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TOPOLOGICAL DERIVATIVE OF THE ENERGY FUNCTIONAL DUE TO FORMATION OF A THIN LIGAMENT ON A SPATIAL BODY

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Abstract. The exterior topological derivative is obtained for the energy functional of three dimensional mixed boundary value problem on the junction of a massive body and a thin rod (a ligament). The asymptotics is constructed and remainder is estimated for general geometrical formulation, namely, the ligament is curvilinear with varying cross-section and the junction zones, in contrast to many other papers, are not supposed to be plane. An example is provided which shows that such variations of shape functionals can be of interest for the topology optimization compared with the standard boundary variations technique.

1. Introduction

The interior topological derivatives are used in shape optimization, in particular for numerical methods of solution. In the paper the new concept of the exterior topological derivative [36] is developped for a three dimensional model problem.

Shape optimization techniques currently used in the domain are closely related to the asymptotic analysis of boundary value problems in singularly perturbed geometrical domains. Classical theory of shape optimization (see e.g., [42], [9]) is based on regular perturbations of the boundary in normal directions, such perturbations depends on a function $h(s), s \in \partial\Omega$, which is small and defines the distance between the reference and perturbed boundaries. In such a case the shape functional (SF) is usually Frechet differentiable and its differential can be evaluated by means of the material derivatives of the boundary value problem under considerations. However, the practical applications of the shape optimization require the topolgy

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changes in the process of numerical computations. The requirement leads to the concept of the internal and external topological derivatives. The internal topological derivative is introduced in [43] and generalized in [37] for broad class of elliptic boundary value problems. Such a derivative is evaluated for the topological variations of geometrical domain in the form of a hole or cavity. The methods of matched and compound asymptotic expansions (cf. [12, 20, 26, 27] and others) is applied in order to derive the explicite form of the internal topological derivative for a specific shape functional. The related results are reported in [38, 39, 40, 41, 37] for sufficiently general class of boundary value problems and shape functionals and lead to new methods of determiation of location of small holes or inhomogeneities in the framework of shape optimization and inverse problems.

The external topological derivative is introduced, for the first time, in [36]. Such a derivative is defined by the topology variation in the form of a thin ligament, connecting two small parts of the boundary, outside of the domain of reference. It is pointed out, on an example given in [36] (Example 1 in Section 4) that in some situations the external topological derivative can furnish better variation of the shape functional compared to the boundary variations or internal topological derivatives. In addition, we could also indicate, that in the level set method described by the Hamilton–Jacoby equation [3, 1], the use of the external topological derivatives would improve the performance of the method by creation of additional holes or cavities. However, it seems that the numerical implementation of such ideas is still an open problem.

Let us return to the external topological derivatives. The associated singular perturbation of geometrical domain can be seen as a junction of singularly degenerated domains with different limit dimensions whose asymptotic analysis was carried out in [15, 28, 16, 29, 17, 18, 30, 31, 32, 4, 2, 5, 6, 35, 11, 7, 33] and others.

Such results cannot be, unfortunately, directly applied to the analysis of shape optimization problems, since the simplified assumption is of the common use in the literature. Namely, that the boundary flattens near the ligament in the vicinity of the junction zones. In [36] such assumptions are not imposed in the case of a plane problem. The general case necessitates in the construction of global asymptotic approximation the introduction of various cut-off functions and rectifying diffeomorphisms. In the present paper the general framework is established in three dimensions. The boundary of the body is smooth, and the ligament takes the form of a thin *cylinder* with the variable cross section and the curved axis. We need to construct and justify, for evaluation of the topological derivative, few terms of the asymptotic expansion – the contribution of the ligament to the energy functional is of the order $O(h^2)$, where $h > 0$ stands for the relative thickness of the ligament.

In Section 2 the boundary value problem in the domain $\Omega(h)$ which is the union of the body Ω_0 with the ligament $\Lambda(h)$ is given. In Sections 3–4 the auxiliary limit boundary value problems are introduced and analised in Ω_0 and $\Lambda(h)$ and the leading terms of asymptotics are constructed in the domains. In Section 6 the asymptotics of solitions to boundary value problem in $\Omega(h)$ are justified. Finally, in Section 7 the obtained asymptotics serves for evaluation of the topological derivative.

2. Formulation of the problem

Let Ω_0 be a domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega_0$ and the compact closure $\overline{\Omega}_0 = \partial\Omega_0 \cup \Omega_0$. We assume that the ends of a smooth simple arc Γ belong to Ω_0 and Γ intersects the boundary $\partial\Omega_0$ at two points $P^{\pm}$. By $\theta^{\pm}$ are denoted the angles between tangent vector to Γ and the tangent planes to $\partial\Omega_0$ at two points $P^{\pm}$, respectively. We assume that $\theta^{\pm} \neq 0$. A point on the arc Γ is denoted by τ. The position of a point τ on Γ is parametrized by the length of the arc. Further, the notation τ means both, a point on Γ and its coordinate along Γ. We suppose that $\tau = \pm l$ at the point $P^{\pm}$, i.e. the part of arc Γ outside Ω_0 is of the length $2l$.

By $\mathbf{t}(\tau), \mathbf{n}(\tau), \mathbf{b}(\tau)$ are denoted tangent, normal and binormal vectors of the arc Γ, respectively. In a neighbourhood of the arc Γ, we introduce the curvilinear coordinates (τ, ν, β), where (ν, β) are Cartesian coordinates with the axes $\mathbf{n}(\tau)$ and $\mathbf{b}(\tau)$ in the plane, normal to Γ at the point τ.

Let $\omega \subset \mathbb{R}^2$ be a domain bounded by a smooth simple closed contour $\partial\omega$ and $\mathfrak{D}_\tau : \mathbb{R}^2 \to \mathbb{R}^2$ a diffeomorphism depending smoothly on the parameter $\tau \in [-l, l]$. We set $\omega(\tau) = \mathfrak{D}_\tau \omega$ and define a thin curvilinear cylinder $L(h)$ as follows

$$L(h) = \{(\tau, \nu, \beta) \in \mathbb{R}^3 : |\tau| < l,\ (\zeta, \eta) := (h^{-1}\nu, h^{-1}\beta) \in \omega(\tau)\}. \tag{1}$$

Here $h \in (0, h_0)$ is a small parameter, $h_0 > 0$ is a fixed number, and $(\zeta,\ \eta)$ are *fast* variables in the cross-section of the cylinder $L(h)$. We assume that the sets ω and $[-l/2, l/2]$ are independent of the parameter h and the bases of $L(h)$ belong to Ω_0 for $h < h_0$. The part $\Lambda(h) = L(h) \setminus \Omega_0$ of the cylinder $L(h)$ is called a thin *ligament*.

The domain $\Omega(h)$ depending on the small parameter h is defined as the union

$$\Omega(h) = \Omega_0 \cup \Lambda(h). \tag{2}$$

For a given h we consider the mixed boundary value problem for the Poisson equation

$$
\begin{aligned}
-\triangle_x u(h,x) &= f(h,x), && x \in \Omega(h),\\
\partial_n u(h,x) &= 0, && x \in \partial\Omega(h) \setminus \overline{\Sigma},\\
u(h,x) &= 0, && x \in \Sigma.
\end{aligned}
\tag{3}
$$

Here ∂_n stands for derivative along the external normal to the surface $\partial\Omega(h)$ and $\Sigma \subset \partial\Omega_0$ is a relatively open set with the positive area $\mathrm{mes}_2\Sigma$. We assume that $P^\pm \notin \Sigma$, and $\overline{\Sigma} \cap \overline{\Lambda}_h = \emptyset$ for $h \in (0, h_0]$ with a sufficiently small $h_0 > 0$.

Let $\mathring{H}^1(\Omega(h); \Sigma)$ be a subspace of vector functions from the Sobolev class $H^1(\Omega(h))$ satisfying the Dirichlet condition $(3)_3$. Problem (3) admits the unique solution $u(h, \cdot) \in \mathring{H}^1(\Omega(h); \Sigma)$ for any right-hand side $f(h, \cdot) \in L_2(\Omega(h))$. As a matter of fact we deal with a family of boundary value problems parameterized by the *relative thickness* $h \in (0, h_0]$ of the ligament $\Lambda(h)$ and with the family of corresponding solutions. However, when we derive the asymptotic formula for solution to problem (3) h is to be regarded as a small but *fixed* positive number. Therefore we speak about the specific boundary value problem with a unique solution $u(h, \cdot) \in \mathring{H}^1(\Omega(h); \Sigma)$.

The main goal of this paper is to construct the asymptotics of the solution $u(x, h)$ as $h \to +0$ and of the Dirichlet integral, or, what is almost the same, of the energy functional for the problem (3). The asymptotic behavior of $u(x, h)$ depends on the right-hand side f. We suppose that

$$
f(h,x) = \widetilde{f}(h,x) + \begin{cases} f_\Omega(x), & x \in \Omega_0,\\ f_\Lambda(\tau), & x \in \Lambda(h). \end{cases}
\tag{4}
$$

Functions f_Ω and f_Λ are defined in the *body* Ω_0 and on the ligament $\Lambda(h)$ with the *axis* $\Upsilon = (-l, l)$, respectively. The principal terms of the asymptotics of $u(x, h)$ are determined by f_Ω and f_Λ. An estimate of a norm of the remainder $\widetilde{f}$ is only used in the justification of asymptotics. The required properties of f_Ω and f_Λ are desribed in more details when dealing with the asymptotic analysis of problem (3). All restrictions we shall have on the terms in (4) seems to be realistic (cf. Remark 1 and Example 1).

3. The first limit problem

Let $v_0 \in \overset{\circ}{H}{}^1(\Omega(h);\Sigma)$ solve the problem

$$
\begin{aligned}
-\triangle_x v_0(x) &= f_\Omega(x), && x \in \Omega_0;\\
\partial_n v_0(x) &= 0, && x \in \partial\Omega_0 \setminus \overline{\Sigma};\\
v_0(x) &= 0, && x \in \Sigma.
\end{aligned} \tag{5}
$$

We derive (5) from (3) by taking $h = 0$, i.e. with $\widetilde{f} = 0$ in (4) and for $\Omega(0) = \Omega_0$ which means that the ligament is removed in (2). Let us assume that $f_\Omega \in L_2(\Omega_0)$. Since the boundary $\partial\Omega_0$ is smooth, the solution v_0 belongs to the Sobolev class H^2 everywhere except of a neighbourhood of the boundary arc $\partial\Sigma$ of the set Σ. We recall that $\partial\Sigma$ separates the part of the boundary with the Dirichlet boundary conditions from the part of the boundary with the Neumann boundary conditions.

It is well known (see, e.g., [13] and [34], [8]) that the inclusion $d_\Sigma \nabla_x^2 v_0 \in L_2(\Omega_0)$ and the estimate

$$\|d_\Sigma \nabla_x^2 v_0; L_2(\Omega_0)\| + \|v_0; H^1(\Omega_0)\| \leqslant c\,\|f_\Omega; L_2(\Omega_0)\|$$

are valid. Here $d_\Sigma(x) = \operatorname{dist}(x,\Sigma)$ is a weight and $\nabla_x^2 v_0$ stands for the collection of second order derivatives of v_0. Since $P^\pm \notin \Sigma$, the Sobolev embedding theorem $H^2 \subset C^0$ in $\mathbb{R}^3$ implies the continuity of solutions at the points $P^\pm$ and therefore, the estimates

$$|v_0(P^\pm)| \leqslant c\,\|f_\Omega; L_2(\Omega_0)\|. \tag{6}$$

Nevertheless, we need much more precise information on the behaviour of v_0 near the ends of the ligament $\Lambda(h)$. We suppose that the inclusion

$$d_P^{-\mu} f_\Omega \in L_2(\Omega_0) \tag{7}$$

holds for a certain $\mu \in (1/2, 3/2)$. Here $d_P(x) = \min\{\operatorname{dist}(x, P^\pm)\}$ is a new weight.

Let us introduce the local coordinate systems $(n^\pm, s^\pm)$, where $(s_1^\pm, s_2^\pm)$ is a local orthogonal coordinate system on the surface $\partial\Omega$ with the origin at $P^\pm$, and $n^\pm$ is the exterior unit normal vector at $P^\pm$. Well known results on the behaviour of solutions of elliptic boundary value problems near the boundary (see [14] and the introductory chapter in book [34]) imply the formula

$$
\begin{aligned}
v_0(x) &= \textstyle\sum_\pm \chi_\Omega(r_\pm)\Big\{v_0(P^\pm) + s_1^\pm \frac{\partial}{\partial s_1^\pm} v_0(P^\pm) + s_2^\pm \frac{\partial}{\partial s_2^\pm} v_0(P^\pm)\Big\} +\\
&+ \widetilde{v}_0(x),\\
|v_0(P^\pm)| &+ |\nabla_s v_0(P^\pm)| + \textstyle\sum_{j=0}^2 \|d^{-\mu-2+j} d_\Sigma^{\delta_{j,2}} \nabla_x^j \widetilde{v}_0; L_2(\Omega_0)\| \leqslant c N_\Omega.
\end{aligned} \tag{8}
$$

Here $\delta_{j,k}$ is a Kronecker symbol and

$$N_\Omega := \|d^{-\mu} f_\Omega; L_2(\Omega_0)\|. \tag{9}$$

The cut-off function $\chi_\Omega \in C_0^\infty(\mathbb{R})$ equals to 1 for $r < r_0/2$ and vanishes for $r > r_0$. We emphasize that for a smooth function f_Ω in $\overline{\Omega_0}$, norm (9) is finite and inclusions (7) are valid.

4. The resultant problem on the ligament

To find the asymptotics of the solution u of problem (3) on the thin ligament $\Lambda(h)$, we introduce further notation. In the orthogonal curvilinear coordinate system (τ, ν, β), the Laplace operator takes the form

$$\begin{aligned}\triangle_{(\tau,\nu,\beta)} U = \tfrac{1}{\sqrt{g}} \Big\{ \tfrac{\partial}{\partial \tau} \Big(\tfrac{\sqrt{g}}{g_{\tau\tau}} \tfrac{\partial U}{\partial \tau} \Big) &+ \tfrac{\partial}{\partial \nu} \Big(\tfrac{\sqrt{g}}{g_{\nu\nu}} \tfrac{\partial U}{\partial \nu} \Big) + \\ &+ \tfrac{\partial}{\partial \beta} \Big(\tfrac{\sqrt{g}}{g_{\beta\beta}} \tfrac{\partial U}{\partial \beta} \Big) \Big\}.\end{aligned} \tag{10}$$

Here $g_{\tau\tau}$, $g_{\nu\nu}$, $g_{\beta\beta}$ denote components of the metric tensor and $g = g_{\tau\tau} g_{\nu\nu} g_{\beta\beta}$. According to the definition of the metric tensor we have

$$\begin{gathered} g_{\nu\nu} = |\mathbf{n}| = 1, \quad g_{\beta\beta} = |\mathbf{b}| = 1, \\ g_{\tau\tau}(\tau,\nu,\beta) = |\mathbf{g}'(\tau) + \nu \mathbf{n}'(\tau) + \beta \mathbf{b}'(\tau)|^2, \end{gathered} \tag{11}$$

where the prime means the differentiation with respect to τ, i.e., $\mathbf{g}' = \partial \mathbf{g} / \partial \tau$. Let $k(\tau)$ and $\varkappa(\tau)$ denote the curvature and the torsion of the arc Γ, respectively. By virtue of the Frenet formula

$$\mathbf{n}'(\tau) = -k(\tau)\mathbf{t}(\tau) + \varkappa(\tau)\mathbf{b}(\tau), \quad \mathbf{b}'(\tau) = -\varkappa(\tau)\mathbf{n}(\tau)$$

it follows that

$$g_{\tau\tau}(\tau,\nu,\beta) = (1 - k(\tau)\nu)^2 + \varkappa(\tau)^2 \nu^2 + \varkappa(\tau)^2 \beta^2. \tag{12}$$

Hence

$$\begin{aligned}\triangle_{(\tau,\nu,\beta)} U = G(\tau,\nu,\beta)^{-1} \Big\{ \partial_\tau G(\tau,\nu,\beta)^{-1} \partial_\tau + \partial_\nu G(\tau,\nu,\beta)^{-1} \partial_\nu + \\ + \partial_\beta G(\tau,\nu,\beta)^{-1} \partial_\beta \Big\},\end{aligned} \tag{13}$$

where $G(\tau,\nu,\beta) = [(1 - k(\tau)\nu)^2 + \varkappa(\tau)^2 \nu^2 + \varkappa(\tau)^2 \beta^2]^{1/2}$.

We search for the asymptotics of the solution u on the ligament $\Lambda(h)$ in the form

$$u(h, x) \sim w_0(\tau) + h w_1(\tau, \zeta, \eta) + h^2 w_2(\tau, \zeta, \eta) + \ldots, \tag{14}$$

where (ζ, η) are the fast transversal variables, τ is the slow longitudinal variable and w_j are functions to be determined. It is the usual form of an asymptotical ansatz for the solutions of boundary value problems in thin domains (see [10], [22], [20] Chapter 18, [23] Chapter 1, *etc.*). In view of

(13) the Laplace operator can be represented in the variables (τ, ζ, η) as follows

$$\begin{aligned} \triangle_{(\tau,\nu,\beta)} \sim h^{-2}\triangle_{(\zeta,\eta)} &+ h^{-1}L_1(\tau,\partial_\zeta) + \\ &+ h^0 L_2(\tau,\zeta,\partial_\tau,\partial_\zeta) + h^1 L_3(\tau,\zeta,\partial_\tau,\partial_\zeta) + \dots \end{aligned} \tag{15}$$

Here

$$\begin{aligned} L_1(\tau,\partial_\zeta) &= -k(\tau)\partial_\zeta, \\ L_2(\tau,\zeta,\partial_\tau,\partial_\zeta) &= \partial_\tau^2 + (\varkappa(\tau)^2 - k(\tau)^2)\zeta\partial_\zeta, \\ L_3(\tau,\zeta,\partial_\tau,\partial_\zeta,\partial_\eta) &= 2\zeta\partial_\tau k(\tau)\partial_\tau + \\ &+ \frac{k}{12}(\tau)\Big\{3\Big(11\varkappa(\tau)^2 - 8k(\tau)^2)\zeta^2 + 8\varkappa(\tau)^2\zeta\eta^2\Big)\triangle_{(\zeta,\eta)} + \\ &+ 16\varkappa(\tau)^2\zeta\eta\partial_\eta + \varkappa(\tau)^2\Big(21\zeta^2 + 2\eta^2\Big)\partial_\zeta\Big\}. \end{aligned} \tag{16}$$

Let us consider the boundary condition $(3)_2$. The normal $\boldsymbol{\mathcal{N}}$ to the lateral surface of $\Lambda(h)$ takes the form

$$\begin{aligned} \boldsymbol{\mathcal{N}}(h,x) &= (1 + h^2 N_0(\tau,\zeta,\eta)^2)^{-1/2} \cdot \\ &\cdot (N_1(\tau,\zeta,\eta), N_2(\tau,\zeta,\eta), hN_0(\tau,\zeta,\eta))^\top. \end{aligned} \tag{17}$$

Here $N = (N_1, N_2)$ is the unit external normal vector to the contour $\partial\omega(\tau) \subset \mathbb{R}^2$, the component N_0 results from the variable cross-section $\omega(\tau)$ (see formula (26) below). By (11), (12) and (17), we have

$$\begin{aligned} (1h \ + \ ^2N_0(\tau,\zeta,\eta)^2)^{1/2}\partial_n = h\tfrac{N_0(\tau,h^{-1}\nu,h^{-1}\beta)}{G(\tau,\nu,\beta)}\tfrac{\partial}{\partial\tau} + \\ + N_1(\tau,h^{-1}\nu,h^{-1}\beta)\tfrac{\partial}{\partial\nu} + N_2(\tau,h^{-1}\nu,h^{-1}\beta)\tfrac{\partial}{\partial\beta}. \end{aligned} \tag{18}$$

Thus, we obtain the representation

$$\begin{aligned} (1 + h^2 N_0(\tau,\zeta,\eta)^2)^{1/2}\partial_n &= h^{-1}\partial_N + \\ &+ h^1 B_1(\tau,\zeta,\eta,\partial_\tau) + h^2 B_2(\tau,\zeta,\eta,\partial_\tau) + \dots, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \partial_N &= N_1(\tau,\zeta,\eta)\partial_\zeta + N_2(\tau,\zeta,\eta)\partial_\eta, \\ B_1(\tau,\zeta,\eta,\partial_\tau) &= N_0(\tau,\zeta,\eta)\partial_\tau, \\ B_2(\tau,\zeta,\eta,\partial_\tau) &= k(\tau)\zeta N_0(\tau,\zeta,\eta)\partial_\tau. \end{aligned} \tag{20}$$

Let us substitute (14), (15) into equation $(3)_1$ on the ligament $\Lambda(h)$ and (14), (19) into the boundary condition $(3)_2$ on the lateral surface of $\Lambda(h)$. We collect the coefficients of the same powers of h which leads to a sequence of Neumann problems posed on the cross-section $\omega(\tau)$. The function w_0 from (14) satisfies the problem

$$-\triangle_{(\zeta,\eta)} w_0(\tau) = 0, \quad (\zeta,\eta) \in \omega(\tau);$$
$$\partial_N w_0(\tau) = 0, \quad (\zeta,\eta) \in \partial\omega(\tau)$$

since w_0 is independent of the variables ζ and η. According to the relations

$$-\triangle_{(\zeta,\eta)} w_1(\tau,\zeta,\eta) = k(\tau)\partial_\zeta w_0(\tau) = 0, \quad (\zeta,\eta) \in \omega(\tau);$$
$$\partial_N w_1(\tau,\zeta,\eta) = 0, \quad (\zeta,\eta) \in \partial\omega(\tau)$$

the second term w_1 of (14) does not depend on ζ, η as well. Taking into account representation (4), we obtain the subsequent Neumann problem for w_2

$$\begin{aligned} -\triangle_{(\zeta,\eta)} w_2(\tau,\zeta,\eta) f_\Lambda(\tau) + \partial_\tau^2 w_0(\tau), &\quad (\zeta,\eta) \in \omega(\tau); \\ \partial_N w_2(\tau,\zeta,\eta) - N_0(\tau,\zeta,\eta)\partial_\tau w_0, &\quad (\zeta,\eta) \in \partial\omega(\tau). \end{aligned} \tag{21}$$

Problem (21) admits a solution if and only if there holds the equality

$$\int\limits_{\omega(\tau)} \left\{ f_\Lambda(\tau) + \partial_\tau^2 w_0(\tau) \right\} d\zeta d\eta - \int\limits_{\partial\omega(\tau)} N_0(\tau,\zeta,\eta)\partial_\tau w_0(\tau)\, ds_{(\zeta,\eta)} = 0. \tag{22}$$

Lemma 1. (cf. [24] §2, [23] Chapter 3). *Let Y be a smooth function on $\overline{\Lambda(1)}$. Then the formula*

$$\begin{aligned} \frac{d}{d\tau} \int\limits_{\omega(\tau)} Y(\tau,\zeta,\eta) d\zeta d\eta &= \int\limits_{\omega(\tau)} \frac{\partial Y}{\partial \tau}(\tau,\zeta,\eta)\, d\zeta d\eta + \\ &- \int\limits_{\partial\omega(\tau)} N_0(\tau,\zeta,\eta) Y(\tau,\zeta,\eta)\, ds_{(\zeta,\eta)} \end{aligned} \tag{23}$$

is valid for $\tau \in [-l/2, l/2]$.

Proof. Let ε be a small number and $\tau \in (\tau_0 - \varepsilon/2, \tau_0 + \varepsilon/2)$. We choose a fixed proper subdomain ω_0 of $\omega(\tau)$ such that the difference $Q(\tau) = \omega(\tau) \setminus \overline{\omega_0}$ implies a sufficiently thin boundary strip. Obviously,

$$\frac{d}{d\tau} \int\limits_{\omega_0} Y(\tau,\zeta,\eta) d\zeta d\eta = \int\limits_{\omega_0} \frac{\partial Y}{\partial \tau}(\tau,\zeta,\eta)\, d\zeta d\eta. \tag{24}$$

Since the strip $Q(\tau)$ is thin, we can divide it into smaller parts $q(\tau)$ such, that domains $q(\tau)$ are determined by the inequalities

$$Z_0 < \zeta < Z_1, \qquad H_0 < \eta < H(\tau,\zeta) \tag{25}$$

in a suitable Cartesian coordinate system (ζ,η) for the sake of brevity, the same notation (ζ,η) is used for the new coordinate system. In the domain defined by (25) equality (17) is equivalent with

$$
\begin{aligned}
N_1(\tau,\zeta) &= -\left(1+|\partial_\zeta H(\tau,\zeta)|^2\right)^{-1/2}\partial_\zeta H(\tau,\zeta),\\
N_2(\tau,\zeta) &= -\left(1+|\partial_\zeta H(\tau,\zeta)|^2\right)^{-1/2},\\
N_0(\tau,\zeta) &= -\left(1+|\partial_\zeta H(\tau,\zeta)|^2\right)^{-1/2}\partial_\tau H(\tau,\zeta).
\end{aligned} \tag{26}
$$

Moreover, on the specific part of the boundary $\partial\omega$, the performed change of variables results in an element of the arc

$$
ds_{(\zeta,\eta)}\left(1+|\partial_\zeta H(\tau,\zeta)|^2\right)^{1/2}d\zeta. \tag{27}
$$

By differentiation of the integral with variable bounds of integration we obtain

$$
\begin{aligned}
\tfrac{d}{d\tau}\int\limits_{q(\tau)} Y(\tau,\zeta,\eta)d\zeta d\eta &= \tfrac{d}{d\tau}\int\limits_0^Z\int\limits_0^{H(\tau,\zeta)} Y(\tau,\zeta,\eta)d\zeta d\eta =\\
&= \int\limits_{q(\tau)}\tfrac{\partial Y}{\partial\tau}(\tau,\zeta,\eta)\,d\zeta d\eta+\int\limits_0^Z Y(\tau,\zeta,\eta)\tfrac{\partial H}{\partial\tau}(\tau,\zeta)d\zeta =\\
&= \int\limits_{q(\tau)}\tfrac{\partial Y}{\partial\tau}(\tau,\zeta,\eta)\,d\zeta d\eta+\\
&\quad -\int\limits_{q(\tau)\cap\partial\omega(\tau)} Y(\tau,\zeta,\eta)N_0(\tau,\zeta)\,ds_{(\zeta,\eta)},
\end{aligned} \tag{28}
$$

where formula (26) and (27) are used in the last equality. Then the desired formula (23) follows by summation of (24) and (28) over all $q(\tau)\in Q(\tau)$. □

According to (23) we can express the solvability condition (22) in the form of an ordinary differential equation for the unknown function w_0

$$
-\frac{\partial}{\partial\tau}|\omega(\tau)|\frac{\partial}{\partial\tau}w_0(\tau)=|\omega(\tau)|f_\Lambda(\tau),\quad \tau\in\Upsilon, \tag{29}
$$

where $|\omega(\tau)|$ denotes the area $\text{meas}_2(\omega(\tau))$ of the domain $\omega(\tau)$. Equation (29) is supplied with the Dirichlet boundary conditions

$$
w_0(\pm l)=v_0(P^\pm), \tag{30}
$$

which are derived from the comparison of ansatz (14) with the elementary ansatz $u(h,x)\sim v_0(x)$ assumed in the body Ω_0 (see Section 2).

The existence of a solution to problem (21) is assured by equation (29), such a solution is determined up to an arbitrary function $\overline{w}_2$ of one variable

τ, i.e.,

$$w_2(\tau,\zeta,\eta) = w_2^{\perp}(\tau,\zeta,\eta) + \overline{w}_2(\tau),$$

where $w_2^{\perp}$ is a unique solution to (21) such that

$$\int\limits_{\omega(\tau)} w_2^{\perp}(\tau,\zeta,\eta)\, d\zeta d\eta = 0. \tag{31}$$

Owing to (15),(16) and (19), (20), the function w_3 is determined by the problem

$$\begin{aligned} -\triangle_{(\zeta,\eta)} w_3(\tau,\zeta,\eta) &= k(\tau)\partial_\zeta w_2^{\perp}(\tau,\zeta,\eta) + \\ &- \partial_\tau^2 w_1(\tau) - 2\zeta\partial_\tau(k(\tau)\partial_\tau w_0(\tau)), \quad (\zeta,\eta) \in \omega(\tau); \\ \partial_N w_3(\tau,\zeta,\eta) &= -N_0(\tau,\zeta,\eta)\partial_\tau w_1 + \\ &- \zeta k(\tau) N_0(\tau,\zeta,\eta)\partial_\tau w_0, \quad (\zeta,\eta) \in \partial\omega(\tau). \end{aligned} \tag{32}$$

In view of (23) problem (32) admits a solution if and only if the function w_1 satisfies the equality

$$\begin{aligned} -\frac{\partial}{\partial\tau}|\omega(\tau)|\frac{\partial}{\partial\tau}w_1(\tau) &= \int\limits_{\omega(\tau)} (k(\tau)\frac{\partial w_2}{\partial\zeta}(\tau,\zeta,\eta) - \zeta\frac{\partial}{\partial\tau}k(\tau)\frac{\partial w_0}{\partial\tau}(\tau)\, d\zeta d\eta + \\ &- \frac{\partial}{\partial\tau}\int\limits_{\omega(\tau)} \zeta\, d\zeta d\eta \frac{\partial w_0}{\partial\tau}, \quad \tau \in \Upsilon. \end{aligned} \tag{33}$$

We impose the boundary conditions

$$w_1(\pm l) = g_1^{\pm}, \tag{34}$$

where the data $g_1^{\pm}$ specified in Section 5 fulfill the inequality

$$|g_1^{\pm}| \leqslant c N_\Omega. \tag{35}$$

Given

$$f_\Lambda \in L_2(\Upsilon), \tag{36}$$

there exist the unique solutions $w_i \in H^2(\Upsilon), i = 0, 1$ of problems (29), (30) and (33), (34). By (6), (33) and (35), the estimates

$$\begin{aligned} \|w_i; H^2(\Upsilon)\| &\leqslant c\big(\|\overline{f}_\Lambda^1; L_2(\Upsilon)\| + \|f_\Omega; L_2(\Omega)\|\big) \leqslant \\ &\leqslant c\big(N_\Lambda + N_\Omega\big), \qquad i = 0, 1 \end{aligned} \tag{37}$$

are valid. Here

$$N_\Lambda := \|f_\Lambda; L_2(\Upsilon)\| \tag{38}$$

and N_Ω is determined by (9) (note that $N_\Omega \geqslant \|f_\Omega; L_2(\Omega)\|$). Moreover, according to an elementary embedding theorem we obtain the representation

$$\begin{aligned} w_0(\tau) &= \textstyle\sum_\pm \chi_\Lambda(\tau \mp l)\Big\{v_0(P^\pm) + (\tau \mp l)\partial_\tau w_0(\pm l)\Big\} + \widetilde{w}_0(\tau), \\ w_1(\tau) &= \textstyle\sum_\pm \chi_\Lambda(\tau \mp l)\Big\{\gamma^\pm + (\tau \mp l)\partial_\tau w_1(\pm l)\Big\} + \widetilde{w}_1(\tau). \end{aligned} \tag{39}$$

The cut-off function $\chi_\Lambda \in C_0^\infty(\mathbb{R})$ is such that $\chi_\Lambda = 1$ for $|t| < l/2$ and $\chi_\Lambda = 0$ for $|t| > l$. Using Hardy's inequalities we obtain

$$\begin{aligned} |\partial_\tau w_i(\pm l)| + \textstyle\sum_{j=0}^2 \|d_\pm^{-2+j}\partial_\tau^j \widetilde{w}_i; L_2(\Upsilon)\| &\leqslant c\,\|w_i; H^2(\Upsilon)\| \leqslant \\ &\leqslant c\,(N_\Omega + N_\Lambda),\ i = 0, 1. \end{aligned} \tag{40}$$

5. Boundary layers and other terms of asymptotics

Asymptotics near the points $P^\pm$ involve boundary layers. In particular, when we wrote the boundary condition for w_0 in the form (30) we have beard silently in mind that the first terms $z_0^\pm$ of the junction layers are the constants $v_0(P^\pm)$. We seek next terms of boundary layers near the points $P^\pm$ in the form

$$u(h, x) \sim z_0^\pm + h z_1^\pm(\xi^\pm) + \dots \tag{41}$$

Here $\xi^\pm = h^{-1}(x - P^\pm)$ are fast variables and $z_1^\pm$ can be determined by means of the special solutions to the Neumann problems in the domains $\Xi^\pm$. The model domains $\Xi^\pm$ are the unions of the half-space $\mathbb{R}^3_+$ with the cylinders $\Pi^\pm$ of the bases $\omega(\pm l)$. The angles $\theta^\pm$ between the axes of the cylinders $\Pi^\pm$ and $\partial\mathbb{R}^3_+$ coincide with the angles between the arc Γ and the surface $\partial\Omega$ at the points $P^\pm$. Let $(\rho_\pm, \varphi_\pm, \psi_\pm)$ denotes the spherical coordinate system with $\rho_\pm = |\xi^\pm|, \varphi_\pm \in (0, \pi)$ and $\varsigma_\pm \in [0, \infty)$ being the coordinates along the axes of cylinders $\Pi^\pm$. Note, that $\rho_\pm \sim h^{-1} r_\pm$ and $\varsigma_\pm \sim h^{-1}(l \mp \tau)$, however we interprete precisely these relations in Section 6.

For our purposes we only need a primitive information on the structure of the solution u in the junction zones. Meanwhile, the second term in (41) can be completely characterized by means of three harmonic functions $\mathbf{z}_0^\pm$, $\mathbf{z}_c^\pm$ and $\mathbf{z}_s^\pm$ with the homogeneous Neumann conditions on $\partial\Xi^\pm$ and with the prescribed asymptotic behaviour at the cylindrical and the conical outlets to infinity

$$\begin{aligned} \mathbf{z}_0^\pm(\xi^\pm) &= \mp|\omega(\pm l)|^{-1}\varsigma_\pm + a_0^\pm + \\ &\quad + O(\exp[-\alpha\varsigma_\pm]), \qquad \xi^\pm \in \Pi^\pm \setminus \mathbb{R}^3_+, \\ \mathbf{z}_0^\pm(\xi^\pm) &= (2\pi)^{-1}\rho_\pm^{-1} + O(\rho_\pm^{-2}), \qquad \xi^\pm \in \mathbb{R}^3_+; \end{aligned} \tag{42}$$

(43)
$$\begin{aligned} \mathbf{z}_c^{\pm}(\xi^{\pm}) &= a_c^{\pm} + O(\exp[-\alpha\varsigma_{\pm}]), && \xi^{\pm} \in \Pi^{\pm} \setminus \mathbb{R}^3_+, \\ \mathbf{z}_c^{\pm}(\xi^{\pm}) &= \rho_{\pm} \sin\varphi_{\pm} \cos\psi_{\pm} + \\ &\quad + O(\rho_{\pm}^{-2}), && \xi^{\pm} \in \mathbb{R}^3_+, \end{aligned}$$

(44)
$$\begin{aligned} \mathbf{z}_s^{\pm}(\xi^{\pm}) &= a_s^{\pm} + O(\exp[-\alpha\varsigma_{\pm}]), && \xi^{\pm} \in \Pi^{\pm} \setminus \mathbb{R}^3_+, \\ \mathbf{z}_s^{\pm}(\xi^{\pm}) &= \rho_{\pm} \sin\varphi_{\pm} \sin\psi_{\pm} + \\ &\quad + O(\rho_{\pm}^{-2}), && \xi^{\pm} \in \mathbb{R}^3_+. \end{aligned}$$

Here $a_i^{\pm}$ are the constants depending only on the cross-section $\omega(\pm l)$ and the angles $\theta^{\pm}$. The terms in the right-hand sides of $(42)_1$, $(43)_1$, $(44)_1$ and $(42)_2$, $(43)_2$, $(44)_2$ verify the homegeneous Neumann problems in the cylinders $\Pi^{\pm}$ and subsets of the half-space $\mathbb{R}^3_+$, respectively. To find the solution $\mathbf{z}_i^{\pm}$, we represent it as a sum of the terms in the right-hand side of (42), (43) or (44) multiplied with proper cut-off functions (cf. formula (67) below) and the solution $\widehat{\mathbf{z}}_i^{\pm}$ with the finite Dirichlet integrals. Such *energy* terms $\widehat{\mathbf{z}}_i^{\pm}$ decay in the half-space $\mathbb{R}^3_+$ and verify the Neumann problems for the Poisson equation with right-hand side, which imply the discrepancies appearing due to the above-mentioned multiplication.

Now, we are going to use the method of matched asymptotic expansions (see e.g. [44, 12]). The first term w_0 of the sum (14) admits asymptotical representation

$$w_0(\tau) \sim v_0(P^{\pm}) + (\tau \mp l)\partial_\tau w_0(\pm l) \sim v_0(P^{\pm}) \mp h\varsigma_{\pm}\partial_\tau w_0(\pm l) \quad \text{on} \quad \Lambda(h).$$

By comparison of the above relation with $(42)_1$ it follows that the term $z_1^{\pm}(\xi^{\pm})$ of ansatz (41) necessarily includes the function

$$\pm|\omega(\pm l)|\mathbf{z}_0^{\pm}(\xi^{\pm})\partial_\tau w_0(\pm l).$$

According to formula $(42)_2$ in the half-space $\mathbb{R}^3_+$, we obtain

$$\pm h|\omega(\pm l)|\mathbf{z}_0^{\pm}(\xi^{\pm})\partial_\tau w_0(\pm l) \sim \pm h^2|\omega(\pm l)|\partial_\tau w_0(\pm l)(2\pi)^{-1}r_{\pm}^{-1} \quad \text{in} \quad \Omega_0.$$

Whence, the asymptotics take the following form in the domain Ω_0:

$$u(h,x) = v_0(x) + h^2 v_2(x) + \dots,$$

where v_0 is a solution to problem (5), and v_2 is a harmonic function with the specific singularities $O(r_{\pm}^{-1})$ at the points $P^{\pm}$. Actually, v_2 solves the boundary value problem

$$-\triangle_x v_2(x) = 0, \qquad x \in \Omega_0,$$

$$\partial_n v_2(x) = \textstyle\sum_\pm \mp|\omega(\pm l)|\partial_\tau w_0(\pm l)\delta(x - P^\pm), \qquad x \in \partial\Omega_0 \setminus \overline{\Sigma}, \tag{45}$$

$$v_2(x) = 0, \qquad x \in \Sigma\ .$$

Furthermore, the asymptotic behaviour of v_0 in the vicinity of two points $P^\pm$ is described by the representation

$$v_2(x) = \sum_\pm \chi_\Omega(r_\pm)\Big\{ \pm \frac{1}{2\pi}|\omega(\pm l)|\partial_\tau w_0(\pm l)\frac{1}{r_\pm} + b_2^\pm \Big\} + \widetilde{v}_2(x). \tag{46}$$

In (45) and (46) δ is the Dirac mass, $(2\pi)^{-1}r_\pm^{-1}$ is the Poisson kernel, $b_2^\pm$ are the constants depending on $|\omega(\pm l)|$, w_0 and Ω_0; $\widetilde{v}_2 \in H^1(\Omega_0)$ is the regular part of v_2 and $\widetilde{v}_2(P^\pm) = 0$. The estimate

$$\begin{aligned} |b_2^\pm| + \sum_{j=0}^{2} \|d_\pm^{-1-\mu+j} d_\Sigma^{\delta_{j,2}} \nabla_x^j \widetilde{v}_2; L_2(\Omega_0)\| \ &\leqslant\ c\sum_\pm |\partial_\tau w_0(\pm l)| \leqslant \\ &\leqslant\ c\,\|w_0; H^2(\Upsilon)\| \leqslant c\,(N_\Lambda + N_\Omega) \end{aligned} \tag{47}$$

holds for $\mu \in (1/2, 3/2)$, note that μ in (47) can be taken the same as in (7). According to $(8)_1$ and (46) we have

$$\begin{aligned} v_0(x) + h^2 v_2(x) \ &\sim\ v_0(P^\pm) + s_1^\pm \partial_{s_1^\pm} v_0(P^\pm) + s_2^\pm \partial_{s_2^\pm} v_0(P^\pm) + \\ &\quad + h^2 \left\{ \pm\frac{1}{2\pi}|\omega(\pm l)|\partial_\tau w_0(\pm l) r_\pm^{-1} + b_2^\pm \right\} \sim \\ &\sim\ v_0(P^\pm) + h \left\{ \frac{1}{2\pi}|\omega(\pm l)|\partial_\tau w_0(\pm l)\rho_\pm^{-1} + \right. \\ &\quad + \rho_\pm \sin\varphi_\pm \cos\psi_\pm \partial_{s_1^\pm} v_0(P^\pm) + \\ &\quad \left. + \rho_\pm \sin\varphi_\pm \sin\psi_\pm \partial_{s_2^\pm} v_0(P^\pm) \right\} + h^2 b_2^\pm \quad \text{on} \quad \Omega_0. \end{aligned} \tag{48}$$

Owing to $(42)_2$, $(43)_2$ and $(44)_2$ we now put

$$\begin{aligned} z_1^\pm(\xi^\pm) \ &=\ \partial_s v_0(P^\pm)\mathbf{z}_c^\pm(\xi^\pm)\partial_t v_0(P^\pm)\mathbf{z}_s^\pm(\xi^\pm) \pm \\ &\quad \pm \tfrac{1}{2\pi}|\omega(\pm l)|\partial_\tau w_0(\pm l)\mathbf{z}_0^\pm(\xi^\pm). \end{aligned} \tag{49}$$

It should be stressed that the difference of the right-hand side (48) and (49) in the half-space $\mathbb{R}^3_+$ becomes $O(\rho_\pm^{-2})$ as $\rho_\pm \to +\infty$.

The constant $v_0(P^\pm)$ determines the right-hand side of boundary condition (30) for w_0. In the same way, we derive boundary conditions for the second term w_1 in expansion (14). First, the constants are determined by taking the limit $\Pi^{\pm} \ni \varsigma_{\pm} \to +\infty$ in the asymptotic representation (49). Then, by an application of the matching procedure we obtain the boundary conditions

$$(50)\qquad w_1(\pm l) = a_1^{\pm} \pm \frac{1}{2\pi}|\omega(\pm l)|\partial_\tau w_0(\pm l)a_0^{\pm}, \qquad \overline{w}_2(\pm l) = b_2^{\pm}.$$

which are already given by (34), and such that estimate (35) is still valid. We may continue the procedure of section 2 and derive an ordinary differential equation for $\overline{w}_2$. However, it is not necessary for the purposes of our paper.

6. Justification of the asymptotics

For any function $u \in \overset{\circ}{H}{}^1(\Omega(h);\Sigma)$, the Friedrichs inequality

$$(51)\qquad \|u; L_2(\Omega(h))\| \leqslant c\,\|\nabla_x u; L_2(\Omega(h))\|$$

holds. The one-dimensional Hardy inequality

$$\int\limits_0^\infty |U(r)|^2 r^{-2}\,dr \leqslant 4\int\limits_0^\infty |U'(r)|^2\,dr \quad \forall\, U \in C_0^\infty[0,\infty)$$

and formula (51) lead to the estimate

$$(52)\qquad \|d_P^{-1}u; L_2(\Omega(h))\| \leqslant c\,\|u; H^1(\Omega(h))\| \leqslant c\,\|\nabla_x u; L_2(\Omega(h))\|.$$

Let us consider the part

$$(53)\qquad \Lambda^\bullet(h) = \{x \in L(h) : |\tau| < l + 2\lambda h\} \supset \Lambda(h)$$

of the curvilinear cylinder (1). We choose the length λ such that the end parts $\Lambda_h^{\pm} = \{x \in \Lambda^\bullet(h) : \pm\tau \in (l+\lambda h, l+2\lambda h)\}$ of the ligament belong to the domain Ω_0. Since, for $x \in \Lambda_h^{\pm}$, the inequality $d_P(x)^{-1} \geqslant ch^{-1}$ is valid with a constant $c > 0$, we derive from (52) the inequality

$$(54)\qquad h^{-1}\|u; L_2(\Lambda_h^{\pm})\| \leqslant c\,\|\nabla_x u; L_2(\Omega(h))\|.$$

Assume that the cut-off function $\mathcal{X} \in C_0^\infty(-l-2\lambda h, l+2\lambda h)$ is equal to 1 for $|\tau| < l + \lambda h$ and satisfies the inequalities

$$(55)\qquad |\partial_\tau^k \mathcal{X}(h,\tau)| \leqslant c_k h^{-k}, \qquad k = 0,1,\ldots$$

By using (54) and (55), we obtain

$$\begin{aligned}\|\nabla_x(\mathcal{X}u); L_2(\Lambda^\bullet(h))\| &\leqslant c\Big\{\|\nabla_x u; L_2(\Lambda^\bullet(h))\| + h^{-1}\sum_{\pm}\|u; L_2(\Lambda_h^{\pm})\|\Big\} \leqslant \\ &\leqslant c\,\|\nabla_x u; L_2(\Omega(h))\|.\end{aligned}$$

To estimate a norm of u on the ligament, we need the Friedrichs-Poincaré inequality

$$(56)\qquad \|(h+d_P)^{-1}\mathcal{X}u; L_2(\Lambda^\bullet(h))\| \leqslant c\,\|\nabla_x(\mathcal{X}u); L_2(\Lambda^\bullet(h))\|.$$

In the proof of the formula, we use the property that the function Xu vanishes at the ends of $\Lambda^\bullet(h)$, therefore the argument used in the proof of the anisotropic weighted Korn's inequality for plates and bars (cf. [25], [23] § 3.3, 3.4 *etc.*) can be applied. In addition, an appropriate diffeomorphism maps $\Lambda^\bullet(h)$ in the straight cylinder, the verification of formula (56) can be performed by an application of a one dimensional Hardy inequality and subsequent integration over the sections. We here give only the final result.

Proposition 1. *The inequality*

$$\|du; L_2(\Omega)\| \leqslant c\,\|\nabla_x u; L_2(\Omega)\| \tag{57}$$

holds with a constant c which is independent of the function $u \in \overset{o}{H}{}^1(\Omega(h);\Sigma)$ *and of the parameter* $h \in (0, h_0]$. *The weight d is defined as follows*

$$d(h,x) = \begin{cases} d_P(x)^{-1}, & x \in \Omega_0, \\ (h + d_P(x))^{-1}, & x \in \Lambda(h). \end{cases} \tag{58}$$

To compose the global asymptotic approximation $\mathcal{U}(h,x)$ for the solution $u(h,x)$ of problem (3), we introduce the cut-off functions

$$\begin{aligned} \mathcal{X}_\Omega(h,x) &= 1 - \textstyle\sum_\pm \chi_\Omega(h^{-1}r_\pm), \\ \mathcal{X}_\Lambda(h,x) &= 1 - \textstyle\sum_\pm \chi_\Lambda(h^{-1}(\tau \mp l)). \end{aligned} \tag{59}$$

The functions $\mathcal{X}_\Omega$ and $\mathcal{X}_\Lambda$ are equal to 1 in Ω_0 and $\Lambda(h)$, respectively, except for small neighbourhoods of the points $P^\pm$. Obviously, for the cut-off functions (59) the estimates of (55) type are valid.

Since we do not use the simplifing assumption on flat junction zones (cf. [15, 17]), the boundary layers constructed in Section 5 are not defined on the domain $\Omega(h)$ but on the sets $\Xi^\pm$ which, after the compression in h^{-1} times, become close to $\Omega(h)$ in the vicinity of the junction zones $\gamma_h^\pm = \partial\Omega_0 \cap \Lambda_h$. Let us construct the mapping $x \mapsto \mathcal{D}_\pm(h,x)$ of the intersection of $\Omega(h)$ and a small neighbourhood of $\overline{\gamma_h^\pm}$,

$$\Omega^\pm(h) = \left\{ x \in \Omega(h) : \begin{array}{ll} r_\pm < r_0, & x \in \Omega_0 \\ l \mp \tau < r_0, & x \in \Lambda(h) \end{array} \right\} \cup \gamma_h^\pm \tag{60}$$

onto the part of the domain $\Xi^\pm$. This mapping is defined in several steps. First, we rectify the surface $\partial\Omega_0$ near $\gamma_h^\pm$, i.e., we use the orthogonal curvilinear coordinate system $(n^\pm, s_1^\pm, s_2^\pm)$ so that the domain Ω_0 is determined by the inequality $n > 0$ near to $\gamma_h^\pm$. Second, we observe that, in coordinates $(n^\pm, s^\pm)$, the parts $\Lambda^\pm(h)\{x \in \Lambda(h) : n^\pm > -2r_0\}$ of ligament (cf. (60)) are defied by

$$\{x \in \Lambda(h) : n^\pm > -2r_0, h^{-1}s^\pm = (h^{-1}s_1^\pm, h^{-1}s_2^\pm) \in \widehat{\omega}^\pm(h,n)\}$$

where the domain $\widehat{\omega}^{\pm}(h,n) \subset \mathbb{R}^2$ is smoothly dependent on small h and n while it is close to the skew cross-section $\widehat{\omega}^{\pm} = \partial\mathbb{R}^3_+ \cap \Pi^{\pm}$ of the cylinder $\Pi^{\pm}$ (see Section 5). Third, we define the family $\{\mathbb{D}^{\pm}(h,n^{\pm},\cdot)\}_{n^{\pm}\in[0,-2r_0],h\in[0,h_0]}$ of diffeomorphisms on the plane $\mathbb{R}^2$ such that $\mathbb{D}^{\pm}(h,n^{\pm},\cdot)\widehat{\omega}^{\pm}(h,n^{\pm}) = \widehat{\omega}^{\pm}$, and put

$$\mathcal{D}^{\pm}(h,x) = \begin{cases} h^{-1}(n^{\pm}\mathbb{D}^{\pm}(h,0,s^{\pm})), & \text{inside} \quad \Omega^{\pm}(h)\cap\Omega_0, \\ h^{-1}(n^{\pm}\mathbb{D}^{\pm}(h,n^{\pm},s^{\pm})), & \text{inside} \quad \Omega^{\pm}(h)\cap\Lambda_h. \end{cases}$$

By a straight-forward verification we see that $x \mapsto \mathcal{D}^{\pm}(h,x)$ is a mapping with the above-mention property and moreover, it is of class H^1_∞ and the following estimates are valid

$$\begin{aligned} \Big|\mathcal{D}_{\pm}(h,x) - h^{-1}(x-P^{\pm})\Big| &\leqslant c\begin{cases} (h+r_{\pm}), & x\in\Omega_0, \\ (h+l\mp\tau), & x\in\Lambda(h), \end{cases} \\ \Big|\nabla_x\mathcal{D}_{\pm}(h,x) - h^{-1}\mathbb{I}_3\Big| &\leqslant c\begin{cases} (h+r_{\pm}), & x\in\Omega_0, \\ (h+l\mp\tau), & x\in\Lambda(h), \end{cases} \end{aligned} \tag{61}$$

where $\mathbb{I}_3$ is the unit matrix of size (3×3).

We determine the asymptotic approximation $\mathcal{U}$ by the formula

$$\begin{aligned} \mathcal{U}(h,x) &= \mathcal{V}(h,x) + h\textstyle\sum_{\pm}\chi_\Omega(r_{\pm})\widetilde{z}_1^{\,\pm}(\mathcal{D}_{\pm}(h,x)), \qquad x\in\Omega_0, \\ \mathcal{U}(h,x) &= \mathcal{W}(h,x) + h\textstyle\sum_{\pm}\chi_\Lambda(\tau\mp l)\widetilde{z}_1^{\,\pm}(\mathcal{D}_{\pm}(h,x)), \quad x\in\Lambda, \end{aligned} \tag{62}$$

where

$$\begin{aligned} \mathcal{V}(h,x) &= \mathcal{X}_\Omega(h,x)\Big\{\widetilde{v}_0(x) + h^2\widetilde{v}_2(x)\Big\} + \\ &\quad + \sum_{\pm}\chi_\Omega(r_{\pm})\Big\{v_0(P^{\pm}) + h^2 b_2^{\pm} + \Big(1-\chi_\Omega(h^{-1}r_{\pm})\Big)\mathcal{Y}^{\pm}(x)\Big\}, \end{aligned} \tag{63}$$

$$\begin{aligned} \mathcal{W}(h,x) &= \mathcal{X}_\Lambda(h,x)\Big\{\widetilde{w}_0(\tau) + h\widetilde{w}_1(\tau) + \widetilde{w}_2(\tau) + h^2 w_2^{\perp}(\tau,\zeta,\eta)\Big\} + \\ &\quad + \textstyle\sum_{\pm}\chi_\Lambda(\tau\mp l)\Big\{w_0(\pm l) + hw_1(\pm l) + h^2\overline{w}_2(\pm l) + \\ &\quad + \textstyle\sum_{i=0}^{1} h^i\Big(1-\chi_\Lambda(h^{-1}(\tau\mp l))\Big)\mathcal{Z}_i^{\pm}(\tau)\Big\}, \\ \mathcal{Y}^{\pm}(x) &:= s_1^{\pm}\partial_{s_1^{\pm}}v_0(P^{\pm}) + s_2^{\pm}\partial_{s_2^{\pm}}v_0(P^{\pm}), \\ \mathcal{Z}_i^{\pm}(\tau) &:= (\tau\mp l)\partial_\tau w_i(\pm l). \end{aligned} \tag{64}$$

Here $\widetilde{v}_0$, $\widetilde{v}_2$ and $\widetilde{w}_0$ are the remainders in representations (8), (46) and (39) while w_1 is an arbitrary linear function of the variable τ with the values $w_1(\pm l)$ as in (50). We suppose that the function $\overline{w}_2$ also satisfies the boundary conditions (50) and therefore

$$\overline{w}_2(\tau) = \widetilde{w}_2(\tau) + \sum_{\pm} \chi_\Lambda(\tau \mp l)\overline{w}_2(\pm l). \tag{65}$$

The cut-off functions are introduced in (62) in order to obtain the sufficiently smooth asymptotic approximations. In view of the boundary conditions (30), (50) the constants $\mathbf{z}_0^\pm$ and $\mathbf{z}_1^\pm$ are actually not multiplied by the cut-off function $1 - \chi_\Omega(h^{-1}r_\pm)$. Since the constants do not decay at infinity, they cannot be included in the terms $\mathbf{z}_0^\pm$ and $\mathbf{z}_1^\pm$ in (49). In this way, the following sum should be taken in (62) instead of (49)

$$\begin{aligned} \widetilde{z}_1^{\,\pm}(\xi^\pm) &= \widetilde{\mathbf{z}}_c^{\,\pm}(\xi^\pm)\partial_s v_0(P^\pm) + \\ &\quad + \widetilde{\mathbf{z}}_s^{\,\pm}(\xi^\pm)\partial_t v_0(P^\pm) \pm \frac{1}{2\pi}|\omega(\pm l)|\partial_\tau w_0(\pm l)\widetilde{\mathbf{z}}_0^{\,\pm}(\xi^\pm), \end{aligned} \tag{66}$$

where, according to (42), (43), (44)

$$\begin{aligned} \widetilde{\mathbf{z}}_0^{\,\pm} &= \mathbf{z}_0^\pm(\xi^\pm)\pm \\ &\quad \pm(1-\chi_\Lambda(\varsigma_\pm))|\omega(\pm l)|^{-1}\varsigma_\pm - a_0^\pm, && \xi^\pm \in \Pi^\pm \setminus \mathbb{R}^3_+, \\ \widetilde{\mathbf{z}}_0^{\,\pm} &= \mathbf{z}_0^\pm(\xi^\pm)+ \\ &\quad -(1-\chi_\Omega(\rho_\pm))(2\pi)^{-1}\rho_\pm^{-1}, && \xi^\pm \in \mathbb{R}^3_+; \\ \widetilde{\mathbf{z}}_c^{\,\pm} &= \mathbf{z}_c^\pm(\xi^\pm) - a_c^\pm, && \xi^\pm \in \Pi^\pm \setminus \mathbb{R}^3_+, \\ \widetilde{\mathbf{z}}_c^{\,\pm} &= \mathbf{z}_c^\pm(\xi^\pm) - (1-\chi_\Omega(\rho_\pm))\cdot \\ &\quad \cdot\partial_s v_0(P^\pm)\rho_\pm \sin\varphi_\pm \cos\psi_\pm, && \xi^\pm \in \mathbb{R}^3_+. \\ \widetilde{\mathbf{z}}_s^{\,\pm} &= \mathbf{z}_s^\pm(\xi^\pm) - a_s^\pm, && \xi^\pm \in \Pi^\pm \setminus \mathbb{R}^3_+, \\ \widetilde{\mathbf{z}}_s^{\,\pm} &= \mathbf{z}_s^\pm(\xi^\pm) - (1-\chi_\Omega(\rho_\pm))\cdot \\ &\quad \cdot\partial_t v_0(P^\pm)\rho_\pm \sin\varphi_\pm \sin\psi_\pm, && \xi^\pm \in \mathbb{R}^3_+. \end{aligned} \tag{67}$$

The only difference between $\widetilde{\mathbf{z}}_i^{\,\pm}$ and the energy solutions $\widehat{\mathbf{z}}_i^{\,\pm}$ mention in Section 5 is the constant $a_i^\pm$ subtracted in the cylindrical outlet: $\widetilde{\mathbf{z}}_i^{\,\pm}$ decays exponentially for $\varsigma_\pm \to +\infty$ (see $(42)_2$–$(44)_2$) while $\widetilde{\mathbf{z}}_i^{\,\pm}$ stabilized to the constant $a_i^\pm$. The functions $\widetilde{\mathbf{z}}_i^{\,\pm}$ are discontinuous, however $\mathcal{U}$ enjoys the required smoothness by means of matching the constants.

Let us calculate the discrepancies of $\mathcal{U} \in \overset{\circ}{H}{}^1(\Omega(h);\Sigma)$ in equation $(3)_1$ and in boundary condition $(3)_2$. For the remainder $\mathcal{R} = u - \mathcal{U}$, we have

$$
\begin{aligned}
-\triangle_x \mathcal{R} &= f - \mathcal{X}_\Omega f_\Omega + \big[\triangle_x, \mathcal{X}_\Omega\big]\big(\widetilde{v}_0 + h^2 \widetilde{v}_2\big) + \\
&+ h \sum_{\pm} \big[\triangle_x, \chi_\Omega(r_\pm)\big]\widetilde{z}_1^{\,\pm} + \\
&+ \sum_{\pm} \chi_\Omega(r_\pm)\Big\{h\triangle_x \widetilde{z}_1^{\,\pm} - \big[\triangle_x, \chi_\Omega(h^{-1}r_\pm)\big]\mathcal{Y}^{\pm}\Big\} \quad \text{on} \quad \Omega(h),
\end{aligned}
$$

$$
\begin{aligned}
-\triangle_x \mathcal{R} &= f + \mathcal{X}_\Lambda \triangle_x\big(w_0 + h w_1 + h^2 w_2\big) + \\
&+ \big[\triangle_x, \mathcal{X}_\Lambda\big]\big(\widetilde{w}_0 + h\widetilde{w}_1 + h^2\widetilde{w}_2 + h^2 w_2^{\perp}\big) + \\
&+ h \textstyle\sum_{\pm} \big[\triangle_x, \chi_\Lambda(\tau \mp l)\big]\widetilde{z}_1^{\,\pm} + \\
&+ \textstyle\sum_{\pm} \chi_\Lambda(\tau \mp l)\Big\{h\triangle_x \widetilde{z}_1^{\,\pm} + \\
&- \textstyle\sum_{i=0}^{1} h^i \big[\triangle_x, \chi_\Lambda(h^{-1}(\tau \mp l))\big]\mathcal{Z}_i^{\pm}\Big\} \text{ on } \Lambda(h).
\end{aligned} \tag{68}
$$

Here $\big[\mathsf{A},\mathsf{B}\big] = \mathsf{AB} - \mathsf{BA}$ is the commutator of the operators A and B, the functions $\mathcal{Y}^{\pm}$ and $\mathcal{Z}^{\pm}$ are introduced in (63), (64). To obtain (67), we, in particular, take into account positions of supports of the cut-off functions. Similarly,

$$
\begin{aligned}
\partial_n \mathcal{R} &= -\big[\partial_n, \mathcal{X}_\Omega\big]\big(\widetilde{v}_0 + h^2 \widetilde{v}_2\big) + \\
&- h \sum_{\pm} \big[\partial_n, \chi_\Omega(r_\pm)\big]\widetilde{z}_1^{\,\pm} + \\
&- \sum_{\pm} \chi_\Omega(r_\pm)\Big\{h\partial_n \widetilde{z}_1^{\,\pm} - \big[\partial_n, \chi_\Omega(h^{-1}r_\pm)\big]\mathcal{Y}^{\pm}\Big\} \quad \text{on} \quad \partial\Omega \setminus \overline{\Lambda}_h,
\end{aligned}
$$

$$
\begin{aligned}
\partial_n \mathcal{R} &= -\mathcal{X}_\Lambda \partial_n\big(w_0 + h w_1 + h^2 w_2\big) + \\
&- \big[\partial_n, \mathcal{X}_\Lambda\big]\big(\widetilde{w}_0 + h\widetilde{w}_1 + h^2\widetilde{w}_2 + h^2 w_2^{\perp}\big) + \\
&- h \textstyle\sum_{\pm} \big[\partial_n, \chi_\Lambda(\tau \mp l)\big]\widetilde{z}_1^{\,\pm} + \\
&- \textstyle\sum_{\pm} \chi_\Lambda(\tau \mp l)\Big\{h\partial_n \widetilde{z}_1^{\,\pm} + \\
&- \textstyle\sum_{i=0}^{1} h^i \big[\partial_n, \chi_\Lambda(h^{-1}(\tau \mp l))\big]\mathcal{Z}_i^{\pm}\Big\} \text{ on } \partial\Lambda(h)
\end{aligned} \tag{69}
$$

for $(\nu,\beta) \in \partial\omega(\tau)$. Multiplication of (68) by $\mathcal{R} \in \mathring{H}^1(\Omega(h);\Sigma)$ and integration by parts in $\Omega(h)$, taking into account boundary conditions (69), leads to the formula with the right hand side to be specified below,

$$
\big(\nabla_x \mathcal{R}, \nabla_x \mathcal{R}\big)_{\Omega(h)} = \ldots \tag{70}
$$

Here $\big(\ ,\ \big)_\Xi$ is the scalar product in $L_2(\Xi)$, and "..." means the integrals linearly dependent on $\mathcal{R}$ which we are going subsequently define and estimate.

On the set $\operatorname{supp}\{1-\mathcal{X}_\Omega\}$ we have inequality

$$d_P(x) \leqslant ch, \tag{71}$$

hence, by Proposition 1 and definition (9) of the term N_Ω we have the estimate

$$\begin{aligned} \left|\left(f-\mathcal{X}_\Omega f_\Omega, \mathcal{R}\right)_\Omega\right| &\leqslant c\Big(\|d_P(1-\mathcal{X}_\Omega)f_\Omega; L_2(\Omega)\| + \\ &\quad + \|d_P\widetilde{f}; L_2(\Omega)\|\Big)\|d_P^{-1}\mathcal{R}; L_2(\Omega)\| \leqslant \\ &\leqslant c\,h^{1+\mu}(N_\Omega+\widetilde{N}_\Omega)\,\|\nabla_x\mathcal{R}; L_2(\Omega)\|. \end{aligned} \tag{72}$$

In order to assure that the remainder in representation (4) is small enough, we have required in (72) that the quantity

$$\widetilde{N}_\Omega := h^{-1-\mu}\|d_P\widetilde{f}; L_2(\Omega)\| \tag{73}$$

is of the order h^0. Taking into account relation (71), Proposition 1, inequalities $(8)_2$ and (47) for the remainders $\widetilde{v}_i$, and the upper bounds (47), (40) for coefficients of linear combination (49), it follows that

$$\begin{aligned} \Big|\big([\triangle_x, \mathcal{X}_\Omega](\widetilde{v}_0+h^2\widetilde{v}_2), \mathcal{R}\big)_\Omega - \big([\partial_n, \mathcal{X}_\Omega](\widetilde{v}_0+h^2\widetilde{v}_2), \mathcal{R}\big)_{\partial\Omega}\Big| &\leqslant \\ \leqslant c\,h^{1+\mu}(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R}; L_2(\Omega(h))\|&, \\ h\Big|\big([\triangle_x, \chi_\Omega(r_\pm)]\,\widetilde{z}_1^{\,\pm}, \mathcal{R}\big)_\Omega - \big([\partial_n, \chi_\Omega(r_\pm)]\,\widetilde{z}_1^{\,\pm}, \mathcal{R}\big)_{\partial\Omega\setminus\Lambda_h}\Big| &\leqslant \\ \leqslant c\,h^3(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R}; L_2(\Omega(h))\|&. \end{aligned}$$

In the second inequality above, we make use of the rate of decay $O(\rho_\pm^{-2}) = O(hr_\pm^{-2})$ for the remainders in the expansions $(42)_2$–$(44)_2$. In the first inequality, formula (55) is used as well as the elementary traces inequalities

$$\|d_P^{i-\mu-3/2}\widetilde{v}_i; L_2(\partial\Omega)\| \leqslant c\sum_{j=0}^{1}\|d_P^{i-\mu-2+j}\nabla_x^j\,\widetilde{v}_i; L_2(\Omega)\|, \quad i=0,1.$$

Since the area of the cross-section ω_h of the ligament $\Lambda(h)$ is of order $O(h^2)$, in view of relations (65), (40), (57) and of estimate (55) for derivatives of the cut-off function $\mathcal{X}_\Lambda$, it follows that

$$\begin{aligned} \Big|\big([\triangle_x, \mathcal{X}_\Lambda]\sum_{i=1}^{2} h^i \widetilde{w}_i, \mathcal{R}\big)_{\Lambda(h)}\Big| \;&\leqslant\; ch\|\nabla_x \mathcal{R}; L_2(\Lambda(h))\| \cdot \\ &\cdot h\sum_{i=1}^{2}\sum_{j=0}^{1} h^i h^{j-2} h^{3-i-j} \cdot \\ &\cdot \|d^{i+j-3}\frac{\partial^j w_i}{\partial \tau^j}; L_2(\Upsilon)\| \leqslant \\ &\leqslant\; c\,h^3 (N_\Omega + N_\Lambda)\,\|\nabla_x \mathcal{R}; L_2(\Omega(h))\|. \end{aligned} \tag{74}$$

Furthermore, by the trace inequalities

$$h^{-1/2}\|U; L_2(\partial\Lambda(h))\| \leqslant c\{h^{-1}\|U; L_2(\Lambda(h))\| + \|\nabla U; L_2(\Lambda(h))\|\}, \tag{75}$$

we have

$$\begin{aligned} \Big|\big([\partial_n, \mathcal{X}_\Lambda]\sum_{i=1}^{2} \widetilde{w}_i, \mathcal{R}\big)_{\partial\Lambda(h)}\Big| \;&\leqslant\; ch^{1/2}\|\nabla_x \mathcal{R}; L_2(\Lambda(h))\| \cdot \\ &\cdot h^{1/2} h\sum_{i=1}^{2}\sum_{j=0}^{1} h^i h^{j-2} h^{3-i-j} \cdot \\ &\cdot \|d^{i+j-3}\frac{\partial^j w_i}{\partial \tau^j}; L_2(\Upsilon)\| \leqslant \\ &\leqslant\; c\,h^2 (N_\Omega + N_\Lambda)\,\|\nabla_x \mathcal{R}; L_2(\Omega(h))\|. \end{aligned} \tag{76}$$

In order to estimate the term containing the remainder $\widetilde{w}_0$ in the right-hand side of (70), and also to justify the inequalities used in the derivation of estimates (74), (76) we need the following estimates, which can be checked up by application, several times, of the Hardy's inequality (for details, see e.g., [23, Lemma 1.2.4]).

Lemma 2. *Assume that $t \mapsto W^k(t) \in H^{k+1}(0,1)$, for $k = 0,1,2$, is such that $\partial_t^j W^k(0) = 0$ for $j = 0,\dots,k-1$. Then, on a small interval $T_{ch} = (0, ch)$, it follows that*

$$h^{-k}\|W^k; L_2(T_{ch})\| \leqslant ch^{1/2}\|W^k; H^{k+1}(0,1)\|. \tag{77}$$

Using (77) we obtain

$$\begin{aligned}\Big|\big([\triangle_x,\mathcal{X}_\Lambda]\widetilde{w}_0,\mathcal{R}\big)_{\Lambda(h)}\Big| &\leqslant ch\|\nabla_x\mathcal{R};L_2(\Lambda(h))\|\cdot\\ &\cdot h\sum_{j=0}^{1}h^{j-2}h^{2-j+1/2}\|w_0;H^{3-j}(\Upsilon)\|\leqslant\\ &\leqslant c\,h^3(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R};L_2(\Omega(h))\|,\\ \Big|\big([\partial_n,\mathcal{X}_\Lambda]\sum_{i=1}^{2}\widetilde{w}_0,\mathcal{R}\big)_{\partial\Lambda(h)}\Big| &\leqslant ch^{1/2}\|\nabla_x\mathcal{R};L_2(\Lambda(h))\|\cdot\\ &\cdot h^{1/2}h\sum_{j=0}^{1}h^ih^{j-2+1/2}h^{2-j+1/2}\,\cdot\\ &\cdot\|w_0;H^{3-j}(\Upsilon)\|\leqslant\\ &\leqslant c\,h^3(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R};L_2(\Omega(h))\|.\end{aligned}$$

The estimate

$$\begin{aligned}h\,\big|\big([\triangle_x,\chi_\Lambda(\tau\mp l)]\widetilde{z}_1^{\,\pm},\mathcal{R}\big)_{\Lambda(h)}-\big([\partial_n,\chi_\Lambda(\tau\mp l)]\widetilde{z}_1^{\,\pm},\mathcal{R}\big)_{\partial\Lambda(h)}\big|\leqslant\\ \leqslant c\,h^{1/2}\exp\{-\delta/h\}(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R};L_2(\Omega(h))\|\end{aligned}$$

with $\delta>0$, is valid due to the exponential decay of remainders in representations $(42)_1$, $(43)_1$, $(44)_1$ and by the inequality $\partial_t\chi_\Lambda(t)=0$ for $t\in[0,r_0/2]$.

For the solution $w_2^\perp$ of problem (21), in view of (36) and $(37)_1$, the estimate is obtained

$$\|w_2^\perp;H^2(\omega)\|\times\|w_2^\perp;H^1(\Upsilon)\|\leqslant c\,(N_\Omega+N_\Lambda). \tag{78}$$

Owing to the orthogonality condition (31), there holds on the cross-section ω_h the Poincaré inequality

$$\|w_2^\perp;L_2(\omega_h)\|\leqslant c\,h\|\nabla_{\nu,\beta}w_2^\perp;H^2(\omega_h)\|, \tag{79}$$

which is combined with relations (78), (75) in order to obtain

$$\begin{aligned}h^2\Big|\big([\triangle_x,\mathcal{X}_\Lambda]w_2^\perp,\mathcal{R}\big)_{\Lambda(h)}\Big| &\leqslant ch^2h\|\nabla_x\mathcal{R};L_2(\Lambda(h))\|\cdot\\ &\cdot\sum_{j=0}^{1}h^{j-2}h^{2-j+1/2+j/2}\,\cdot\\ &\cdot\|w_2^\perp;H^1(\Upsilon\to H^2(\omega))\|\leqslant\\ &\leqslant c\,h^{5/2}(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R};L_2(\Omega(h))\|,\end{aligned}$$

$$\begin{aligned} h^2\Big|\big(\big[\partial_n,\mathcal{X}_\Lambda\big]w_2^\perp,\mathcal{R}\big)_{\partial\Lambda(h)}\Big| &\leqslant ch^2h^{1/2}\|\nabla_x\mathcal{R};L_2(\Lambda(h))\|\cdot \\ &\quad\cdot h^{1/2}h^1h^{-1}\|w_2^\perp;H^1(\Upsilon\to H^2(\omega))\|\leqslant \\ &\leqslant c\,h^{5/2}(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R};L_2(\Omega(h))\|. \end{aligned}$$

Here $H^1(\Upsilon\to H^2(\omega))$ denotes the space with the norm

$$\|w;H^1(\Upsilon\to H^2(\omega))\|=\int_{-l}^{l}(\|w;H^2(\omega)\|^2+\|\partial_s w;H^2(\Upsilon)\|)^{1/2}\,ds.$$

On the other hand, in view of $(37)_1$, (78) (75), (57) and the relation

$$|G(\tau,\nu,\beta)-1|\leqslant ch^1,\quad\text{for}\quad(\tau,\nu,\beta)\in\Lambda(h),$$

it follows that

$$\begin{aligned} &\Big|\big(\mathcal{X}_\Lambda(h^2\triangle_x w_2^\perp-h^2\triangle_{(\nu,\beta)}w_2^\perp+\triangle_x w_0-\partial_\tau^2 w_0),\mathcal{R}\big)_{\Lambda(h)}+ \\ &+\big(\mathcal{X}_\Lambda(h^2\partial_n w_2^\perp-h^3\partial_N w_2^\perp+h^2\partial_n w_0-N_0\partial_\tau w_0,\mathcal{R}\big)_{\partial\Lambda(h)}\Big|\leqslant \\ &\leqslant c\,h^3(N_\Omega+N_\Lambda)\,\|\nabla_x\mathcal{R};L_2(\Omega(h))\|. \end{aligned}$$

The function $\overline{w}_2$ depends only on one variable τ, thus, the trace inequality (75), Proposition 1 and formula (13), (18) yield

$$\begin{aligned} h^2\ &\big|\big(\mathcal{X}_\Lambda\triangle_x\overline{w}_2,\mathcal{R}\big)_{\Lambda(h)}-\big(\mathcal{X}_\Lambda\partial_n\overline{w}_2,\mathcal{R}\big)_{\partial\Lambda(h)}\big|\leqslant \\ \leqslant\ & c\,h^2\Big(\mathrm{meas}_2(\omega_h)^{1/2}(N_\Omega+N_\Lambda)\|\mathcal{R};L_2(\Lambda(h))\|+ \\ &+h^1\,\mathrm{meas}_1(\partial\omega_h)^{1/2}h^1\,(N_\Omega+N_\Lambda)\,\|\mathcal{R};L_2(\partial\Lambda(h))\|\Big)\leqslant \\ \leqslant\ & c\,h^3(N_\Omega+N_\Lambda)\|\nabla_x\mathcal{R};L_2(\Omega(h))\|. \end{aligned}$$

To derive upper bounds for the term $(\mathcal{X}_\Lambda\triangle_x w_1,\mathcal{R})_{\Lambda(h)}$ and the term $(\mathcal{X}_\Lambda\partial_n w_1,\mathcal{R})_{\partial\Lambda(h)}$, we define

$$\overline{\mathcal{R}}(\tau):h^{-2}|\omega|(\tau)^{-1}\int_{\omega(\tau)}\mathcal{R}(\nu,\tau)\,d\nu.$$

Since the difference $\mathcal{R}-\overline{\mathcal{R}}$ has zero mean over the cross-section $\omega(h)$, Poincaré inequality (79) provides the estimate

$$\begin{aligned} \|\mathcal{X}_\Lambda\big(\mathcal{R}-\overline{\mathcal{R}}\big);L_2(\Lambda(h))\| &\leqslant ch^1\|\mathcal{X}_\Lambda\nabla_{(\nu,\beta)}\mathcal{R};L_2(\Lambda(h))\|\leqslant \\ &\leqslant Ch^1\|\nabla_x\mathcal{R};L_2(\Lambda(h))\|. \end{aligned}\tag{80}$$

Thus

$$h^1 \quad \Big|\big(\mathcal{X}_\Lambda \triangle_x w_1, \mathcal{R}\big)_{\Lambda(h)} - \big(\mathcal{X}_\Lambda \triangle_x w_1, \overline{\mathcal{R}}\big)_{\Lambda(h)}\Big| \leqslant$$
$$\leqslant h^2 |\omega_h|^{1/2} \|w_1; H^2(\Upsilon)\| \|\mathcal{X}_\Lambda\big(\mathcal{R} - \overline{\mathcal{R}}\big); L_2(\Lambda(h))\| \leqslant$$
$$\leqslant c\, h^3 \left(N_\Omega + N_\Lambda\right) \|\nabla_x \mathcal{R}; L_2(\Omega(h))\|.$$

In the same way, taking into account representation (19) and inequalities (75), (80) we obtain

$$h \quad \Big|\big(\mathcal{X}_\Lambda \partial_n w_1, \mathcal{R}\big)_{\partial\Lambda(h)} - \big(\mathcal{X}_\Lambda \partial_n w_1, \overline{\mathcal{R}}\big)_{\partial\Lambda(h)}\Big| \leqslant$$
$$\leqslant h\Big(\mathrm{mes}\{\partial\Lambda(h)\}\Big)^{1/2} \|w_1; H^2(\Upsilon)\| \|\mathcal{X}_\Lambda\big(\mathcal{R} - \overline{\mathcal{R}}\big); L_2(\partial\Lambda(h))\| \leqslant$$
$$\leqslant c\, h^3 \left(N_\Omega + N_\Lambda\right) \|\nabla_x \mathcal{R}; L_2(\Omega(h))\|.$$

For the analysis of the term $h(\mathcal{X}_\Lambda \partial_n w_1, \overline{\mathcal{R}})_{\partial\Lambda(h)}$ we list several basic relations for this curvilinear cylinders.

Lemma 3. *1. Assume that Y is a smooth function defined on $\overline{\Lambda(1)}$. Then, for $\tau \in [-l, l]$ we have the equality*

$$(81) \qquad \int\limits_{\partial\Lambda(1)} Y(\tau,\zeta,\eta)\, dS = \int\limits_{-l}^{l} \Big\{ \int\limits_{\partial\omega(\tau)} Y(\tau,\zeta,\eta) \sqrt{1 + N_0(\tau,\zeta,\eta)^2}\, ds_{(\zeta,\eta)} \Big\} d\tau.$$

2. For a smooth function Y on $\overline{\Lambda(h)}$ with $\tau \in [-l, l]$ we have

$$\int\limits_{\partial\Lambda(h)} Y(\tau, h^{-1}\nu, h^{-1}\beta) dS =$$
$$(82) \quad = \int\limits_{-l}^{l} \Big\{ \int\limits_{\partial\omega_h(\tau)} Y(\tau, h^{-1}\nu, h^{-1}\beta) \sqrt{1 + h^2 N_0(\tau, h^{-1}\nu, h^{-1}\beta)^2}\, ds_{(\nu,\beta)} \Big\} d\tau.$$

3. The derivative of the area of the cross-section is given by

$$(83) \qquad \frac{d}{d\tau}|\omega_h| - h^1 \int\limits_{\partial\omega_h(\tau)} N_0(\tau, h^{-1}\nu, h^{-1}\beta)\, ds_{(\nu,\beta)}.$$

Proof. **1.** For small $\varepsilon > 0$ the interval $[-l; l]$ is divided into small intervals denoted by l_ε. In the same way as in the proof of Lemma 1, the domains $q(\tau)$ are defined for $\tau \in l_\varepsilon$ by (25). The well known formula for evaluation of a surface integral can be rewritten in the following way

$$\int\limits_{\partial\Lambda(1)} Y(\tau,\zeta,\eta)dS =$$
$$= \int\limits_{\tau_0-\varepsilon}^{\tau_0+\varepsilon}\int\limits_{Z_0}^{Z_1}\int\limits_{H_0}^{H(\tau,\zeta)} Y(\tau,\zeta,\eta)\left(1+\frac{\partial H}{\partial\tau}(\tau,\zeta)^2+\frac{\partial H}{\partial\zeta}(\tau,\zeta)^2\right)^{1/2} d\tau d\zeta d\eta.$$

By (26) we have the equality $(1+N_0(\tau,\zeta)^2)^{1/2}(1+\partial H/\partial\zeta(\tau,\zeta)^2)^{-1/2}(1+\partial H/\partial\tau(\tau,\zeta)^2+\partial H/\partial\zeta(\tau,\zeta)^2)^{1/2}$, which combined with (27), (28) leads to

$$\int\limits_{\tau_0-\varepsilon}^{\tau_0+\varepsilon}\int\limits_{\partial\Lambda(1)\cap q(\tau)} Y(\tau,\zeta,\eta)dS = \tag{84}$$
$$= \int\limits_{\tau_0-\varepsilon}^{\tau_0+\varepsilon}\Big\{\int\limits_{\partial\omega(\tau)\cap q(\tau)} Y(\tau,\zeta,\eta)\sqrt{1+N_0(\tau,\zeta)^2}ds_{(\zeta,\eta)}\Big\}d\tau.$$

Summing up (84) over all l_ε and q results in (81).

2. follows by taking $(\zeta,\eta)=(h^{-1}\nu,h^{-1}\beta)$ and applying the same argument as in the proof of part **1**.

3. follows from (23) for $Y=1$ and $\omega(\tau)=\omega_h(\tau)$, $(\zeta,\eta)=(h^{-1}\nu,h^{-1}\beta)$. □

By a transformation of $h(\mathcal{X}_\Lambda\partial_n w_1,\overline{\mathcal{R}})_{\partial\Lambda(h)}$, applying (18) and (82) leads to

$$\begin{aligned} h(\mathcal{X}_\Lambda\partial_n w_1,\overline{\mathcal{R}})_{\partial\Lambda(h)} &= h((1+h^2N_0)^{-1/2}\mathcal{X}_\Lambda\frac{hN_0}{G}\frac{dw_1}{d\tau},\overline{\mathcal{R}})_{\partial\Lambda(h)} = \\ &= -h^2\int\limits_{-l}^{l}\Big\{\int\limits_{\partial\omega_h(\tau)}(1+h^2N_0(\tau,h^{-1}\nu,h^{-1}\beta)^2)^{-1/2}\cdot \\ &\quad\cdot\mathcal{X}_\Lambda(\tau)N_0(\tau,h^{-1}\nu,h^{-1}\beta)G(\tau,\nu,\beta)^{-1}\cdot \\ &\quad\cdot\frac{dw_1}{d\tau}(\tau)\overline{\mathcal{R}}(\tau)\mathcal{X}_\Lambda(\tau)\cdot \\ &\quad\cdot 1+h^2N_0(\tau,h^{-1}\nu,h^{-1}\beta)^2)^{1/2}ds_{(\nu,\beta)}\Big\}d\tau = \dots \end{aligned}\tag{85}$$

$$\begin{aligned} \ldots \quad = \quad & -h^2 \int_{-l}^{l} \overline{\mathcal{R}}(\tau) \frac{dw_1}{d\tau}(\tau) \cdot \\ & \cdot \int_{\partial\omega_h(\tau)} N_0(\tau, h^{-1}\nu, h^{-1}\beta) ds_{(\nu,\beta)} d\tau + \\ & + h^2 \int_{-l}^{l} \overline{\mathcal{R}}(\tau) \mathcal{X}_\Lambda(\tau) \frac{dw_1}{d\tau}(\tau) \cdot \\ & \cdot \int_{\partial\omega_h(\tau)} N_0(\tau, h^{-1}\nu, h^{-1}\beta)(1 - G(\tau,\nu,\beta)^{-1}) ds_{(\nu,\beta)} d\tau. \end{aligned}$$

From

$$|1 - G(\tau,\nu,\beta)^{-1}| \leqslant c(|\nu| + |\beta^2|) \quad \text{for} \quad x \in \overline{\omega_h}, \tag{86}$$

we derive the inequality $|1 - G(\tau,\nu,\beta)^{-1}| \leqslant ch^2$ on the boundary $\partial\Lambda(h)$. Therefore, the last integral in (85) can be estimated, taking into account (37) and the inequality

$$h^1 \|\overline{\mathcal{R}}; L_2(\Upsilon)\| \leqslant c\|\mathcal{R}; L_2(\Lambda(h))\|.$$

Hence, from (85), in view of (83), we obtain

$$\begin{aligned} h\Big|(\mathcal{X}_\Lambda \partial_n w_1, \overline{\mathcal{R}})_{\partial\Lambda(h)} \quad & - \int_{-l}^{l} \int_{\omega_h(\tau)} \overline{\mathcal{R}}(\tau) \mathcal{X}_\Lambda(\tau) \frac{\partial w_1}{\partial \tau}(\tau) \frac{\partial |\omega_h(\tau)|}{\partial \tau} \, d\tau \Big| \leqslant \\ & \leqslant \; ch^4 \|w_1; H^2(\Upsilon)\| \, \|\mathcal{X}_\Lambda \overline{\mathcal{R}}; L_2(\Upsilon)\| \leqslant \\ & \leqslant \; ch^3 (N_\Omega + N_\Lambda) \, \|\nabla_x \mathcal{R}; L_2(\Omega(h))\|. \end{aligned} \tag{87}$$

Since by (86) we have the inequality

$$|1 - G(\tau,\nu,\beta)^{-1}| \leqslant ch^1, \qquad \text{for} \quad (\tau,\nu,\beta) \in \overline{\omega_h}, \tag{88}$$

therefore, by (13), (37) and

$$\|\overline{\mathcal{R}}; L_2(\Lambda(h))\| \leqslant c\|\mathcal{R}; L_2(\Lambda(h))\|$$

we conclude that

$$
\begin{aligned}
&h\Big|(\mathcal{X}_\Lambda \triangle_{(\tau,\nu,\beta)} w_1, \overline{\mathcal{R}})_{\Lambda(h)} - \int\limits_{-l}^{l}\int\limits_{\omega_h(\tau)} \overline{\mathcal{R}}(\tau)\mathcal{X}_\Lambda(\tau)|\omega_h(\tau)|\frac{\partial^2 w_1}{\partial \tau^2}(\tau) d\nu d\beta d\tau\Big| \leqslant \\
&\leqslant h\int\limits_{-l}^{l}\int\limits_{\omega_h(\tau)} \Big|\mathcal{X}_\Lambda(\tau)\overline{\mathcal{R}}(\tau)G(\tau,\nu,\beta)^{-1}\cdot \\
&\cdot\frac{\partial}{\partial\tau}G(\tau,\nu,\beta)^{-1}\frac{\partial w_1}{\partial\tau}(\tau) - \frac{\partial^2 w_1}{\partial\tau^2}(\tau)\Big| d\nu d\beta d\tau \leqslant \\
&\leqslant hc\,\|w_0; H^2(\Upsilon)\|\, h^1\|(|\nu| + |\beta^2|)\mathcal{X}_\Lambda\overline{\mathcal{R}}; L_2(\Lambda(h))\| \leqslant \\
&\leqslant c\,h^3(N_\Omega + N_\Lambda)\,\|\nabla_x\mathcal{R}; L_2(\Omega(h))\|.
\end{aligned}
\tag{89}
$$

On the other hand, using (29), (87) and (89), we obtain

$$
\begin{aligned}
&\Big|(\mathcal{X}_\Lambda \triangle_{(\tau,\nu,\beta)} w_1, \overline{\mathcal{R}})_{\Lambda(h)} - (\mathcal{X}_\Lambda \partial_n w_1, \overline{\mathcal{R}})_{\partial\Lambda(h)} + \\
&+\int\limits_{-l}^{l}\int\limits_{\omega_h(\tau)} \mathcal{X}_\Lambda(\tau)\overline{\mathcal{R}}(\tau)|\omega_h|(\tau)\overline{f}^1_\Lambda(\tau)\, d\nu d\beta d\tau\Big| \leqslant \\
&\qquad\qquad \leqslant c\,h^3(N_\Omega + N_\Lambda)\,\|\nabla_x\mathcal{R}; L_2(\Omega(h))\|.
\end{aligned}
$$

By $(37)_1$ and Proposition 1 the following estimate is valid:

$$
|(\mathcal{R}, \chi_\Lambda f_\Lambda)_{\Lambda(h)}| \leqslant c\,h^{5/2}N_\Lambda\,\|\nabla_x\mathcal{R}; L_2(\Omega(h))\|.
$$

We make the precision in what sense the remainder in $(4)_2$ can be made arbitrarily small

$$
\begin{aligned}
&\widetilde{f}(h,x) = \widetilde{f}_0(h,x) + \widetilde{f}_\perp(h,\tau,\nu,\beta), \\
&\int\limits_{\omega_h(\tau)} \widetilde{f}_\perp(h,\tau,\nu,\beta)\, d\nu d\beta = 0, \qquad \tau \in \big(-l-2h\lambda, l+2h\lambda\big), \\
&\widetilde{N}_\Lambda : \big\{h^{-3}\|\widetilde{f}_0; L_2(\Lambda(h))\| + h^{-2}\|\widetilde{f}_\perp; L_2(\Lambda(h))\|\big\},
\end{aligned}
\tag{90}
$$

where the term $\widetilde{N}_\Lambda$ is of order $O(1)$, and the length $\lambda > 0$ is introduced in $(90)_2$ for the same reason as before in (53).

On the basis of assumptions (90), from (57), (80), (88) it follows that

$$\begin{aligned}\left|\big(f-f_\Lambda,\mathcal{R}\big)_{\Lambda(h)}\right| &= \Big|(\widetilde{f}_0,\mathcal{R}\big)_{\Lambda(h)}+\big(\widetilde{f}_\perp,\mathcal{X}_\Lambda(\mathcal{R}-\overline{\mathcal{R}})\big)_{\Lambda(h)}+\\ &\quad+\int\limits_{-l}^{l}\int\limits_{\omega_h(\tau)}\mathcal{X}_\Lambda(\tau)\overline{\mathcal{R}}(\tau)(G(\tau,\nu,\beta)^{-1}-1)\cdot\\ &\quad\cdot\widetilde{f}_\perp(h,\tau,\nu,\beta)\,d\nu d\beta d\tau\Big|\leqslant\\ &\leqslant c\left(N_\Lambda+\widetilde{N}_\Lambda\right)\Big\{h^3\|d_p^{-1}\mathcal{R};L_2(\Lambda(h))\|+\\ &\quad+h^2\|\mathcal{X}_\Lambda(\mathcal{R}-\overline{\mathcal{R}});L_2(\Lambda(h))\|+h^3\|\overline{\mathcal{R}};L_2(\Upsilon)\|\Big\}\leqslant\\ &\leqslant c\,h^3\big(N_\Lambda+\widetilde{N}_\Lambda\big)\|\nabla_x\mathcal{R};L_2(\Omega(h))\|.\end{aligned}$$

Finally, we are going to establish the estimates for the last terms in $(68)_{1,2}$ and $(69)_{1,2}$.

Let us observe that the function $\widetilde{z}_1^{\pm}(\mathcal{D}_\pm(h,\cdot))$ can be made continuous on $\mathbb{R}^3_+$ by an addition of a constant, which we are allowed to make without any problem, see the comments to formula (67). By an integration by parts, the terms in (70) form the following sum

$$\begin{aligned}&h\big(\nabla_x\widetilde{z}_1^{\pm}(\mathcal{D}_\pm(h,\cdot)),\nabla_x(\chi^\pm\mathcal{R})\big)_{\Omega(h)}+\\ &-\big(\big[\triangle_x,\chi_\Omega(h^{-1}r_\pm)\big]\mathcal{Y}^\pm,\chi^\pm\mathcal{R}\big)_\Omega+\\ &+\big(\big[\partial_n,\chi_\Omega(h^{-1}r_\pm)\big]\mathcal{Y}^\pm,\chi^\pm\mathcal{R}\big)_{\partial\Omega\setminus\Lambda_h}+\\ &-\big(\big[\triangle_x,\chi_\Lambda(h^{-1}(\tau\mp l))\big]\mathcal{Z}^\pm,\chi^\pm\mathcal{R}\big)_{\Lambda(h)}+\\ &+\big(\big[\partial_n,\chi_\Lambda(h^{-1}(\tau\mp l))\big]\mathcal{Z}^\pm,\chi^\pm\mathcal{R}\big)_{\partial\Lambda_h\setminus\Omega},\end{aligned}\tag{91}$$

where $\chi^\pm(x)=\chi^\pm(r_\pm)$ in Ω and $\chi^\pm(x)=\chi^\pm(\tau\mp l)$ on $\Lambda(h)$, the functions $\mathcal{Y}^\pm$ and $\mathcal{Z}^\pm$ are defined in $(64)_3$. Let us perform the change of variables $\xi^\pm=\mathcal{D}_\pm(h,x)$ in (91) and, therefore, replace in (91) integrals over $\Omega(h)$, Λ_h and $\partial\Omega\setminus\Lambda_h$, $\partial\Lambda_h\setminus\Omega$ by integrals over $\mathbb{R}^3_+$, $\Pi^\pm$ and $\partial\mathbb{R}^3_+\setminus\Pi^\pm$, $\partial\Pi^\pm\setminus\mathbb{R}^3_+$, recpectively. In other worlds we obtain the integrals over the domain $\Xi^\pm$ and its boundary with the *test* function $\xi^\pm\mapsto\chi^\pm(\mathcal{D}_\pm^{-1}(h,\xi^\pm))\mathcal{R}(\mathcal{D}_\pm^{-1}(h,\xi^\pm))$. Let us note that the value $\widetilde{z}_1^{\pm}(\xi^\pm)$ decays with $|\xi^\pm|\to\infty$ not slower than $O(|\xi^\pm|^{-2})$ (see $(42)_2$, $(43)_2$, $(44)_2$ and (67)), and the supports of the remaining integrands are included in the set $\{x:r_\pm<ch\}$. At the same time,

the changes of variables

$$\nabla_x \mapsto h^{-1}\nabla_{\xi^\pm}, \qquad \partial_n \mapsto h^{-1}\nabla_{\xi^\pm};$$
$$h^{-1}(\tau \mp l) \mapsto \mp\xi^\pm \qquad \text{on} \quad \Pi^\pm \setminus \mathbb{R}^3_+,$$
$$h^{-1}r_\pm \mapsto \rho^\pm \qquad \text{on} \quad \mathbb{R}^3_+,$$

lead to a perturbation of differential operators in (91) and, in particular, the appearance of the Jacobian. In view of (61) and the above information on the integrands, the errors due to neglecting the Jacobian, can be estimated by the expression

$$c\,h^3\big(N_\Omega + N_\Lambda\big)\|\nabla_x\mathcal{R}; L_2(\Omega(h))\|.$$

The sum of remained integrals equals zero, since it is a part of the integral identity satisfied by the function $\widetilde{z}_1^\pm$ (compare with the definitions given in (66) and (67)). We verify the above established estimate on an example of the first scalar product in (91)

$$\begin{aligned} &h\Big|\Big(\nabla_x\widetilde{z}_1^{\,\pm}(\mathcal{D}_\pm(h,\cdot)), \nabla_x(\chi^\pm\mathcal{R})\Big)_{\Omega(h)} - \textstyle\sum_\pm \Big(\nabla_\xi\widetilde{z}_1^{\,\pm}, \nabla_\xi\mathcal{R}^\pm\Big)_{\Xi^\pm}\Big| \leqslant \\ &\leqslant ch\Big|\textstyle\sum_\pm \Big(\Big\{\big|\det X_\pm\big|^{-1}X_\pm^\top X_\pm - \mathbb{I}_3\Big\}\nabla_\xi\widetilde{z}_1^{\,\pm}, \nabla_\xi\mathcal{R}^\pm\Big)_{\Xi^\pm}\Big|. \end{aligned} \tag{92}$$

Here $X_\pm(h,\xi^\pm)$ is the Jacobian of the mapping $x \mapsto \xi^\pm$, $\top$ denotes the transposition and $\mathcal{R}^\pm(\xi^\pm) = \chi^\pm(x)\mathcal{R}(x)$. From formula $(61)_2$ and (42), (43), (44), (66), (67) we obtain

$$\Big|\big|\det X_\pm(h,\xi^\pm)\big|^{-1}X_\pm(h,\xi^\pm)^\top X_\pm(h,\xi^\pm) - \mathbb{I}_3\Big| \leqslant ch^2(1+|\xi^\pm|)^{-1},$$

$$\big|\nabla_\xi\widetilde{z}_1^{\,\pm}(\xi^\pm)\big| \leqslant c\,\big(N_\Omega + N_\Lambda\big)(1+|\xi^\pm|)^{-1},$$

therefore, the expression (92) is dominated by

$$c\,h^3\Big|\sum_\pm \int\limits_{\Xi^\pm}(1+|\xi^\pm|)^1\big|\nabla_\xi\widetilde{z}_1^{\,\pm}(\xi^\pm)\big|\;\big|\nabla_\xi\mathcal{R}^\pm(\xi^\pm)\big|\,d\,\xi^\pm\Big| \leqslant$$
$$\leqslant c\,h^4\big(N_\Omega + N_\Lambda\big)\sum_\pm \|\nabla_\xi\mathcal{R}^\pm; L_2(\Xi^\pm)\|.$$

Now, the required estimate follows by using the inverse mappings $\xi^\pm \mapsto x$ in the latter norms.

Therefore, the upper bounds are derived for all terms of equality (70). It is easy to see, that for the choice of $\mu = 3/2 - \varepsilon$, with a small $0 < \varepsilon < 1/2$, the *worst* term is of order $h^{5/2-\varepsilon}$. We recall that $\mathcal{R} = u - \mathcal{U}$, and establish the result on the error of approximation by taking into account Proposition 1.

Theorem 1. *Assume that the function f satisfies conditions* (4), (36), $(90)_{1,2}$ *and* (7) *with $\mu = 1/2$. Then the solution $u \in \overset{o}{H}{}^1(\Omega(h);\Sigma)$ of problem* (3) *and its asymptotic approximation* (62) *are related by the inequality*

$$\begin{aligned} \|d(u-\mathcal{U});L_2(\Omega)\| + \|\nabla_x(u-\mathcal{U});L_2(\Omega)\| \leqslant \\ \leqslant C\,h^{5/2-\varepsilon}\big\{N_\Omega + \widetilde{N}_\Omega + N_\Lambda + \widetilde{N}_\Lambda\big\}, \end{aligned} \tag{93}$$

where ε is a arbitrarily small positive number, d is the weight in (58), *N_Ω and $\widetilde{N}_\Omega$ are the parameters from* (9) *and* (73) *for $\mu = 1/2$, and N_Λ, $\widetilde{N}_\Lambda$ are defined by* (38), $(90)_3$, *respectively. The constant C is independent either of $h \in (0, h_0]$, or the terms in the decompositions of f in* (4) *and* $(90)_1$.

7. Asymptotics of the energy functional

The energy functional for solutions u of problem (3)

$$E_h(u) = \int\limits_{\Omega(h)} \Big\{\Big|\nabla_x u(h,x)\Big|^2 - 2u(h,x)f(h,x)\Big\}dx, \tag{94}$$

can be rewritten, using the Green's formula, as follows

$$E_h(u) = -\int\limits_{\Omega(h)} \Big|\nabla_x u(h,x)\Big|^2 dx = -\int\limits_{\Omega(h)} u(h,x)f(h,x)dx. \tag{95}$$

In the last integral, on the basis of estimate (93), the exact solution u can be replaced by its approximation $\mathcal{U}$. Furthermore, the boundary layers $\widetilde{z}_1^{\pm}$ and all cut-off functions can be neglected in such a substitution which uses form (62) with representations $(63)_1$, $(63)_2$ of the approximate solution $\mathcal{U}$. The error committed is of the same order as with the error due to the precision of the asymptotic approximation. As an example let us check the error for the boundary layer. To this end, we observe that the inequalities $|\widetilde{z}_1^{\pm}(\xi^{\pm})| \leqslant c\,|\xi^{\pm}|^{-2}$ (see (42), (43), (44)) lead to the estimate

$$\begin{aligned} h^1\Big|\int\limits_{\Omega(h)} f\chi^{\pm}\widetilde{z}_1^{\pm}\,d\,x\Big| \;&\leqslant\; h^1\Big|\int\limits_{\Omega(h)} \chi^{\pm}(h+d_P)^{-2}f^2\,d\,x\Big|^{1/2}\cdot \\ &\cdot\Big|\int\limits_{\Omega(h)} \chi^{\pm}(h+d_P)^2\big|\widetilde{z}_1^{\pm}\big|^2 d\,x\Big|^{1/2} \leqslant \\ &\leqslant\; c\,h^3\big\{N_\Omega + \widetilde{N}_\Omega + N_\Lambda + \widetilde{N}_\Lambda\big\}^2. \end{aligned}$$

The remaining terms are estimated in the same way as in the previous section. Therefore,

$$
\begin{aligned}
E_h(u) &= -\int_\Omega \{v_0(x) + h^2 v_2(x)\} f_\Omega(x)dx + \\
&\quad - \int_{\Lambda(h)} w_0(\tau) f_\Lambda^0(\tau)dx + O(h^{5/2-\varepsilon}) = \\
&= E_0(v_0) - h^2 \int_\Omega v_2(x) f_\Omega(x)dx + \\
&\quad -h^2 \int_\Upsilon |\omega(\tau)| w_0(\tau) f_\Lambda^0(\tau)d\tau + O(h^{5/2-\varepsilon}).
\end{aligned} \tag{96}
$$

Taking into account that the functions w_0 and v_2 are given by solutions of problems (29), (30) and (45), respectively, it follows that

$$
\begin{aligned}
&-\int_\Upsilon w_0(\tau)|\omega(\tau)| f_\Lambda^0(\tau)d\tau = -\int_\Upsilon w_0(\tau)\partial_\tau|\omega(\tau)|\partial_\tau w_0(\tau)d\tau = \\
&\quad = -\int_\Upsilon |\omega(\tau)||\partial_\tau w_0(\tau)|^2 d\tau + \sum_\pm \pm v_0(P^\pm)|\omega(\pm l)|\partial_\tau w_0(\pm l), \\
&-\int_\Omega v_2 f_\Omega dx = -\int_\Omega v_2 \triangle_x v_0 dx - \sum_\pm \pm|\omega(\pm l)|\partial_\tau w_0(\pm l) v_0(P^\pm).
\end{aligned}
$$

Recall that $|\omega(\tau)|$ stands for the area of the cross-section $\omega(\tau)$. The last equality has only a meaning in the sense of distributions, δ being the Dirac mass in the boundary conditions $(45)_2$. The equivalent approach in the derivation consists in integration by parts over the domain $\{x \in \Omega : r_\pm > d\}$ followed by the limit passage with $d \to +0$. Finally, we substitute the obtained formula in (96) and come to the next statement.

Proposition 2. *Under the assumptions of Theorem 1 on the right-hand side f in problem* (3), *energy functional* (94) *enjoys the asymptotic representation*

$$
E_h(u) = E_0(v_0) - h^2 \int_\Upsilon |\omega(\tau)||\partial_\tau w_0(\tau)|^2 d\tau + \widetilde{E}_h(u), \tag{97}
$$

where

$$
\big|\widetilde{E}_h(u)\big| \leqslant c\, h^{5/2-\varepsilon}\big\{N_\Omega + \widetilde{N}_\Omega + N_\Lambda + \widetilde{N}_\Lambda\big\}^2,
$$

and the constant c *is independent of* h *and* f. *The expression*

$$E_0(v_0) = \int_\Omega \Big\{ \big|\nabla_x v_0(x)\big|^2 - 2v_0(x) f_\Omega(x) \Big\} dx = - \int_\Omega \big|\nabla_x v_0(x)\big|^2 dx \tag{98}$$

is the energy functional for the limit problem (5).

Remark 1. *There is no characteristics of boundary layer included in the asymptotic formula* (97). *Such a form remains valid for the case of small perturbations of the boundary, of the size* $O(h)$, *in the regions of the junction of the ligament* $\Lambda(h)$ *to the body* Ω_0 *(this kind of perturbations can be compared with those considered in* [20]*). In particular, the edges of the boundary formed by the set* (1) *with the boundary* $\partial\Omega_0$ *can be smoothed so the new boundary by* $\partial\Omega(h)$ *becomes smooth. Such a procedure does not change the topological derivative of the energy functional.*

Let us point out, that in view of representation (10) the subtrahend in the right-hand side of (97) is the leading term of the Dirichlet integral of the solution $u(h, x)$ projected on the ligament. The corresponding integral over the domain Ω_0 is included in the leading term (98) of asymptotics (97). Formula (97) can be rewritten as follows

$$\frac{dE_h}{dh}(u)\bigg|_{h=0} = - \int_\Upsilon |\omega(\tau)| \partial_\tau |w_0(\tau)|^2 d\tau, \tag{99}$$

and, according to the previous investigations (see [43, 37, 36] and others), expression (99) can be seen as the *exterior topological derivative* of functional (94) due to formation of a ligament of *thickness* h.

Example 1. *Let* $\Omega = \{x : x_1^2 + x_2^2 + x_3^2 \in (1, 4)\}$ *be a spherical annulus included between two spheres* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *of radii* $R = 1, 2$, *respectively. The Dirichlet conditions* $(3)_3$ *are prescribed on* $\Sigma = \mathbb{S}_2$, *and the right-hand side of the equation* $(3)_2$ *in* Ω *is independent of parameter* h *and takes the form*

$$f(r) = 30\frac{2}{3} - 50\frac{2}{3}r - 20r^2,$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. *The function* f *is selected in such a way that the solution* v_0 *to the limit problem* (5), *is given by*

$$v_0(r) = \frac{1}{9}(14r^{-1} - 31 + 46r^2 - 38r^3 + 9r^4)$$

and at the same time vanishes on the smaller sphere $\mathbb{S}_1$ *with the first order and the second order derivatives. Such a behaviour of the function* v_0

combined with the results of [42] *imply that the energy functional* (94) *in the domain*

$$\widehat{\Omega}(h) = \{x : r(x) \in (1 - h\mathcal{R}(s), 2)\},$$

obtained by the smooth perturbation of the boundary i.e., $\mathcal{R} \in C^\infty(\mathbb{S}_1)$ *and* $\mathcal{R} > 0$*, differs from the limit value of the functional* (98) *in* $\Omega = \widehat{\Omega}(0)$ *by the term of order* $O(h^6)$*. Beside that, for nonsmooth perturbations of the boundary, we refer to* [20], *Chapter 2 for the details, functional* (94) *becomes greater, with an increment of order* $O(h^3)$ *due to a singular perturbation in the vicinity of a point at the boundary* $\mathbb{S}_1$*. Such a singular perturbation is obtained by gluing to the interior sphere of a tubercle of the diameter* $h^{2/3}$ *and the volume* $O(h^3)$ *of the same order as the volume of* $\mathrm{mes}_3(\widehat{\Omega}(h) \setminus \Omega)$*. However, the formation of the ligament* $\Lambda(h) = \Lambda_h \cap \Omega$*,*

$$\Lambda_h = \{x : |x_1^2 + x_2^2| < h, |x_3| < 3/2\},$$

results in the increment of the functional of order h^2*, by Proposition 2, since the example we have problem* (29), (30) *with*

$$l = 2, \quad \tau \in (0, 2), \quad f_\Lambda(\tau) = f(r), \quad v_0(P^\pm) = 0,$$

which means that

$$w_0(\tau) = \frac{\tau}{9}\Big\{ - 15\tau^3 + 76\tau^2 + 552\tau - 1288\Big\}$$

and the topological derivatives is different from zero.

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