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STOCHASTIC PROCESSES OF VECTOR VALUED PETTIS AND MCSHANE INTEGRABLE FUNCTIONS

VALERIA MARRAFFA[‡]

Abstract. Convergence theorems of Pettis integrable martingales and more general stochastic processes taking values in a Banach space without the weak Radon–Nikodym property, are considered. Also McShane integrable martingales are studied.

1. Introduction

In this paper we study properties of stochastic processes, consisting of weakly measurable functions, taking values in a Banach space. For a Banach–valued martingale of Pettis integrable functions, the weak Radon–Nikodym property is equivalent to the convergence in Pettis norm (see [8]). Without assuming this property we ask for which smaller class T of functionals f the a.s. convergence of fX_n to fX for $f \in T$ implies the convergence of X_n to X in Pettis norm. For Bochner integrable stochastic processes it was shown that T can be a total set (see [2] and [7]). In Section 3 we prove analogous results for Pettis – integrable martingales. In [6] convergence theorems for more general sequences are proved under the hypothesis that the range is relatively weakly compact or has the weak Radon-Nikodym property. Using decomposition theorems we obtain convergence results for stochastic processes taking values in a space which does not possess the weak Radon–Nikodym property (see Theorem 4 and Theorem 5).

In the fourth section martingales of McShane integrable functions are considered. In particular it is proved that a closed martingale converges in the McShane seminorm (Proposition 1). For a Pettis integrable closed martingale the same is true if the range of the indefinite integral is norm relatively compact (see [8] Corollary 1). Also a convergence theorem for McShane integrable martingales is proved (see Theorem 7).

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[‡] *University of Palermo, Department of Mathematics*, Archirafi 34 St., 90123 Palermo, Italy. E-mail: marraffa@math.unipa.it.

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2. Definitions and notations

Let E be a Banach space with norm $\|\cdot\|$, $B(E)$ its unit ball and E^* its dual. A subset T of E^* is called a *total set* over E, if $f(x) = 0$ for each $f \in T$ implies $x = 0$.

Throught this note $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_{n\in I\!N}$ a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$, if $m < n$. Moreover, without loss of generality, we will assume that $\mathcal{F}$ is the completion of $\sigma(\cup_n \mathcal{F}_n)$.

Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$, then a function $X : \Omega \to E$ is called *weakly $\mathcal{F}_0$-measurable*, if the function fX is $\mathcal{F}_0$-measurable for every $f \in E^*$. A weakly $\mathcal{F}$-measurable function is called *weakly measurable*. A function $X : \Omega \to E$ is said to be *Pettis integrable*, if fX is Lebesgue integrable on Ω for each $f \in E^*$ and there exists a set function $\nu : \mathcal{F} \to E$ such that

$$f\nu(A) = \int_A fX$$

for all $f \in E^*$ and $A \in \mathcal{F}$. In this case we write $\nu(A) = P\int_A X$ and we call $\nu(\Omega)$ the *Pettis integral* of X *over* Ω and ν is the *indefinite Pettis integral of* X.

The space of all E-valued Pettis integrable functions is denoted by $\mathcal{P}(\Omega, \mathcal{F}, P; E)$ or simply $\mathcal{P}(E)$. The Pettis norm of a Pettis integrable functions is

$$|X|_P = \sup\left\{\int_\Omega |fX| : f \in B(E^*)\right\}.$$

It is well known that

$$\sup\left\{\left\|P\int_A X\right\| : A \in \mathcal{F}\right\}$$

defines an equivalent norm in $P(E)$.

We say that $(X_n, \mathcal{F}_n)$ is a *stochastic process of Pettis integrable functions*, if for each $n \in I\!N$, $X_n : \Omega \to E$ is Pettis integrable, X_n is weakly $\mathcal{F}_n$-measurable and the *Pettis conditional expectation* $E(X_n|\mathcal{F}_m)$ of X_n exists for all $n \geq m$. It should be noted that, in general, if X is only Pettis integrable, even it is strongly measurable, there is no *Pettis conditional expectation* of X with respect to a sub-σ-algebra of $\mathcal{F}$. We say that the stochastic process $(X_n, \mathcal{F}_n)$ is a *martingale*, if $E(X_n|\mathcal{F}_m) = X_m$ for $n \geq m$.

We say that a martingale $(X_n, \mathcal{F}_n)$ is

(i) *convergent* in $\mathcal{P}(E)$, if there exists a function $X \in \mathcal{P}(E)$ such that

$$\lim_{n\to\infty} |X_n - X|_P = 0;$$

(ii) *uniformly bounded*, if $\sup_n |X_n|_P < +\infty$;

(iii) *uniformly continuous*, if $\lim_{P(A)\to 0} P\int_A X_n = 0$ uniformly with respect to n;
(iv) *uniformly integrable*, if it is uniformly bounded and uniformly continuous.

3. Convergence of Pettis integrable stochastic processes

We deal with functions which are Pettis integrable. To prove the following theorem we use an idea of [5].

Theorem 1. *Let $(X_n, \mathcal{F}_n)$ be a uniformly integrable martingale of Pettis integrable functions and let X be a weakly measurable function. Moreover let T be a weak*-sequentially dense subset of E^*, and assume that fX_n converges to fX a.s. for each $f \in T$ (the null set depends on f). Then $X \in \mathcal{P}(E)$ and for each $A \in \mathcal{F}$*

$$\lim_n P\int_A X_n = P\int_A X.$$

Moreover if the martingale $(X_n, \mathcal{F}_n)$ is Cauchy in $\mathcal{P}(E)$, then X_n converges to X in the Pettis norm.

Proof. Let $A \in \cup_n \mathcal{F}_n$ and denote by

$$\nu(A) = \lim_n P\int_A X_n.$$

Since $(X_n, \mathcal{F}_n)$ is uniformly integrable, $\nu : \cup_n \mathcal{F}_n \to E$ is of bounded variation and countably additive, hence it is strongly additive. By the Caratheodory–Hahn–Kluvanek extension theorem, ν can be extended to a countably additive measure on $\mathcal{F}$, absolutely continuous with respect to P and of bounded variation. Moreover

$$\nu(A) = \lim_n P\int_A X_n$$

exists for all $A \in \mathcal{F}$. Indeed let $A \in \mathcal{F}$ and fix $\varepsilon > 0$. Since (X_n) is uniformly continuous there is $\delta > 0$ such that, if $P(D) < \delta$ then $\|P\int_D X_n\| < \varepsilon$ uniformly with respect to n. Therefore there is $n_1 \in I\!N$ and $B \in \mathcal{F}_{n_1}$ such that $P(A \triangle B) < \delta$ (see [4], p. 56). If $n_2 \geq n_1$, then $\|P\int_B X_n - P\int_B X_m\| < \varepsilon$ for each $n, m \geq n_2$. Thus

$$\begin{aligned} \|P\textstyle\int_A X_n - P\int_A X_m\| &\leq \|P\textstyle\int_B X_n - P\int_B X_m\| + \\ + \quad \|P\textstyle\int_{B\setminus A} X_n\| + \|P\int_{B\setminus A} X_m\| &+ \|P\textstyle\int_{A\setminus B} X_n\| + \|P\int_{A\setminus B} X_m\| < 5\varepsilon. \end{aligned}$$

Therefore $\nu(A) = \lim_n P\int_A X_n$ exists for all $A \in \mathcal{F}$. For each $f \in T$ and $A \in \mathcal{F}$, by Vitali Theorem, we get

$$f\nu(A) = \lim_n \int_A fX_n = \int_A fX. \tag{1}$$

Also, for each $f \in E^*$, $(fX_n, \mathcal{F}_n)$ is a real valued L^1-bounded martingale, then there exists a real random variable X_f such that

$$f\nu(A) = \lim_n \int_A fX_n = \int_A X_f.$$

Let $f \in E^*$ and choose $(f_k)_k \subset T$ such that, for each $x \in E$, $f_k(x)$ converges to $f(x)$. For each $k \in I\!N$

$$\left|\int_A f_k X\right| = |f_k\nu(A)| \leq \|f_k\|\|\nu(A)\|$$

and since by Banach–Steinhaus theorem $\sup_k \|f_k\| < \infty$, the sequence $(f_kX)_k$ is uniformly L^1-integrable. Then by Vitali theorem

$$\lim_k \int_A f_kX = \int_A fX \tag{2}$$

and X is scalarly integrable. Now by (1), for each $k \in I\!N$ and $A \in \mathcal{F}$

$$f_k\nu(A) = \int_A f_kX.$$

Also

$$\lim_k f_k\nu(A) = f\nu(A). \tag{3}$$

Then (1), (2) and (3) imply

$$f\nu(A) = \int_A fX,$$

for all $f \in E^*$ and $A \in \mathcal{F}$. Thus X is Pettis integrable and $\nu(A) = P\int_A X$ for all $A \in \mathcal{F}$. Also we have

$$\lim_n P\int_A X_n = \nu(A) = P\int_A X.$$

Since $(X_n)_n$ is a martingale this implies that $E(X|\mathcal{F}_n) = X_n$. Assume now that (X_n) is Cauchy in $\mathcal{P}(E)$. Applying the Doob–Helms theorem for the scalar valued case we get the required convergence. □

Corollary 1. *Let $(X_n, \mathcal{F}_n)$ be a uniformly integrable martingale of Pettis integrable strongly measurable functions, X a strongly measurable function. Let T be a total subset of X^*, and assume that fX_n converges to fX a.s. for* *each $f \in T$ (the null set depends on f). Then $X \in \mathcal{P}(E)$ and X_n converges to X in the Pettis norm.*

Proof. By Pettis measurability theorem we can assume that E is separable, then since T is closed and *weak**-dense, from the previous theorem we get that X is Pettis integrable and $E(X|\mathcal{F}_n) = X_n$ for $n \in I\!N$. Therefore the assertion follows from [13], Lemma 1.4. $\square$

Now we consider more general sequences $(X_n, \mathcal{F}_n)$ and we assume that X_n is strongly measurable for each $n \in I\!N$.

Definition 1. *A stochastic process $(X_n, \mathcal{F}_n)$ of Bochner integrable functions is said to be L^1-bounded, if $\sup_n \int_\Omega \|X_n\| < \infty$.*

Definition 2. *A stochastic process $(X_n, \mathcal{F}_n)$ of strongly measurable functions is said to be a* game which becomes fairer with time *(briefly a P-*martingale*), if for each $\varepsilon > 0$*

$$\lim_n \sup_{m \geq n} P(\|E(X_m|\mathcal{F}_n) - X_n\| > \varepsilon) = 0.$$

If for each $\varepsilon > 0$

$$\lim_n \sup_{m \geq n} P(\sup_{n \leq q \leq m} \|E(X_m|\mathcal{F}_q) - X_q\| > \varepsilon) = 0$$

the sequence $(X_n, \mathcal{F}_n)$ is called a mil*.*

We need the following decomposition theorems:

Theorem 2. ([6], Theorem 1.2) *Let $(X_n, \mathcal{F}_n)$ be an L^1-bounded P-martingale consisting of Bochner integrable functions. Then (X_n) can be written in a unique form as*

$$X_n = M_n + P_n, \quad n \in I\!N,$$

where for each $n \in I\!N$, M_n and P_n are Bochner integrable functions, (M_n) is a uniformly integrable martingale and (P_n) goes to zero in probability.

Theorem 3. ([12], Theorem 8) *Let $(X_n, \mathcal{F}_n)$ be a mil consisting of Bochner integrable functions such that $\liminf_n \int_\Omega \|X_n\| < \infty$. Then (X_n) can be written in a unique form as*

$$X_n = M_n + Z_n, \quad n \in I\!N,$$

where for each $n \in I\!N$, M_n and Z_n are Bochner integrable functions, (M_n) is a uniformly integrable martingale and (Z_n) is a mil converging to zero a.s.

Definition 3. *A stochastic process $(X_n, \mathcal{F}_n)$ of Pettis integrable functions is* σ-bounded*, if there exists an increasing sequence $(B_n)_n$, $B_n \in \mathcal{F}_n$, such that $\lim_n P(B_n) = 1$ and the sequence (X_n) restricted to each B_m, $m = 1, 2, \ldots$, is L^1-bounded.*

Theorem 4. *Let $(X_n, \mathcal{F}_n)$ be a σ-bounded P-martingale of Pettis integrable functions and X a strongly measurable function. Let T be a total subset of E^*, and assume that fX_n converges to fX a.s. for each $f \in T$ (the null set depends on f). Then X_n converges to X in probability (i.e. for every $\varepsilon > 0$ we have $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$).*

Proof. Since $(X_n, \mathcal{F}_n)$ is σ-bounded, there is an increasing sequence (B_n), $B_n \in \mathcal{F}_n$, such that $\lim_n P(B_n) = 1$ and $(X_n)_n$ restricted to each B_m, $m = 1, 2, \dots$, is L^1-bounded. For $m \in I\!\!N$, define

$$X_n^m = \begin{cases} X_n I_{B_n}, & \text{if } n \le m, \\ X_n I_{B_m}, & \text{if } n \ge m, \end{cases}$$

where the symbol I_B denotes the characteristic function of the set B. For each m the sequence $(X_n^m, n \in I\!\!N)$ is an L^1-bounded P-martingale in $L^1(E)$, therefore by Theorem 2, for $n \in I\!\!N$,

$$X_n^m = M_n^m + P_n^m, \tag{4}$$

where $(X_n^m)_n$ is a uniformly integrable martingale and $(P_n^m)_n$ goes to zero in probability. Moreover for each $f \in T$, $fX_n^m(\omega) \to fXI_{B_m}(\omega)$ for each $\omega \notin N_f$, $P(N_f) = 0$. Indeed fixed $\omega \in \Omega \setminus N_f$ and $\varepsilon > 0$ there exists n_ε such that for each $n > n_\varepsilon$ it follows

$$|f(X_n(\omega)) - f(X(\omega))| < \varepsilon.$$

If $\omega \in B_m$, then $X_n(\omega) = X_n^m(\omega)$ and $X(\omega) = XI_{B_m}(\omega)$. If $\omega \notin B_m$ $X_n^m(\omega) = 0$, $XI_{B_m}(\omega) = 0$, and

$$|f(X_n^m(\omega)) - f(XI_{B_m}(\omega))| < \varepsilon.$$

Since $(P_n^m)_n$ converges to zero in probability there exists a subsequence of $(P_n^m)_n$, still denoted by $(P_n^m)_n$ converging to zero a.s. Then for each $f \in T$, $fP_n^m \to 0$ a.s. and since

$$fX_n^m = fM_n^m + fP_n^m$$

we get

$$\lim_{n\to\infty} fM_n^m = fXI_{B_m}.$$

Since the sequence $(M_n^m)_n$ is a uniformly integrable martingale, it is L^1-bounded. Then it follows from [2] Theorem 3 or [7] Theorem 4 that

$$\lim_{n\to\infty} M_n^m = XI_{B_m} \text{ a.s.}$$

in the strong topology. Moreover $(M_n^m)_n$ converges to XI_{B_m} in probability. We want to prove that (X_n) converges to X in probability. By hypothesis there is $m_0 \in I\!\!N$ such that

$$P(B_{m_0}) > 1 - \frac{\varepsilon}{2}.$$

Also $(X_n^{m_0})_n$ converges in probability to $XI_{B_{m_0}}$, then there exists $p \geq m_0$ so that for all $n \geq p$,

$$P(\|X_n^{m_0} - XI_{B_{m_0}}\| > \varepsilon) < \frac{\varepsilon}{2}.$$

Thus we get

$$P(\|X_n - X\| > \varepsilon) < P(\omega \in B_{m_0} : \|X_n - X\| > \varepsilon) + P(\omega \notin B_{m_0}) \leq$$

$$\leq P(\|X_n^{m_0} - XI_{B_{m_0}}\| > \varepsilon) + \tfrac{\varepsilon}{2} < \varepsilon.$$

Then X_n converges to X in probability and the assertion follows. □

Applying Talagrand decomposition theorem of mils (Theorem 3) we get the following result:

Theorem 5. *Let $(X_n, \mathcal{F}_n)$ be a σ-bounded mil of Pettis integrable strongly measurable functions and X a strongly measurable function. Moreover let T be a total subset of E^*, and assume that fX_n converges to fX a.s. for each $f \in T$ (the null set depends on f). Then X_n converges to X a.s. in the strong topology.*

Proof. Find $(B_n)_n$ and define $(X_n^m : n, m \in I\!N)$ as in the previous theorem. The sequence $(X_n^m, n \in I\!N)$ is an L^1-bounded mil in $L^1(E)$. Therefore by Theorem 3, for $n \in I\!N$,

$$X_n^m = M_n^m + Z_n^m, \tag{5}$$

where $(M_n^m)_n$ is a uniformly integrable martingale and $(Z_n^m)_n$ is a mil converging to zero a.s. Moreover for each $f \in T$

$$fX_n^m = fM_n^m + fZ_n^m.$$

Then $f(M_n^m(\omega)) \to f(XI_{B_m}(\omega))$ for each $\omega \notin N_f$, $P(N_f) = 0$. As before it follows that $(M_n^m(\omega))_n$ converges in the strong topology to $XI_{B_m}(\omega)$ for all $m = 1, 2, \ldots$, and $\omega \in \Omega \setminus N$, $P(N) = 0$. By (5) also $(X_n^m)_n$ converges to XI_{B_m} a.s. in the strong topology. We want to prove that X_n converges to X a.s.. Fix $\omega \in B_{m_0}$ and $\omega \notin N$. For every positive number ε there exists $n_\varepsilon > m_0$ such that for all $n > n_\varepsilon$ we get

$$\|X_n(\omega) - X(\omega)\| = \|X_n^{m_0}(\omega) - XI_{B_{m_0}}(\omega)\| < \varepsilon.$$

Since $P(N \cup (\Omega \setminus \cup_m B_m)) = 0$ the assertion follows. □

Assuming a weaker strongly measurability condition on the martingale $(X_n, \mathcal{F}_n)$ and on the random variable X, in Theorem 1 we can weaken the hypothesis on the set T.

Theorem 6. *Let $(X_n, \mathcal{F}_n)$ be a uniformly integrable martingale of Pettis integrable functions. Assume that there exists an increasing sequence of measurable sets $(B_m)_m$, $B_m \in \mathcal{F}_m$, such that $\lim_m P(B_m) = 1$ and that the function X and the function X_n restricted to each B_m are strongly measurable, $n = 1, 2, \ldots$ Assume, moreover, that for each $f \in T$, where T is a total set, fX_n converges to fX a.s. (the null depends on f). Then $X \in \mathcal{P}(E)$ and for each $A \in \mathcal{F}$*

$$\lim_n P\int_A X_n = P\int_A X.$$

Furthermore if the martingale $(X_n, \mathcal{F}_n)$ is Cauchy in $\mathcal{P}(E)$, then X_n converges to X in the Pettis norm.

Proof. As in Theorem 1 let $\nu : \mathcal{F} \to E$ the absolutely continuous extension to $\mathcal{F}$ of bounded variation of the measure

$$\nu(A) = \lim_{n\to\infty} P\int_A X_n, \quad A \in \cup\mathcal{F}_n.$$

For each $m \in \mathbb{N}$ define

$$X_n^m = \begin{cases} X_n I_{B_n}, & \text{if } n \leq m, \\ X_n I_{B_m}, & \text{if } n \geq m. \end{cases}$$

Then $(X_n^m)_n$ is a uniformly integrable martingale of strongly measurable Pettis integrable functions such that for all $f \in T$

$$fX_n^m \to fXI_{B_m} \text{ a.s.}$$

By Corollary 1 it follows that

$$X_n^m \to XI_{B_m} \text{ a.s.}$$

Also the function XI_{B_m} is Pettis integrable and, for each $f \in E^*$, fXI_{B_m} converges to fX. The family $\{fXI_{B_m} : f \in E^*,\ m \in \mathbb{N}\}$ is uniformly integrable, since so is $\{fX_n^m : f \in E^*,\ n, m \in \mathbb{N}\}$. Then by Vitali theorem

$$\lim_{m\to\infty} \int_\Omega fXI_{B_m} = \int_\Omega fX$$

and by the Vitali theorem for Pettis integral due to Musial ([9], Theorem 1), X is Pettis integrable. For $f \in T$ and $A \in \mathcal{F}$ we have

$$f\nu(A) = \lim_n \int_A fX_n = \int_A fX = f\,P\int_A X,$$

where the last equality is justified by the fact that X is Pettis integrable. Since T is a total set we get that

$$\nu(A) = P\int_A X.$$

Also

$$\nu(A) = \lim_n P\int_A X_n = P\int_A X.$$

Since $(X_n)_n$ is a martingale this implies that $E(X|\mathcal{F}_n) = X_n$. Now assume that (X_n) is Cauchy in $\mathcal{P}(E)$, then as in Theorem 1 the required convergence follows applying the Doob–Helms theorem to the scalar valued case. □

Theorem 1, Corollary 1 and Theorem 6 hold also for amarts, changing the proofs as in [14] Theorem 2.

4. Martingales of McShane integrable functions

In this section we consider sequences of McShane integrable functions.

Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a probability space which is a Radon, outer regular and compact probability space. A *McShane partition* of Ω is a set $\{(S_i, \omega_i), i = 1, \dots, p\}$, where $(S_i)_i$ is a disjoint family of measurable sets of finite measure, $P(\Omega \setminus \cup_{i=1}^p S_i) = 0$ and $\omega_i \in \Omega$ for each $i = 1, \dots, p$. A *gauge* on Ω is a function $\Delta : \Omega \to \mathcal{A}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. A McShane partition $\{(S_i, \omega_i), i = 1, \dots, p\}$ is *subordinate to a gauge* Δ, if $S_i \subset \Delta(\omega_i)$ for $i = 1, \dots, p$.

A function $f : \Omega \to E$ is *McShane integrable* (briefly M-integrable), with *McShane integral* $z \in E$, if for each $\varepsilon > 0$ there exists a gauge $\Delta : \Omega \to \mathcal{A}$, such that

$$\left\| \sum_{i=1}^{p} P(S_i) f(\omega_i) - z \right\| < \varepsilon$$

for each McShane partition $\{(S_i, \omega_i) : i = 1, \dots, p\}$ subordinate to Δ.

It is known that if $f : \Omega \to E$ is M-integrable, then

$$\nu_f(\Omega) = \left\{ (M)\int_A f : A \in \mathcal{F} \right\}$$

is totally bounded (see [1], Theorem B, and [3], Corollary 3E), hence it is norm relatively compact. Denote by $M(\Omega, \mathcal{A}, \mathcal{F}, P; E)$ or simply by $M(E)$ the set of all M-integrable functions endowed with the seminorm

$$|X|_M = \sup\left\{ \int_\Omega |fX| : f \in B(E^*) \right\},$$

which is equivalent to the seminorm ([11])

$$\sup\left\{ \left\| M\int_A X \right\| : A \in \mathcal{F} \right\}.$$

If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, X is McShane integrable and Y is McShane integrable on $(\Omega, \mathcal{A}, \mathcal{G}, P)$, then Y is called the *McShane conditional expectation* of X with respect to $\mathcal{G}$, if

(i) Y is weakly $\mathcal{G}$-measurable;
(ii) for every $A \in \mathcal{G}$, $M\int_A Y = M\int_A X$.

The symbol $Y = E_M(X|\mathcal{G})$ will denote the McShane conditional expectation of X with respect to $\mathcal{G}$.

We say that $(X_n, \mathcal{F}_n)$ is a *stochastic process of M-integrable functions*, if for each $n \in I\!N$, X_n is M-integrable, X_n is weakly measurable with respect to $\mathcal{F}_n$ and the McShane conditional expectation $E_M(X_n|\mathcal{F}_m)$ of X_n exists for all $n \geq m$. Also we observe that the conditional expectation of an M-integrable function does not always exist, indeed the same is true for strongly measurable Pettis integrable functions and a strongly measurable Pettis integrable function is McShane integrable.

Also in case of a stochastic process of M-integrable functions, we say that $(X_n, \mathcal{F}_n)$ is a martingale, if X_n is an M-integrable function for each n, and if for all $n \geq m$ $E_M(X_n|\mathcal{F}_m) = X_m$ or equivalently for all $A \in \mathcal{F}_m$

$$M\int_A X_m = M\int_A X_n.$$

If X is M-integrable and $E_M(X|\mathcal{F}_n)$ exists for all n, then $X_n = E_M(X|\mathcal{F}_n)$ is called a closed martingale. Since an M-integrable function is Pettis integrable ([3], Theorem 1Q) and $\nu_f(\Omega) = \left\{(M)\int_A f : A \in \mathcal{F}\right\}$ is norm relatively compact, there exists a sequence of simple functions $f_n : \Omega \to E$, converging to f in $|\cdot|_M$, i.e. $\lim |f_n - f|_M = 0$ (see [10], Theorem 9.1).

Now we are extending Lemma 1.4 of [13] to a martingale of McShane integrable functions.

Proposition 1. *Let $(X_n, \mathcal{F}_n)$ be a martingale of M-integrable functions. Then the following are equivalent*

(i) *there exists an M-integrable function X such that X_n is $|\cdot|_M$ convergent to X;*
(ii) *there exists an M-integrable function X such that $E_M(X|\mathcal{F}_n) = X_n$ for each $n \in I\!N$;*
(iii) *there exists an M-integrable function X such that for each $A \in \cup_n \mathcal{F}_n$*

$$\lim_n M\int_A X_n = M\int_A X.$$

Proof. Assume (i) holds. Then there exists a function $X : \Omega \to E$, which is M-integrable and such that $|X_n - X|_M \to 0$. Since

$$|X_n - X|_M = \sup\left\{\left\|M\int_A (X_n - X)\right\| : A \in \mathcal{F}\right\}$$

we have that

$$\lim_n M\int_A X_n = M\int_A X$$

for all $A \in \mathcal{F}$, and (iii) holds. Moreover if $A \in \mathcal{F}_m$, we get, by the martingale condition, that for all $n > m$,

$$M\int_A X_n = M\int_A X_m,$$

then $M\int_A X = M\int_A X_m$, i.e.

$$E_M(X|\mathcal{F}_n) = X_n,$$

that implies (ii).

Conversely assume (iii).
Since (X_n) is a martingale it follows that $M\int_A X_n = M\int_A X_m$ for all $n > m$ and $A \in \mathcal{F}_m$, therefore $M\int_A X = M\int_A X_n$, that is

$$E_M(X|\mathcal{F}_m) = X_m.$$

Since X is M-integrable there exists a simple function $X_\varepsilon = \sum_{i=1}^{l} x_i I_{A_i}$ for which $|X - X_\varepsilon|_M < \frac{\varepsilon}{2}$. We can assume that, for $i = 1, \ldots, l$, $A_i \in \mathcal{F}_{m_0}$. Hence for $n > m_0$ considering that $E_M(X_\varepsilon|\mathcal{F}_{m_0}) = X_\varepsilon$ and that the conditional expectation is a contraction

$$\begin{aligned} |X - X_n|_M &\le |X - X_\varepsilon|_M + |X_\varepsilon - X_n|_M < \\ &< \tfrac{\varepsilon}{2} + |E_M((X_\varepsilon - X)|\mathcal{F}_n)|_M \le \\ &\le \tfrac{\varepsilon}{2} + |X_\varepsilon - X|_M < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus X_n is $|\cdot|_M$ convergent to X, then (i) holds. Trivially (ii) implies (iii) and the proposition follows. □

The condition $(ii) \Rightarrow (i)$ in the previous proposition says that a closed martingale is $|\cdot|_M$ convergent. We have the following:

Proposition 2. *Let $(X_n, \mathcal{F}_n)$ be a martingale of M-integrable functions. Then, for all $A \in \cup_n \mathcal{F}_n$, the set function $\mu(A) = \lim_n M\int_A X_n$ is absolutely continuous and has norm relatively compact range, if and only if the martingale $(X_n, \mathcal{F}_n)$ is $|\cdot|_M$ Cauchy.*

Proof. The proof follows as in [8], Proposition 2 with small changes. □

Proposition 1 and 2 hold also for M-integrable martingales indexed by a directed set.

We prove a convergence theorem for M-integrable martingales.

Theorem 7. *Let $(X_n, \mathcal{F}_n)$ be a uniformly integrable martingale of M-integrable functions and suppose that there exists a weakly measurable function $X : \Omega \to E$ such that X_n converges to X a.s. in the weak topology. Then X_n is $|\cdot|_M$ convergent to X.*

Proof. Since $(X_n)_n$ is uniformly integrable the set function $\nu : \cup_n \mathcal{F}_n \to E$ defined as

$$\nu(A) = \lim_n M \int_A X_n$$

is an absolutely continuous measure of bounded variation and it can be extended to the whole $\mathcal{F}$ to an absolutely continuous measure of bounded variation. Also as in Theorem 1 for each $A \in \mathcal{F}$

$$\nu(A) = \lim_n M \int_A X_n.$$

Moreover for each $\omega \notin N$ with $P(N) = 0$, $f(X_n(\omega))$ converges to $f(X(\omega))$ for each $f \in E^*$. Hence it follows from [3, Theorem 4A] that X is M-integrable and $\nu(\Omega) = M\int_\Omega X$. Then for each $A \in \cup_n \mathcal{F}_n$

$$\lim_n M \int_A X_n = M \int_A X$$

and the assertion follows from Proposition 1. $\square$

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