shall construct two sequences \( \{A_n\}_{n \in \mathbb{N}} \) and \( \{\epsilon_n\}_{n \in \mathbb{N}} \) such that \( A_n \subset [0, 1], \) \( \lim_{n \to \infty} \epsilon_n = 0, \mu(\limsup_n (A_n)) = 0 \) and \( \mu(\limsup_n (A_n + \epsilon_n)) = 1. \) The sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) will be here strictly decreasing.

We start with \( A_1 = [0, \frac{1}{2}]. \) Next are \( A_2 = A_3 = A_4 = (0, \frac{1}{8}] \cup (\frac{1}{8}, \frac{1}{2}] \cup \frac{1}{8} \). They are followed by \( (A_5 = A_6 = \ldots = A_{11} = (0, \frac{1}{64}] \cup (\frac{1}{2}, \frac{1}{8}] \cup \ldots \cup (\frac{5}{2}, \frac{5}{8}] \cup \frac{1}{64}, \ldots, A_{11} = \frac{1}{64}, \ldots, \epsilon_{11} = \frac{1}{64} \) and so on. Appropriate \( \epsilon_n \)'s are \( \epsilon_1 = \frac{1}{2}, \epsilon_2 = \frac{2}{8}, \epsilon_3 = \frac{2}{8}, \epsilon_4 = \frac{3}{8}, \epsilon_5 = \frac{7}{64}, \epsilon_6 = \frac{6}{64}, \ldots, \epsilon_{11} = \frac{1}{64} \) and so on.

In general we define by induction a sequence \( \{a_k\}_{k \in \mathbb{N} \cup \{0\}} \) in the following way: \( a_0 = 0, a_{k+1} = a_k + k \) for \( k \in \mathbb{N} \cup \{0\}. \) Let
\[
E_k = \bigcup_{i=0}^{2^{2k-1}-1} \left( \frac{i}{2^{2k}}, \frac{i+1}{2^{2k}} \right)
\]
and for \( n \) of the form
\[
n = \sum_{i=1}^{k} (2^i - 1) + j, \text{ where } j = 1, \ldots, 2^k - 1
\]
put
\[
A_n = E_k \quad \text{and} \quad \epsilon_n = \frac{2^k - j}{2^{2k+1}}.
\]
The sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) is strictly decreasing. Observe that
\[
\mu(E_k) = 2^{2k} \cdot \frac{1}{2^{2k+1}} = 2^{-1} = 2^{-(a_{k+1} - a_k)} = 2^{-k},
\]
which implies that $\lim_{n \to \infty} \mu(\bigcup_{m=n}^{\infty} A_m) = 0$, and consequently we have $\mu(\limsup_n (A_n)) = 0$.

On the other hand we have

$$\mu\left(\bigcup_{n \in M} (A_n + \epsilon_n)\right) = 1 - 2^{-k},$$

where

$$M = \left\{ n \in N : \sum_{i=1}^{k} (2^{i-1} - 1) + 1 \leq n \leq \sum_{i=1}^{k+1} (2^{i-1} - 1) \right\}$$

and therefore $\mu(\bigcup_{m=n}^{\infty} (A_m + \epsilon_m)) = 1$ for every $n$. We thus get

$$\mu(\limsup_n (A_n + \epsilon_n)) = 1.$$

The lower limit of sequence of measurable sets behaves similarly with respect to “small” translations.

**Theorem 2.** For each sequence $\{A_n\}_{n \in N}$ of measurable subsets of $[0, 1]$ such that $\mu(\liminf_n A_n) = 0$ there exists a sequence $\{\epsilon_n\}_{n \in N}$ of numbers different from zero such that $\lim_{n \to \infty} \epsilon_n = 0$ and $\mu(\liminf_n (A_n + \epsilon_n)) = 0$.

**Proof.** First observe that for every two families of sets $\{A_t\}_{t \in T}$ and $\{B_t\}_{t \in T}$ we have

$$\left(\bigcap_{t \in T} A_t\right) \triangle \left(\bigcap_{t \in T} B_t\right) \subset \bigcup_{t \in T} (A_t \triangle B_t).$$

Similarly as in proof of Theorem 1 for each $n \in N$ there exists $\epsilon_n \in (0, 1)$ such that $\mu(A_n \triangle (A_n + \epsilon_n)) < \frac{1}{2^k}$. From the assumption concerning the lower limit it follows that there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N} \cup \{0\}}$ of natural numbers such that

$$\mu\left(\bigcap_{m=n_{k-1}+1}^{n_k} A_m\right) < \frac{1}{k} \text{ for } k \in \mathbb{N}.$$

Since by virtue of the above observation

$$\bigcap_{m=n_{k-1}+1}^{n_k} (A_m + \epsilon_m) \subset \bigcap_{m=n_{k-1}+1}^{n_k} A_m \cup \bigcup_{m=n_{k-1}+1}^{n_k} (A_m \triangle (A_m + \epsilon_m)),$$

then we have

$$\mu\left(\bigcap_{m=n_{k-1}+1}^{n_k} (A_m + \epsilon_m)\right) < \frac{1}{k} + \frac{1}{2^{n_{k-1}}}.$$
This clearly implies $\mu(\bigcap_{m=n}^{\infty}(A_m + \epsilon_m)) = 0$ for each $n \in \mathbb{N}$ and consequently $\mu(\liminf_n (A_n + \epsilon_n)) = 0$. \hfill \square

**Remark 2.** Observe that if $\liminf_n \mu(A_n) = 0$, then for arbitrary sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of real numbers $\mu(\liminf_n (A_n + \epsilon_n)) = 0$. The situation again is different if $\liminf_n \mu(A_n) > 0$ and $\mu(\liminf_n A_n) = 0$. Here also can happen either that $\mu(\liminf_n (A_n + \epsilon_n)) = 0$ for each sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ convergent to zero, or only for some sequences.

**Example 3.** We shall construct here the sequence $\{A_n\}_{n \in \mathbb{N}}$ of measurable subsets of $[0,1]$ such that $\mu(A_n) = \frac{1}{2}$ for each $n \in \mathbb{N}$ and $\mu(\liminf_n (A_n + \epsilon_n)) = 0$ for each sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\liminf_n |\epsilon_n| = 0$.

Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of sets

$$A_n = \bigcup_{i=0}^{2^{n-1} - 1} \left( \frac{2i}{2^n}, \frac{2i + 1}{2^n} \right)$$

and $\{\epsilon_n\}_{n \in \mathbb{N}}$ a sequence of positive numbers with $\liminf_n |\epsilon_n| = 0$. We denote $B_n = A_n + \epsilon_n$.

We shall show that for arbitrary $n \in \mathbb{N}$, $\mu\left( \bigcap_{m=n}^{\infty} (A_m + \epsilon_m) \right) = 0$. This will imply that

$$\mu\left( \liminf_n (A_m + \epsilon_m) \right) = \mu\left( \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (A_m + \epsilon_m) \right) \leq \sum_{n=1}^{\infty} \mu\left( \bigcap_{m=n}^{\infty} (A_m + \epsilon_m) \right) = \sum_{n=1}^{\infty} \mu(0) = 0.$$

It is enough to define for arbitrary $n \in \mathbb{N}$, a subsequence $\{m_k\}_{k \in \mathbb{N}}$, such that $m_1 \geq n$ and $\mu\left( \bigcap_{k=1}^{\infty} (A_{m_k} + \epsilon_{m_k}) \right) = 0$.

Fix $n \in \mathbb{N}$. We shall define $\{m_k\}_{k \in \mathbb{N}}$ by induction. Put $m_1 = n$. Assume $m_1, m_2, ..., m_{k-1}$ are already defined, such that $m_1 < m_2 < ... < m_{k-1}$ and for arbitrary $j \in \{1,2,...,k-2\}$ we have inequality

$$\mu\left( \bigcap_{i=1}^{j+1} (A_{m_i} + \epsilon_{m_i}) \right) \leq \frac{2}{3} \mu\left( \bigcap_{i=1}^{j} (A_{m_i} + \epsilon_{m_i}) \right).$$

We shall choose $m_k$ to have

$$\mu\left( \bigcap_{i=1}^{k} (A_{m_i} + \epsilon_{m_i}) \right) \leq \frac{2}{3} \mu\left( \bigcap_{i=1}^{k-1} (A_{m_i} + \epsilon_{m_i}) \right).$$
Observe, that the set \( \bigcap_{i=1}^{k-1} (A_{m_i} + \epsilon_{m_i}) \) is a sum of finite number of closed intervals and finite number of points. Let \( d_{k-1} \) be the length of the shortest interval. We choose \( m_k \) big enough to have
\[
\frac{1}{2^{m_k}} < \frac{1}{3} d_{k-1}.
\]

Then for arbitrary closed, non-degenerate component \( I \) of the set
\[
\bigcap_{i=1}^{k-1} (A_{m_i} + \epsilon_{m_i}),
\]
we have
\[
\mu(I \cap A_{m_k}) \leq \frac{2}{3} \mu(I),
\]
which ensures (*).

**Example 4.** We shall construct here two sequences \( \{B_n\}_{n \in N} \) and \( \{\epsilon_n\}_{n \in N} \) such that \( B_n \subset [0,1], \mu(\liminf_n B_n) = 0, \lim_{n \to \infty} \epsilon_n = 0 \) and \( \mu(\liminf_n (B_n + \epsilon_n)) = 1 \). The sequence \( \{\epsilon_n\}_{n \in N} \) will be here strictly decreasing. We shall write \( \hat{A} = \{1 - x : x \in A\} \).

Consider the sets \( C_n = A_n + \epsilon_n, n \in N \) where \( A_n, \epsilon_n, n \in N \) are as defined in Example 2. Put \( B_n = [0,1] - \hat{C}_n \). Let us notice that \( B_n + \epsilon_n = [0,1] - \hat{A}_n \).

From what was established in Example 2 we have \( \mu(\liminf_n B_n) = 0 \) as \( \mu(\limsup_n (A_n + \epsilon_n)) = 1 \) and \( \mu(\liminf_n (B_n + \epsilon_n)) = 1 \) as \( \mu(\limsup_n A_n) = 0 \).

Now we shall discuss similar properties of upper and lower limits of sequences with the Baire property.

Let \( S \) be a \( \sigma \)-algebra of sets with the Baire property and \( I \) the \( \sigma \)-ideal of sets of the first category. If \( A \in S \) we say that \( A \) is \( S \)-measurable.

**Theorem 3.** For each sequence \( \{A_n\}_{n \in N} \), of \( S \)-measurable (Baire measurable) subsets of \([0,1]\) such that \( \liminf_n A_n \in I \) we have also \( \limsup_n (A_n + \epsilon_n) \in I \) for every sequence \( \{\epsilon_n\}_{n \in N} \) of numbers different from zero such that \( \lim_{n \to \infty} \epsilon_n = 0 \).

**Proof.** We may assume that all sets \( A_n, n \in N \) are open here. From the assumption we have \( \bigcap_{n \geq m} A_n \in I \) for every \( m \in N \). Since \( \bigcap_{n \geq m} A_n \) is a \( G_\delta \) set it has to be nowhere dense. We shall show that also \( \bigcap_{n \geq m} (A_n + \epsilon_n) \) is nowhere dense for every \( m \in N \). Fix \( m_0 \in N \) and let \( (a,b) \subset [0,1] \). We shall show that there exists interval \( (c,d) \subset (a,b) \) such that \( (c,d) \cap \bigcap_{n \geq m_0} (A_n + \epsilon_n) = \phi \).

Let \( (a_1, b_1) = (a + \frac{1}{4}(b-a), b - \frac{1}{4}(b-a)) \), and let \( m_1 \geq m_0 \) be a natural number such that \( |\epsilon_n| < \frac{1}{4}(b-a) \) for every \( n \geq m_1 \). So \( (a_1 + \epsilon_n, b_1 + \epsilon_n) \subset (a_1 + \epsilon_n, b_1 + \epsilon_n) \cap (a_2 + \epsilon_n, b_2 + \epsilon_n) \cap \ldots \cap (a_k + \epsilon_n, b_k + \epsilon_n) \subseteq (a, b) \).
(a, b), n \geq m_1. Since \( \bigcap_{n \geq m} A_n \) is nowhere dense, there exists \( n_1 \geq m_1 \) such that \( A_{n_1} \) is not dense in \((a_1, b_1)\). Hence there exists interval \((c_1, d_1) \subset (a_1, b_1)\) such that \((c_1, d_1) \cap A_{n_1} = \emptyset\). Consequently \((c_1 + \epsilon_{n_1}, d_1 + \epsilon_{n_1}) \cap (A_{n_1} + \epsilon_{n_1}) = \emptyset\) and therefore \((c_1 + \epsilon_{n_1}, d_1 + \epsilon_{n_1}) \cap \bigcap_{n \geq m_0} (A_n + \epsilon_n) = \emptyset\).

We define now \( (c, d) = (c_1 + \epsilon_{n_1}, d_1 + \epsilon_{n_1}) \subset (a_1 + \epsilon_{n_1}, b_1 + \epsilon_{n_1}) \subset (a, b)\) and have \((c, d) \cap \bigcap_{n \geq m_0} (A_n + \epsilon_n) = \emptyset\).

Therefore \( \bigcap_{n \geq m_0} (A_n + \epsilon_n) \) is nowhere dense set and so is \( \bigcap_{n \geq m} (A_n + \epsilon_n) \) for every \( m \in \mathbb{N} \), hence the theorem is proved. \( \square \)

**Theorem 4.** For each sequence \( \{A_n\}_{n \in \mathbb{N}} \) of \( S \)-measurable subsets of \([0, 1]\) such that \( \lim \sup_n A_n = I \), we have also \( \lim \sup_n (A_n + \epsilon_n) \in I \) for every sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) of numbers different from zero such that \( \lim_{n \to \infty} \epsilon_n = 0 \).

**Proof.** Again we may assume that all sets \( A_n \), \( n \in \mathbb{N} \) are open here. Observe that \( \bigcap_{n=1}^{\infty} \bigcup_{n \geq m} A_n \) as \( \mathcal{G}_s \) set of first category is easily seen to be nowhere dense. We shall show that \( \lim \sup_n (A_n + \epsilon_n) \in I \) is also nowhere dense, hence of first category. Let \((a, b) \subset [0, 1]. We shall show that there exists an interval \((c, d) \subset (a, b)\) such that \( \lim \sup_n (A_n + \epsilon_n) \cap (c, d) = \emptyset\). There exists \( m_0 \in \mathbb{N} \) such that \( \bigcup_{n \geq m_0} A_n \) is not dense in \((a, b)\). Hence there exists interval \((c_0, d_0) \subset (a, b)\) such that \((c_0, d_0) \cap \bigcup_{n \geq m_0} A_n = \emptyset\). Let now \( m_1 = m_0 \) be a natural number such that \( |\epsilon_n| < \frac{1}{3} (d_0 - c_0) \). Define \((c, d) = (c_0 + \frac{1}{3} (d_0 - c_0), d_0 - \frac{1}{3} (d_0 - c_0)) \subset (a, b)\). We have \((c, d) \cap (A_n + \epsilon_n) = \emptyset\), for \( n \geq m_1 \), which implies \((c, d) \cap \bigcup_{n \geq m_0} (A_n + \epsilon_n) = \emptyset\). From arbitrariness of \((a, b)\), \( \lim \sup_n (A_n + \epsilon_n) \) is nowhere dense, hence the theorem is proved. \( \square \)

K. P. Rath in [2] has proved the following theorem:

**Theorem 5.** Let \( \{\epsilon_n\}_{n \in \mathbb{N}} \) be a sequence of positive numbers converging to 0 and \( \{f_n\}_{n \in \mathbb{N}} \) a sequence of real valued measurable functions on the unit interval \([0, 1]\) given by

\[
f_n(t) = (-1)^k \text{ if } k \epsilon_n \leq t < (k + 1) \epsilon_n, \quad k = 0, 1, 2, \ldots
\]

Let \( \{f_{n_k}\}_{k \in \mathbb{N}} \) be any subsequence of \( \{f_n\}_{n \in \mathbb{N}} \). Then the set of points at which \( \{f_{n_k}\}_{k \in \mathbb{N}} \) converges has Lebesgue measure zero.

We shall prove the theorem describing much wider class of sequences of measurable functions divergent in measure. We shall write \( A^c = [0, 1] - A \).

**Theorem 6.** Suppose that a sequence \( \{A_n\}_{n \in \mathbb{N}} \) of measurable subsets of \([0, 1]\) fulfills the following condition: there exists a positive number \( \epsilon > 0 \) such that for each open interval \((a, b) \subset [0, 1]\) and for each natural number \( m \) there exists a natural number \( n > m \) for which

\[
\frac{m(A_n \cap (a, b))}{m((a, b))} > \epsilon.
\]
Then \( m(\limsup_n A_n) = 1 \).

Proof. Suppose that \( m(\liminf_n A_n^c \cap [0, 1]) > 0 \). Since \( \liminf_n A_n^c \cap [0, 1] = \bigcup_{m=1}^\infty \bigcap_{n \geq m} A_n^c \), then there exists \( m \in \mathbb{N} \) such that \( m\left(\bigcap_{n \geq m} A_n^c\right) > 0 \).

Let \( x_0 \in (0, 1) \) be a point of density of \( \bigcap_{n \geq m} A_n^c \). There exists an interval \((a, b) \subset [0, 1]\) such that \( x_0 \in (a, b) \) and

\[
m \left(\bigcap_{n \geq m} A_n^c \cap (a, b)\right) \frac{b-a}{b-a} > 1 - \epsilon_1.
\]

Hence

\[
m \left(\bigcap_{n \geq m} A_n^c \cap (a, b)\right) > 1 - \frac{\epsilon}{2}
\]

for each \( n > m \), so

\[
m \left(\bigcap_{n \geq m} A_n^c \cap (a, b)\right) \frac{b-a}{b-a} < \frac{\epsilon}{2}
\]

for each \( n > m \) - a contradiction. \( \square \)

As a simple consequence of the above we obtain:

**Theorem 7.** Suppose that a sequence \( \{A_n\}_{n \in \mathbb{N}} \) of measurable subsets of \([0, 1]\) fulfills the following condition: there exist two positive numbers \( \epsilon_1, \epsilon_2 \) such that for each open interval \((a, b) \subset [0, 1]\) and for each natural number \( m \) there exist \( n, p > m \) such that

\[
m \left(\bigcap_{n \geq m} A_n^c \cap (a, b)\right) \frac{b-a}{b-a} > \epsilon_1 \quad \text{and} \quad m \left(\bigcap_{n \geq m} A_n^c \cap (a, b)\right) \frac{b-a}{b-a} < \epsilon_2.
\]

Then \( m(\limsup_n A_n) = 1 \) and \( m(\liminf_n A_n) = 0 \).

**Corollary 1.** Under the assumptions of Theorem 6 the sequence \( \{\chi_{A_n}\}_{n \in \mathbb{N}} \) of measurable functions is divergent in measure.

**Remark 3.** Observe that if \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence from theorem of Rath, then for each subsequence \( \{f_{n_k}\}_{k \in \mathbb{N}} \) the condition from our Theorem 6 is fulfilled for \( A_k = \{x \in [0, 1] : f_{n_k}(x) = 1\} \).

**Remark 4.** Observe that the order of quantifiers in condition from Theorem 6 is important. Really, let \( \delta < 1 \) be arbitrary small real number. We shall define a sequence \( \{A_n\}_{n \in \mathbb{N}} \) of measurable subsets of \([0, 1]\) fulfilling the condition: for each open interval \((a, b) \subset [0, 1]\) there exists a positive number \( \epsilon > 0 \) such that for each natural number \( m \) there exists a natural number \( n > m \) for which

\[
m \left(\bigcap_{n > m} A_n \cap (a, b)\right) \frac{b-a}{b-a} > \epsilon.
\]

but \( m(\limsup_n A_n) < \delta \).
Let \( C^n \) be a Cantor set in \([0,1]\) of measure \( \frac{1}{2^n} \). Put \( C_{(a,b)}^n = (b-a)C^n + a \).

Take \( n_0 \in \mathbb{N} \), such that \( \frac{1}{2^n} < \delta \). We define the sequence \( \{A_n\}_{n \in \mathbb{N}} \) by induction

\[
A_1 = C^{n_0}.
\]

Suppose that all \( A_1, A_2, ..., A_{n-1} \) are already defined, we put

\[
A_n = A_{n-1} \cup \bigcup_{(a,b) \text{ is component of } A_{n-1}} C_{(a,b)}^{n_0 + n - 1}.
\]

We have \( m(\limsup_n A_n) < \delta \), and the set \( \limsup_n A_n \) is of first category.

References