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ON A THREE-DIMENSIONAL COMPETITIVE SYSTEM

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Abstract. Explicite conditions guaranteeing that all (except three) solutions of a three-dimensional competitive system have periodic trajectories as its ω-limit set are given.

1. Introduction

We deal with a simple model of evolution of three species competiting mutually for the same source in an environment

$$(1)\qquad \begin{cases} \dot{x} = x(1 - a_1x - b_1y - c_1z), \\ \dot{y} = y(1 - a_2x - b_2y - c_2z), \\ \dot{z} = z(1 - a_3x - b_3y - c_3z). \end{cases}$$

This system is considered only in the set $\{(x, y, z) : x, y, z \geq 0\}$ which is invariant. We do not consider th location of species in space (in most cases, this gives a nonlinear parabolic system) but a capacity of the environment is emphasized. For example, if species y and z are extinct: $y = z = 0$, all solutions of the logistic equation obtained then from (1): $\dot{x} = x(1 - a_1x)$ tend to the stationary solution $x = a_1^{-1}$ – this number of individuals of x's exploits the environment completely.

All coefficients a_i, b_i, c_i, $i = 1, 2, 3$ are supposed to be positive. By changing variables one can set some coefficients equal 1. Especially, $a_1 = b_2 = c_3 = 1$ if x, y, z are relative number of individuals with respect to the capacity of the environment. One can remove one more of parameters but the symmetry of the system will be broken. We believe that the system with all nine parameters being arbitrary shows the best symmetry properties and its studies contains the same complications as systems with some coefficients equal 1, thus we do not change variables. It is worth noticing that from

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the ecological point of view the system with $a_1 = b_2 = c_3 = 1$ is the most natural. Moreover, the most interesting will be solutions with all values x, y, z that do not exceed the capacity of the environment, although the model includes the situation when initially we have greater populations, for example, if some pieces of populations are artificially imported to the environment. Similar models are studied for cooperating (symbiotic) species or three species with one (or two) of them being predators and remaining ones being preys. In both cases some coefficients are negative. For example, if z stands for the number of predators and two remaining species are preys, then a_3, b_3 are negative.

This model (and more general) has been studied by several authors since 1975, when R. M. May and W. J. Leonard [7] examined the case: $a_1 = b_2 = c_3 = 1$, $b_1 = c_2 = a_3 =: \alpha$, $c_1 = a_2 = b_3 =: \beta$, especially with $\alpha + \beta = 2$. They proved the existence of a periodic trajectory for this system.

From the mathematical point of view, the equation (1) is a special case of a dynamical system given by an autonomous differential equation

$$\dot{\xi} = f(\xi), \tag{2}$$

where $f : \mathbb{R}^k \supset D \to \mathbb{R}^k$ is a C^1-function with the property

$$\frac{\partial f_i}{\partial \xi_j} \leq 0 \quad \text{for} \quad i \neq j. \tag{3}$$

Such dynamical systems are extensively studied from a series of papers by M. Hirsch [1]–[6] where they are called competitive systems. Parallely M. Hirsch considered cooperative systems where all inequalities in the property (3) are reversed. In 90-es, H. L. Smith introduced the abstract notions of a monotone dynamical system (see [9]) and an abstract competitive system (see [10]), where the evolution is given in an abstract form (it is not governed by ODEs). The most interesting result of this theory seems the theorem that, roughly speaking, any k-dimensional monotone system behaves as (nonmonotone) $k-1$-dimensional one. For example, trajectories of 2-dimensional monotone systems tend towards fixed points (or to infinity) and 3-dimensional monotone systems satisfy the Poincaré–Bendixson Theorem. We use notions from [10], but one can find another condition than (3) for the competitive systems in the biological literature

$$\frac{\partial}{\partial \xi_j}\left(\frac{1}{\xi_i} f_i(\xi)\right) \leq 0 \quad \text{for} \quad i \neq j. \tag{4}$$

On the set D of positive triples (x, y, z) both conditions are equivalent.

We shall find conditions on coefficients a_i, b_i, c_i guaranteeing that all positive trajectories except exactly two special ones tend to a limit cycle as $t \to +\infty$. The situation when all solutions tend to periodic trajectories is impossible as we shall see below. The existence of periodic trajectories for general competitive systems 2 has been obtained earlier by H. L. Smith [8] and Z. Hsiu-rong and H. L. Smith [12] but they have only one fixed point in a closed invariant set. We have many fixed points. R. M. May and W. J. Leonard proved the existence of a periodic trajectory for their particular competitive system. Our system is more general.

2. Fixed points and their stability

We study the system (1) in the open set

$$D := \{(x, y, z) : \ x, y, z > 0\},$$

which is obviously invariant and the same is true for its closure $\overline{D}$. Fixed points of the system can be easily found – four of them always lie in $\overline{D}$:

$$P_0 = (0,0,0), \quad P_x = (a_1^{-1}, 0, 0), \quad P_y = (0, b_2^{-1}, 0), \quad P_z = (0, 0, c_3^{-1}),$$

next four are

$$P_{xy} = \left(\frac{b_2 - b_1}{a_1 b_2 - a_2 b_1}, \frac{a_1 - a_2}{a_1 b_2 - a_2 b_1}, 0\right), P_{xz} = \left(\frac{c_3 - c_1}{a_1 c_3 - a_3 c_1}, 0, \frac{a_1 - a_3}{a_1 c_3 - a_3 c_1}\right),$$

$$P_{yz} = \left(0, \frac{c_3 - c_2}{b_2 c_3 - b_3 c_2}, \frac{b_2 - b_3}{b_2 c_3 - b_3 c_2}\right)$$

and $P_1 = (\alpha, \beta, \gamma)$ being the solution of the equation

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{where} \quad M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

We assume that $W := \det M \neq 0$ what means that none of two of planes

$$H_i : \quad a_i x + b_i y + c_i z = 1, \quad i = 1, 2, 3$$

are parallel.

Denote by $f(x, y, z)$ the right hand side of (1). Then $f'(P_0) = I$ and its unique eigenvalue 1 is positive, thus P_0 is a source.

Similarly

$$f'(P_x) = \begin{bmatrix} -1 & -b_1/a_1 & -c_1/a_1 \\ 0 & 1 - a_2/a_1 & 0 \\ 0 & 0 & 1 - a_3/a_1 \end{bmatrix}$$

and, if $\max(a_2, a_3) < a_1$, then one eigenvalue -1 is negative and two remaining ones are positive. The corresponding eigenspaces are: $\{(x, 0, 0) : \ x \in \mathbb{R}\}$ and $\{(0, y, z) : \ y, z \in \mathbb{R}\}$. The stable manifold of P_x is 1-dimensional

$$\{(x, 0, 0) : \ x > 0\}$$

and by the Stable and Unstable Manifold Theorem all trajectories except starting in the stable manifold cannot tend to P_x as $t \to +\infty$.

Similar arguments work for P_y and P_z under conditions

$$\max(b_1, b_3) < b_2, \quad \max(c_1, c_2) < c_3.$$

These assumptions can be weakened if we apply another argumentation. Notice that the first (resp. second, third) coordinate of a point $P(t)$ – $x(t)$ (resp. $y(t)$, $z(t)$) – are incresing functions of t when $P(t)$ lies under the plane H_1 (resp. H_2, H_3) and decreasing when it is over this plane. If $\min(a_2, a_3) < a_1$, say $a_2 < a_1$, then P_x is below H_2 and none trajectory in D can tend to P_x since its second coordinate $y(t)$ is increasing near $P_x = (a_1^{-1}, 0, 0)$. We have proved even more:

Lemma 1. *If*

$$\min(a_2, a_3) < a_1, \quad \min(b_1, b_3) < b_2, \quad \min(c_1, c_2) < c_3, \tag{5}$$

then P_x, P_y and P_z do not belong to the ω-limit set $\omega(P)$ of any point $P \in D$.

3. Using the theory of monotone systems

Let us divide the set $\overline{D}$ into three pieces

$$D_+ := \{(x, y, z) \in \overline{D} : \min_i (a_i x + b_i y + c_i z) > 1\},$$

$$A := \{(x, y, z) \in \overline{D} : \min_i (a_i x + b_i y + c_i z) \leq 1 \leq \max_i (a_i x + b_i y + c_i z)\},$$

$$D_- := \{(x, y, z) \in \overline{D} : \max_i (a_i x + b_i y + c_i z) < 1\}.$$

D_+ (resp. D_-) is the set of points sitting under (resp. over) all three planes H_i, $i = 1, 2, 3$, $A = \overline{D} \setminus (D_+ \cup D_-)$. The following result is an immediate consequence (reformulation) of [9], p. 34, Prop. 2.1.

Lemma 2. *The set A is positively invariant and all trajectories in D eventually come into A. Moreover, A contains any compact invariant set that contains no fixed points.*

Thus, $\omega(P) \subset A$ for any $P \in D$. Notice that P_x, P_y, $P_z \in A$ and similarly P_{xy}, P_{xz}, P_{yz}, if they belong to $\overline{D}$.

The last fixed point $P_1 = (\alpha, \beta, \gamma)$ can lie in D (and then in A) or outside this cone. The following theorem excludes the existence of a periodic trajectory in D, if $P_1 \in D$.

Lemma 3. ([9], p. 44, Prop. 4.3) *Let Γ be a nontrivial periodic trajectory of a competitive system in $D \subset \mathbb{R}^3$ and*

$$\Gamma \subset [p, q] := \{\xi : p_i \leq \xi_i \leq q_i, \ i = 1, 2, 3\} \subset D.$$

Then the set K of all points x which are not related to any point $y \in \Gamma$ (relation $x \leq y$ means $x_i \leq y_i$ for any i) has two components, one of them is bounded and contains a fixed point.

Hence, if we want to have a nontrivial periodic trajectory, then P_1 must belong to D and we have two options:
(i) $W > 0$ and three other determinants

$$W_x := \det \begin{bmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{bmatrix},$$

$$W_y := \det \begin{bmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{bmatrix},$$

$$W_z := \det \begin{bmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{bmatrix}$$

are positive or
(ii) $W < 0$ and the above three determinants are negative.
The last theorem that we shall use describes our situation. Our version sligtly differs from the one in the cited book. A trajectory starting in D can tend to its boundary, hence one should exclude this possibility.

Lemma 4. ([9], p. 43, Th. 4.2) *Let D be an invariant open set of a competitive system and for any $P \in D$, $\omega(P) \subset D$. Suppose that D contains a unique fixed point P_1 and that the stable manifold of P_1, $W^s(P_1)$ is 1-dimensional and its tangent space $T_{P_1}(W^s(P_1))$ is spanned by a vector with all positive coordinates. Then, for any $P \in D \setminus W^s(P_1)$, the ω-limit set $\omega(P)$ is a nontrivial periodic trajectory.*

4. Linearization near P_1

One can compute the Jacobi matrix of the r.h.s f in the point P_1

$$f'(P_1) = - \begin{bmatrix} a_1\alpha & b_1\alpha & c_1\alpha \\ a_2\beta & b_2\beta & c_2\beta \\ a_3\gamma & b_3\gamma & c_3\gamma \end{bmatrix},$$

where $\alpha = W_x/W$, $\beta = W_y/W$, $\gamma = W_z/W$ in notations of the preceding section. The characteristic polynomial of this matrix

$$Q(\lambda) = \lambda^3 + (a_1\alpha + b_2\beta + c_3\gamma)\lambda^2 +$$
$$+ \left(\alpha\beta \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \alpha\gamma \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + \beta\gamma \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \right) \lambda + \alpha\beta\gamma W,$$

is so complicated that we cannot find its roots in a readable form by using some software for symbolic computations (MAPLE and MATHEMATICA). It is caused by so large number of parameters that the Cardano formulas give too complicated answer. In order to get one eigenvalue, we use a different way.

Let us write down the differential equation of trajectories of (1)

$$\left\{\begin{array}{l} \frac{dy}{dx} = \frac{y(1-a_2x-b_2y-c_2z)}{x(1-a_1x-b_1y-c_1z)}, \\ \frac{dz}{dx} = \frac{z(1-a_3x-b_3y-c_3z)}{x(1-a_1x-b_1y-c_1z)}. \end{array}\right. \tag{6}$$

One of its solution can be guessed:

$$y = \beta\left(\frac{x}{\alpha}\right), \quad z = \gamma\left(\frac{x}{\alpha}\right). \tag{7}$$

The authors found it first experimenting with numerical solutions. A plot of an initial value problem with data taken accidentally proved to be a segment starting from the origin and ending at P_1. Simple calculations show that (7) solves equation (6). It suggests that one of eigenvalues λ_1 of the matrix $f'(P_1)$ satisfies the following equation

$$f'(P_1)\cdot[\alpha,\beta,\gamma]^T = \lambda_1[\alpha,\beta,\gamma]^T.$$

It follows that $\lambda_1 = -1$ which is real negative. Thus, the characteristic polynomial has the factorization

$$Q(\lambda) = (\lambda+1)(\lambda^2+p\lambda+q),$$

where

$$p := a_1\alpha + b_2\beta + c_3\gamma - 1,$$
$$q := \alpha\beta\gamma W.$$

Notice that $\operatorname{sgn} q = \operatorname{sgn} W$ under our assumption $P_1 \in D$. Since we can compare this factorization with the previous formula for Q, the following equality is true:

$$a_1\alpha+b_2\beta+c_3\gamma-1+\alpha\beta\gamma W = \alpha\beta\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \alpha\gamma\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + \beta\gamma\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

5. Main result

Now, we are prepared to the proof of our main theorem.

Theorem 1. *Assume (5). Let all four determinants W, W_x, W_y, W_z be positive and $p < 0$. Then, for any point $P \in D$ that does not belong to the half-line (7) with $x > 0$, the ω-limit set $\omega(P)$ is a periodic trajectory. For P from this half-line, $\omega(P) = P_1$.*

Proof. Due to Lemma 1, none of fixed points P, P_y, P_z belongs to ω-limit sets of points from D. One can compute the Jacobi matrix of the r.h.s. of (1) at other fixed points. For example,

$$f'(P_{xy}) = \frac{1}{a_1b_2 - a_2b_1} \begin{bmatrix} a_1(b_1 - b_2) & b_1(b_1 - b_2) & c_1(b_1 - b_2) \\ a_2(a_2 - a_1) & b_2(a_2 - a_1) & c_2(a_2 - a_1) \\ 0 & 0 & W_z \end{bmatrix}.$$

Since the plane xy is invariant and W_z – one of eigenvalues is positive, the fixed point P_{xy} cannot be a limit point of any trajectory from D. The same argument holds for P_{xz} and P_{yz}.

Take any $P \in D$. If $\omega(P) \cap \partial D \neq \emptyset$, then $\omega(P) \subset \partial D$ by the invariance of the last set. But this set is a sum of three planar sets, where the induced system is also competitive and, therefore, $\omega(P)$ is a fixed point. This is impossible by our previous considerations.

We have $q = \alpha\beta\gamma W > 0$, thus two remaining eigenvalues of the matrix $f'(P_1)$ are

$$\lambda_{2,3} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

have positive real parts. It follows that the unstable manifold $W^u(P_1)$ is 2-dimensional and the assertion is obtained by Lemma 4. $\square$

One can look at our assumptions for the case of May–Leonard system

$$\begin{aligned} \dot{x} &= x(1 - x - \alpha y - \beta z), \\ \dot{y} &= y(1 - \beta x - y - \alpha z), \\ \dot{z} &= z(1 - \alpha x - \beta y - z). \end{aligned} \tag{8}$$

First, we need $\min(\alpha, \beta) < 1$ corresponding assumption 5. Then, we compute

$$W = \alpha^3 + \beta^3 - 3\alpha\beta + 1, \quad W_x = W_y = W_z = \alpha^2 + \beta^2 - \alpha\beta - \alpha - \beta + 1$$

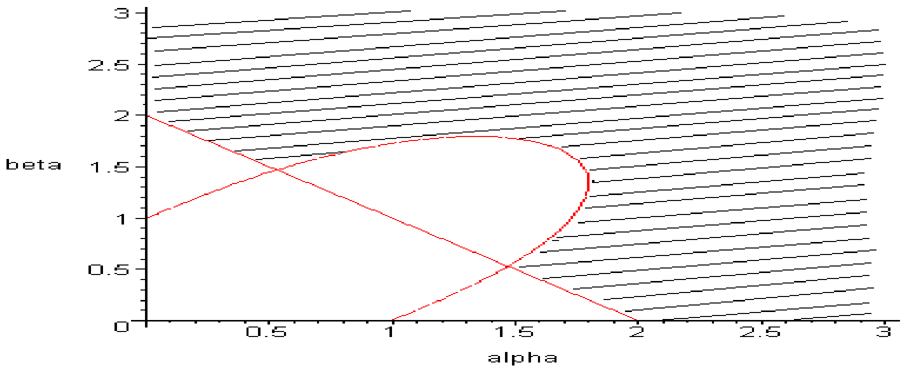
and

$$p = \frac{3}{\alpha + \beta + 1} - 1.$$

Thus the assertion of the Theorem holds if

$$\begin{cases} \alpha^3 + \beta^3 - 3\alpha\beta + 1 > 0, \\ \alpha + \beta > 2. \end{cases} \tag{9}$$

The inequality $W_x > 0$ holds for any $\alpha, \beta > 0$. The set of all $(\alpha, \beta) \in \mathbb{R}_+^2$ is unbounded – see the picture below.



This implies that the set of all parameters $(a_i, b_i, c_i) \in \mathbb{R}^9_+$ is unbounded, as well.

Notice that for system 8 with $\alpha + \beta = 2$, one has that $N = x + y + z$ satisfies $\dot{N} = N(1 - N)$. All positive solutions of the last equation tend to $N = 1$. This means that the attractor of the system is contained in the plane $x + y + z = 1$.

We consider the opposite case $W < 0$. Then, if at least one of the determinants W_x, W_y, W_z is positive, we have $P_1 \notin \overline{D}$ and each trajectory tends to a fixed point. If one of them, say W_z, is negative, then one can study the linearization near P_{xy}. Since the plane $\{(x, y, 0) : \; x, y \in \mathbb{R}\}$ is invariant, the behaviour of the system near P_{xy} in D is determined by the real eigenvalue W_z corresponding to the eigenspace $\{(0, 0, z) : \; z \in \mathbb{R}\}$. Since this number is negative, then at least one trajectory in D tends toward P_{xy}. Therefore, if $W < 0$, then they are trajectories in D which do not start from the curve (7) and tend to one of fixed points: P_{xy}, P_{yz}, P_{xz}.

We do not study all degenerate cases when there are infinite number of fixed points or at least two of planes H_1, H_2, H_3 are parallel.

6. Final remarks

It is known that the periodic trajectory $\omega(P)$ can depend on the point $P \in D$. There are at least two possible cases:

- there exists exactly one periodic trajectory Γ for (1) being the global attractor of the system in D;
- there exists an infinite number of periodic trajectories forming an annulus. This situation appears in the paper [7], but it corresponds to the degenerate case $p = 0$. Below, we present a plot of a numerical solution of

the system (1) for the matrix

$$M = \begin{bmatrix} 2 & 1.1 & 3.1 \\ 3.1 & 2 & 0.9 \\ 0.95 & 2.9 & 2 \end{bmatrix}$$

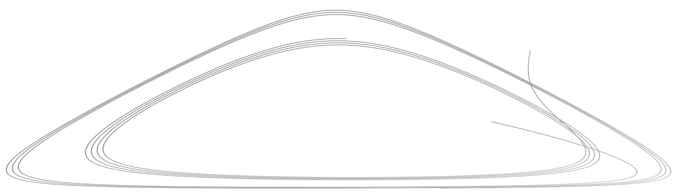
with two different initial conditions

$$x(0) = 0.08, \quad y(0) = 0.32, \quad z(0) = 0.51$$

and

$$x(0) = 0.01, \quad y(0) = 0.2, \quad z(0) = 0.05.$$

Here, $W = 18.8795$, $\alpha = 0.157313$, $\beta = 0.186446$, $\gamma = 0.154930$ and $p = -0.081534$. The assumptions of the theorem are satisfied but we can see that these trajectories have different ω-limit sets.



Under assumptions of the theorem, typical behaviour of the system is periodic. It is completely opposite to the principle of competitive exclusion which says that all (positive) solutions tend to fixed points where at least one species is extinct. This situation is typical for the system of two competiting species and three species provided that $W < 0$. All periodic solutions of our system are orbitally stable. It means that, in practice, only periodic behaviour of all three populations are observed.

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