STOCHASTIC PROCESSES OF VECTOR VALUED PETTIS AND MCSHANE INTEGRABLE FUNCTIONS

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Abstract. Convergence theorems of Pettis integrable martingales and more general stochastic processes taking values in a Banach space without the weak Radon–Nikodym property, are considered. Also McShane integrable martingales are studied.

1. Introduction

In this paper we study properties of stochastic processes, consisting of weakly measurable functions, taking values in a Banach space. For a Banach–valued martingale of Pettis integrable functions, the weak Radon–Nikodym property is equivalent to the convergence in Pettis norm (see [8]). Without assuming this property we ask for which smaller class $T$ of functionals $f$ the a.s. convergence of $fX_n$ to $fX$ for $f \in T$ implies the convergence of $X_n$ to $X$ in Pettis norm. For Bochner integrable stochastic processes it was shown that $T$ can be a total set (see [2] and [7]). In Section 3 we prove analogous results for Pettis – integrable martingales. In [6] convergence theorems for more general sequences are proved under the hypothesis that the range is relatively weakly compact or has the weak Radon-Nikodym property. Using decomposition theorems we obtain convergence results for stochastic processes taking values in a space which does not possess the weak Radon–Nikodym property (see Theorem 4 and Theorem 5).

In the fourth section martingales of McShane integrable functions are considered. In particular it is proved that a closed martingale converges in the McShane seminorm (Proposition 1). For a Pettis integrable closed martingale the same is true if the range of the indefinite integral is norm relatively compact (see [8] Corollary 1). Also a convergence theorem for McShane integrable martingales is proved (see Theorem 7).

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2. Definitions and notations

Let $E$ be a Banach space with norm $|\cdot|$, $B(E)$ its unit ball and $E^*$ its dual. A subset $T$ of $E^*$ is called a total set over $E$, if $f(x) = 0$ for each $f \in T$ implies $x = 0$.

Throughout this note $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_{n \in \mathbb{N}}$ a family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$, if $m < n$. Moreover, without loss of generality, we will assume that $\mathcal{F}$ is the completion of $\sigma(\bigcup_n \mathcal{F}_n)$.

Let $\mathcal{F}_0$ be a sub-$\sigma$-algebra of $\mathcal{F}$, then a function $X: \Omega \to E$ is called weakly $\mathcal{F}_0$-measurable, if the function $fX$ is $\mathcal{F}_0$-measurable for every $f \in E^*$. A weakly $\mathcal{F}$-measurable function is called weakly measurable. A function $X: \Omega \to E$ is said to be Pettis integrable, if $fX$ is Lebesgue integrable on $\Omega$ for each $f \in E^*$ and there exists a set function $\nu: \mathcal{F} \to E$ such that $f \nu(A) = \int_A fX$ for all $f \in E^*$ and $A \in \mathcal{F}$. In this case we write $\nu(A) = P\int_A X$ and we call $\nu(\Omega)$ the Pettis integral of $X$ over $\Omega$ and $\nu$ is the indefinite Pettis integral of $X$.

The space of all $E$-valued Pettis integrable functions is denoted by $P(\Omega, \mathcal{F}, P; E)$ or simply $P(E)$. The Pettis norm of a Pettis integrable functions is

$$|X|_P = \sup \left\{ \int_\Omega |fX| : f \in B(E^*) \right\}.$$ 

It is well known that

$$\sup \left\{ \left\| P\int_A X \right\| : A \in \mathcal{F} \right\}$$

defines an equivalent norm in $P(E)$.

We say that $(X_n, \mathcal{F}_n)$ is a stochastic process of Pettis integrable functions, if for each $n \in \mathbb{N}$, $X_n: \Omega \to E$ is Pettis integrable, $X_n$ is weakly $\mathcal{F}_n$-measurable and the Pettis conditional expectation $E(X_n|\mathcal{F}_m)$ of $X_n$ exists for all $n \geq m$. It should be noted that, in general, if $X$ is only Pettis integrable, even it is strongly measurable, there is no Pettis conditional expectation of $X$ with respect to a sub-$\sigma$-algebra of $\mathcal{F}$. We say that the stochastic process $(X_n, \mathcal{F}_n)$ is a martingale, if $E(X_n|\mathcal{F}_m) = X_m$ for $n \geq m$.

We say that a martingale $(X_n, \mathcal{F}_n)$ is

(i) convergent in $P(E)$, if there exists a function $X \in P(E)$ such that

$$\lim_{n \to \infty} |X_n - X|_P = 0;$$

(ii) uniformly bounded, if $\sup_n |X_n|_P < +\infty;$
(iii) uniformly continuous, if \( \lim_{P(A) \to 0} P \int_A X_n = 0 \) uniformly with respect to \( n \);
(iv) uniformly integrable, if it is uniformly bounded and uniformly continuous.

3. Convergence of Pettis integrable stochastic processes

We deal with functions which are Pettis integrable. To prove the following theorem we use an idea of [5].

**Theorem 1.** Let \((X_n, \mathcal{F}_n)\) be a uniformly integrable martingale of Pettis integrable functions and let \( X \) be a weakly measurable function. Moreover let \( T \) be a weak*-sequentially dense subset of \( E^* \), and assume that \( fX_n \) converges to \( fX \) a.s. for each \( f \in T \) (the null set depends on \( f \)). Then \( X \in \mathcal{P}(E) \) and for each \( A \in \mathcal{F} \)

\[
\lim_n P \int_A X_n = P \int_A X.
\]

Moreover if the martingale \((X_n, \mathcal{F}_n)\) is Cauchy in \( \mathcal{P}(E) \), then \( X_n \) converges to \( X \) in the Pettis norm.

**Proof.** Let \( A \in \cup_n \mathcal{F}_n \) and denote by

\[
\nu(A) = \lim_n P \int_A X_n.
\]

Since \((X_n, \mathcal{F}_n)\) is uniformly integrable, \( \nu : \cup_n \mathcal{F}_n \to E \) is of bounded variation and countably additive, hence it is strongly additive. By the Caratheodory–Hahn–Kluvanek extension theorem, \( \nu \) can be extended to a countably additive measure on \( \mathcal{F} \), absolutely continuous with respect to \( P \) and of bounded variation. Moreover

\[
\nu(A) = \lim_n P \int_A X_n
\]

exists for all \( A \in \mathcal{F} \). Indeed let \( A \in \mathcal{F} \) and fix \( \varepsilon > 0 \). Since \((X_n)\) is uniformly continuous there is \( \delta > 0 \) such that, if \( P(D) < \delta \) then \( \|P \int_D X_n\| < \varepsilon \) uniformly with respect to \( n \). Therefore there is \( n_1 \in \mathbb{N} \) and \( B \in \mathcal{F}_{n_1} \) such that \( P(A \Delta B) < \delta \) (see [4], p. 56). If \( n_2 \geq n_1 \), then \( \|P \int_B X_n - P \int_B X_m\| < \varepsilon \) for each \( n, m \geq n_2 \). Thus

\[
\|P \int_A X_n - P \int_A X_m\| \leq \|P \int_B X_n - P \int_B X_m\| + \|P \int_{B \Delta A} X_n\| + \|P \int_{B \Delta A} X_m\| + \|P \int_{A \Delta B} X_n\| + \|P \int_{A \Delta B} X_m\| < 5\varepsilon.
\]
Therefore \( \nu(A) = \lim_n P \int_A X_n \) exists for all \( A \in \mathcal{F} \). For each \( f \in T \) and \( A \in \mathcal{F} \), by Vitali Theorem, we get

\[
f \nu(A) = \lim_n \int_A f X_n = \int_A f X.
\]

Also, for each \( f \in E^* \), \((fX_n, \mathcal{F}_n)\) is a real valued \( L^1 \)-bounded martingale, then there exists a real random variable \( X_f \) such that

\[
f \nu(A) = \lim_n \int_A f X_n = \int_A X_f.
\]

Let \( f \in E^* \) and choose \((f_k)_{k} \subset T\) such that, for each \( x \in E \), \( f_k(x) \) converges to \( f(x) \). For each \( k \in \mathbb{N} \)

\[
\left| \int_A f_k X \right| = |f_k \nu(A)| \leq \|f_k\| \|\nu(A)\|
\]

and since by Banach–Steinhaus theorem \( \sup_k \|f_k\| < \infty \), the sequence \((f_kX)_k\) is uniformly \( L^1 \)-integrable. Then by Vitali theorem

\[
\lim_k \int_A f_k X = \int_A f X
\]

and \( X \) is scalarly integrable. Now by (1), for each \( k \in \mathbb{N} \) and \( A \in \mathcal{F} \)

\[
f_k \nu(A) = \int_A f_k X.
\]

Also

\[
\lim_k f_k \nu(A) = f \nu(A).
\]

Then (1), (2) and (3) imply

\[
f \nu(A) = \int_A f X,
\]

for all \( f \in E^* \) and \( A \in \mathcal{F} \). Thus \( X \) is Pettis integrable and \( \nu(A) = P \int_A X \) for all \( A \in \mathcal{F} \). Also we have

\[
\lim_n P \int_A X_n = \nu(A) = P \int_A X.
\]

Since \((X_n)_n\) is a martingale this implies that \( E(X|\mathcal{F}_n) = X_n \). Assume now that \((X_n)\) is Cauchy in \( \mathcal{P}(E) \). Applying the Doob–Helms theorem for the scalar valued case we get the required convergence. \( \square \)

**Corollary 1.** Let \((X_n, \mathcal{F}_n)\) be a uniformly integrable martingale of Pettis integrable strongly measurable functions, \( X \) a strongly measurable function. Let \( T \) be a total subset of \( E^* \), and assume that \( f X_n \) converges to \( f X \) a.s. for
Each \( f \in T \) (the null set depends on \( f \)). Then \( X \in \mathcal{P}(E) \) and \( X_n \) converges to \( X \) in the Pettis norm.

**Proof.** By Pettis measurability theorem we can assume that \( E \) is separable, then since \( T \) is closed and weak*-dense, from the previous theorem we get that \( X \) is Pettis integrable and \( E(X|\mathcal{F}_n) = X_n \) for \( n \in \mathbb{N} \). Therefore the assertion follows from [13], Lemma 1.4. \( \square \)

Now we consider more general sequences \((X_n, \mathcal{F}_n)\) and we assume that \( X_n \) is strongly measurable for each \( n \in \mathbb{N} \).

**Definition 1.** A stochastic process \((X_n, \mathcal{F}_n)\) of Bochner integrable functions is said to be \( L^1 \)-bounded, if
\[
\sup_n \int_\Omega \| X_n \| < \infty.
\]

**Definition 2.** A stochastic process \((X_n, \mathcal{F}_n)\) of strongly measurable functions is said to be a game which becomes fairer with time (briefly a \( P \)-martingale), if for each \( \varepsilon > 0 \)
\[
\lim_n \sup m \geq n P(\sup_{n \leq q \leq m} \| E(X_m|\mathcal{F}_q) - X_q \| > \varepsilon) = 0.
\]

If for each \( \varepsilon > 0 \)
\[
\lim_n \sup m \geq n P(\sup_{n \leq q \leq m} \| E(X_m|\mathcal{F}_q) - X_q \| > \varepsilon) = 0
\]
the sequence \((X_n, \mathcal{F}_n)\) is called a mil.

We need the following decomposition theorems:

**Theorem 2.** ([6], Theorem 1.2) Let \((X_n, \mathcal{F}_n)\) be an \( L^1 \)-bounded \( P \)-martingale consisting of Bochner integrable functions. Then \((X_n)\) can be written in a unique form as
\[
X_n = M_n + P_n, \quad n \in \mathbb{N},
\]
where for each \( n \in \mathbb{N} \), \( M_n \) and \( P_n \) are Bochner integrable functions, \((M_n)\) is a uniformly integrable martingale and \((P_n)\) goes to zero in probability.

**Theorem 3.** ([12], Theorem 8) Let \((X_n, \mathcal{F}_n)\) be a mil consisting of Bochner integrable functions such that \( \liminf_n \int_\Omega \|X_n\| < \infty \). Then \((X_n)\) can be written in a unique form as
\[
X_n = M_n + Z_n, \quad n \in \mathbb{N},
\]
where for each \( n \in \mathbb{N} \), \( M_n \) and \( Z_n \) are Bochner integrable functions, \((M_n)\) is a uniformly integrable martingale and \((Z_n)\) is a mil converging to zero \( a.s. \).

**Definition 3.** A stochastic process \((X_n, \mathcal{F}_n)\) of Pettis integrable functions is \( \sigma \)-bounded, if there exists an increasing sequence \((B_n)_n\), \( B_n \in \mathcal{F}_n \), such that \( \lim_n P(B_n) = 1 \) and the sequence \((X_n)\) restricted to each \( B_m, m = 1, 2, \ldots \), is \( L^1 \)-bounded.
Theorem 4. Let \((X_n, \mathcal{F}_n)\) be a \(\sigma\)-bounded \(P\)-martingale of Pettis integrable functions and \(X\) a strongly measurable function. Let \(T\) be a total subset of \(E^*\), and assume that \(fX_n\) converges to \(fX\) a.s. for each \(f \in T\) (the null set depends on \(f\)). Then \(X_n\) converges to \(X\) in probability (i.e. for every \(\varepsilon > 0\) we have \(\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0\)).

Proof. Since \((X_n, \mathcal{F}_n)\) is \(\sigma\)-bounded, there is an increasing sequence \((B_n)\), \(B_n \in \mathcal{F}_n\), such that \(\lim_{n \to \infty} P(B_n) = 1\) and \((X_n)\) restricted to each \(B_m\), \(m = 1, 2, \ldots\), is \(L^1\)-bounded. For \(m \in \mathbb{N}\), define

\[
X^m_n = \begin{cases} 
X_n I_{B_n}, & \text{if } n \leq m, \\
X_n I_{B_m}, & \text{if } n \geq m,
\end{cases}
\]

where the symbol \(I_B\) denotes the characteristic function of the set \(B\). For each \(m\) the sequence \((X^m_n, n \in \mathbb{N})\) is an \(L^1\)-bounded \(P\)-martingale in \(L^1(E)\), therefore by Theorem 2, for \(n \in \mathbb{N}\),

\[
(4)\quad X^m_n = M^m_n + P^m_n,
\]

where \((X^m_n)\) is a uniformly integrable martingale and \((P^m_n)\) goes to zero in probability. Moreover for each \(f \in T\), \(fX^m_n(\omega) \to fXI_{B_m}(\omega)\) for each \(\omega \notin N_f\), \(P(N_f) = 0\). Indeed fixed \(\omega \in \Omega \setminus N_f\) and \(\varepsilon > 0\) there exists \(n_\varepsilon\) such that for each \(n > n_\varepsilon\) it follows

\[
|f(X^m_n(\omega)) - f(X(\omega))| < \varepsilon.
\]

If \(\omega \in B_m\), then \(X_n(\omega) = X^m_n(\omega)\) and \(X(\omega) = XI_{B_m}(\omega)\). If \(\omega \notin B_m\), \(X^m_n(\omega) = 0\), \(XI_{B_m}(\omega) = 0\), and

\[
|f(X^m_n(\omega)) - f(XI_{B_m}(\omega))| < \varepsilon.
\]

Since \((P^m_n)\) converges to zero in probability there exists a subsequence of \((P^m_n)\), still denoted by \((P^m_n)\) converging to zero a.s. Then for each \(f \in T\), \(fP^m_n \to 0\) a.s. and since

\[
fX^m_n = fM^m_n + fP^m_n
\]

we get

\[
\lim_{n \to \infty} fM^m_n = fXI_{B_m}.
\]

Since the sequence \((M^m_n)\) is a uniformly integrable martingale, it is \(L^1\)-bounded. Then it follows from \([2]\) Theorem 3 or \([7]\) Theorem 4 that

\[
\lim_{n \to \infty} M^m_n = XI_{B_m} \text{ a.s.}
\]

in the strong topology. Moreover \((M^m_n)\) converges to \(XI_{B_m}\) in probability. We want to prove that \((X_n)\) converges to \(X\) in probability. By hypothesis there is \(m_0 \in \mathbb{N}\) such that

\[
P(B_{m_0}) > 1 - \frac{\varepsilon}{2}.
\]
Also \((X_{n}^{m_{0}})_{n}\) converges in probability to \(XI_{B_{m_{0}}}\), then there exists \(p \geq m_{0}\) so that for all \(n \geq p\),

\[
P(\|X_{n}^{m_{0}} - XI_{B_{m_{0}}}\| > \varepsilon) < \frac{\varepsilon}{2}.
\]

Thus we get

\[
P(\|X_{n} - X\| > \varepsilon) < P(\omega \in B_{m_{0}} : \|X_{n} - X\| > \varepsilon) + P(\omega \notin B_{m_{0}}) \leq \frac{\varepsilon}{2} < \varepsilon.
\]

Then \(X_{n}\) converges to \(X\) in probability and the assertion follows. \(\square\)

Applying Talagrand decomposition theorem of mils (Theorem 3) we get the following result:

**Theorem 5.** Let \((X_{n}, F_{n})\) be a \(\sigma\)-bounded mil of Pettis integrable strongly measurable functions and \(X\) a strongly measurable function. Moreover let \(T\) be a total subset of \(E^{*}\), and assume that \(fX_{n}\) converges to \(fX\) a.s. for each \(f \in T\) (the null set depends on \(f\)). Then \(X_{n}\) converges to \(X\) a.s. in the strong topology.

**Proof.** Find \((B_{n})_{n}\) and define \((X_{n}^{n} : n, m \in \mathbb{N})\) as in the previous theorem. The sequence \((X_{n}^{m}, n \in \mathbb{N})\) is an \(L^{1}\)-bounded mil in \(L^{1}(E)\). Therefore by Theorem 3, for \(n \in \mathbb{N}\),

\[
X_{n}^{m} = M_{n}^{m} + Z_{n}^{m},
\]

where \((M_{n}^{m})_{n}\) is a uniformly integrable martingale and \((Z_{n}^{m})_{n}\) is a mil converging to zero a.s. Moreover for each \(f \in T\)

\[
fX_{n}^{m} = fM_{n}^{m} + fZ_{n}^{m}.
\]

Then \(f(M_{n}^{m}(\omega)) \rightarrow f(XI_{B_{m}}(\omega))\) for each \(\omega \notin N_{f}, P(N_{f}) = 0\). As before it follows that \((M_{n}^{m}(\omega))_{n}\) converges in the strong topology to \(XI_{B_{m}}(\omega)\) for all \(m = 1, 2, \ldots\), and \(\omega \in \Omega \setminus N, P(N) = 0\). By (5) also \((X_{n}^{m})_{n}\) converges to \(XI_{B_{m}}\) a.s. in the strong topology. We want to prove that \(X_{n}\) converges to \(X\) a.s.. Fix \(\omega \in B_{m_{0}}\) and \(\omega \notin \Omega\). For every positive number \(\varepsilon\) there exists \(n_{\varepsilon} > m_{0}\) such that for all \(n > n_{\varepsilon}\) we get

\[
\|X_{n}(\omega) - X(\omega)\| = \|X_{n}^{m_{0}}(\omega) - XI_{B_{m_{0}}}(\omega)\| < \varepsilon.
\]

Since \(P(N \cup (\Omega \setminus \cup_{m} B_{m})) = 0\) the assertion follows. \(\square\)

Assuming a weaker strongly measurability condition on the martingale \((X_{n}, F_{n})\) and on the random variable \(X\), in Theorem 1 we can weaken the hypothesis on the set \(T\).
Theorem 6. Let \( (X_n, \mathcal{F}_n) \) be a uniformly integrable martingale of Pettis integrable functions. Assume that there exists an increasing sequence of measurable sets \( (B_m)_m, B_m \in \mathcal{F}_m \), such that \( \lim_n P(B_m) = 1 \) and that the function \( X \) and the function \( X_n \) restricted to each \( B_m \) are strongly measurable, \( n = 1, 2, \ldots \). Assume, moreover, that for each \( f \in T \), where \( T \) is a total set, \( fX_n \) converges to \( fX \) a.s. (the null depends on \( f \)). Then \( X \in \mathcal{P}(E) \) and for each \( A \in \mathcal{F} \)

\[
\lim_n P \int_A X_n = P \int_A X.
\]

Furthermore if the martingale \( (X_n, \mathcal{F}_n) \) is Cauchy in \( \mathcal{P}(E) \), then \( X_n \) converges to \( X \) in the Pettis norm.

Proof. As in Theorem 1 let \( \nu : \mathcal{F} \to E \) the absolutely continuous extension to \( \mathcal{F} \) of bounded variation of the measure

\[
\nu(A) = \lim_{n \to \infty} P \int_A X_n, \quad A \in \bigcup \mathcal{F}_n.
\]

For each \( m \in \mathbb{N} \) define

\[
X^m_n = \begin{cases} 
X_nI_{B_m}, & \text{if } n \leq m, \\
X_nI_{B_m}, & \text{if } n \geq m.
\end{cases}
\]

Then \( (X^m_n)_n \) is a uniformly integrable martingale of strongly measurable Pettis integrable functions such that for all \( f \in T \)

\[
fX_n^m \to fXI_{B_m} \text{ a.s.}
\]

By Corollary 1 it follows that

\[
X^m_n \to XI_{B_m} \text{ a.s.}
\]

Also the function \( XI_{B_m} \) is Pettis integrable and, for each \( f \in E^* \), \( fXI_{B_m} \) converges to \( fX \). The family \( \{fXI_{B_m} : f \in E^*, m \in \mathbb{N}\} \) is uniformly integrable, since so is \( \{fX^m_n : f \in E^*, n, m \in \mathbb{N}\} \). Then by Vitali theorem

\[
\lim_{m \to \infty} \int \Omega fXI_{B_m} = \int \Omega fX
\]

and by the Vitali theorem for Pettis integral due to Musial ([9], Theorem 1), \( X \) is Pettis integrable. For \( f \in T \) and \( A \in \mathcal{F} \) we have

\[
f\nu(A) = \lim_n \int_A fX_n = \int_A fX = f P \int_A X,
\]

where the last equality is justified by the fact that \( X \) is Pettis integrable. Since \( T \) is a total set we get that

\[
\nu(A) = P \int_A X.
\]
Also

\[ \nu(A) = \lim_n P \int_A X_n = P \int_A X. \]

Since \((X_n)_n\) is a martingale this implies that \(E(X|\mathcal{F}_n) = X_n\). Now assume that \((X_n)_n\) is Cauchy in \(\mathcal{P}(E)\), then as in Theorem 1 the required convergence follows applying the Doob–Helms theorem to the scalar valued case. □

Theorem 1, Corollary 1 and Theorem 6 hold also for amarts, changing the proofs as in [14] Theorem 2.

4. Martingales of McShane integrable functions

In this section we consider sequences of McShane integrable functions.

Let \((\Omega, \mathcal{A}, \mathcal{F}, P)\) be a probability space which is a Radon, outer regular and compact probability space. A McShane partition of \(\Omega\) is a set \(\{(S_i, \omega_i), i = 1, \ldots, p\}\), where \((S_i)_i\) is a disjoint family of measurable sets of finite measure, \(P(\Omega \setminus \bigcup_{i=1}^{p} S_i) = 0\) and \(\omega_i \in \Omega\) for each \(i = 1, \ldots, p\). A gauge on \(\Omega\) is a function \(\Delta : \Omega \to \mathcal{A}\) such that \(\omega \in \Delta(\omega)\) for each \(\omega \in \Omega\).

A McShane partition \(\{(S_i, \omega_i), i = 1, \ldots, p\}\) is subordinate to a gauge \(\Delta\), if \(S_i \subset \Delta(\omega_i)\) for \(i = 1, \ldots, p\).

A function \(f : \Omega \to E\) is McShane integrable (briefly \(M\)-integrable), with McShane integral \(z \in E\), if for each \(\varepsilon > 0\) there exists a gauge \(\Delta : \Omega \to \mathcal{A}\), such that

\[ \left\| \sum_{i=1}^{p} P(S_i)f(\omega_i) - z \right\| < \varepsilon \]

for each McShane partition \(\{(S_i, \omega_i) : i = 1, \ldots, p\}\) subordinate to \(\Delta\).

It is known that if \(f : \Omega \to E\) is \(M\)-integrable, then

\[ \nu_f(\Omega) = \left\{ (M) \int_A f : A \in \mathcal{F} \right\} \]

is totally bounded (see [1], Theorem B, and [3], Corollary 3E), hence it is norm relatively compact. Denote by \(M(\Omega, \mathcal{A}, \mathcal{F}, P; E)\) or simply by \(M(E)\) the set of all \(M\)-integrable functions endowed with the seminorm

\[ |X|_M = \sup \left\{ \int_{\Omega} |fX| : f \in B(E^*) \right\}, \]

which is equivalent to the seminorm ([11])

\[ \sup \left\{ \left\| M \int_A X \right\| : A \in \mathcal{F} \right\}. \]

If \(\mathcal{G}\) is a sub-\(\sigma\)-algebra of \(\mathcal{F}\), \(X\) is McShane integrable and \(Y\) is McShane integrable on \((\Omega, \mathcal{A}, \mathcal{G}, P)\), then \(Y\) is called the McShane conditional expectation of \(X\) with respect to \(\mathcal{G}\), if
(i) $Y$ is weakly $\mathcal{G}$-measurable;
(ii) for every $A \in \mathcal{G}$, $M \int_A Y = M \int_A X$.

The symbol $Y = E_M(X|\mathcal{G})$ will denote the McShane conditional expectation of $X$ with respect to $\mathcal{G}$.

We say that $(X_n, \mathcal{F}_n)$ is a stochastic process of $M$-integrable functions, if for each $n \in \mathbb{N}$, $X_n$ is $M$-integrable, $X_n$ is weakly measurable with respect to $\mathcal{F}_n$ and the McShane conditional expectation $E_M(X_n|\mathcal{F}_m)$ of $X_n$ exists for all $n \geq m$. Also we observe that the conditional expectation of an $M$-integrable function does not always exist, indeed the same is true for strongly measurable Pettis integrable functions and a strongly measurable Pettis integrable function is McShane integrable.

Also in case of a stochastic process of $M$-integrable functions, we say that $(X_n, \mathcal{F}_n)$ is a martingale, if $X_n$ is an $M$-integrable function for each $n$, and if for all $n \geq m$ $E_M(X_n|\mathcal{F}_m) = X_m$ or equivalently for all $A \in \mathcal{F}_m$

$$M \int_A X_m = M \int_A X_n.$$  

If $X$ is $M$-integrable and $E_M(X|\mathcal{F}_n)$ exists for all $n$, then $X_n = E_M(X|\mathcal{F}_n)$ is called a closed martingale. Since an $M$-integrable function is Pettis integrable ([3], Theorem 1Q) and $\nu_f(\Omega) = \{(M) \int_A f : A \in \mathcal{F}\}$ is norm relatively compact, there exists a sequence of simple functions $f_n : \Omega \to E$, converging to $f$ in $| \cdot |_M$, i.e. $\lim |f_n - f|_M = 0$ (see [10], Theorem 9.1).

Now we are extending Lemma 1.4 of [13] to a martingale of McShane integrable functions.

**Proposition 1.** Let $(X_n, \mathcal{F}_n)$ be a martingale of $M$-integrable functions. Then the following are equivalent

(i) there exists an $M$-integrable function $X$ such that $X_n$ is $| \cdot |_M$ convergent to $X$;
(ii) there exists an $M$-integrable function $X$ such that $E_M(X|\mathcal{F}_n) = X_n$ for each $n \in \mathbb{N}$;
(iii) there exists an $M$-integrable function $X$ such that for each $A \in \cup_n \mathcal{F}_n$

$$\lim_n M \int_A X_n = M \int_A X.$$  

Proof. Assume (i) holds. Then there exists a function $X : \Omega \to E$, which is $M$-integrable and such that $|X_n - X|_M \to 0$. Since

$$|X_n - X|_M = \sup \left\{ \left\| M \int_A (X_n - X) \right\| : A \in \mathcal{F} \right\}$$

we have that

$$\lim_n M \int_A X_n = M \int_A X.$$
for all \( A \in \mathcal{F} \), and (iii) holds. Moreover if \( A \in \mathcal{F}_m \), we get, by the martingale condition, that for all \( n > m \),
\[
M \int_A X_n = M \int_A X_m,
\]
then \( M \int_A X = M \int_A X_m \), i.e.
\[
E_M(X|\mathcal{F}_n) = X_n,
\]
that implies (ii).

Conversely assume (iii).
Since \((X_n)\) is a martingale it follows that \( M \int_A X_n = M \int_A X_m \) for all \( n > m \) and \( A \in \mathcal{F}_m \), therefore \( M \int_A X = M \int_A X_n \), that is
\[
E_M(X|\mathcal{F}_m) = X_m.
\]
Since \( X \) is \( M \)-integrable there exists a simple function \( X_{\varepsilon} = \sum_{i=1}^l x_i I_{A_i} \) for which \( |X - X_{\varepsilon}|_M < \frac{\varepsilon}{2} \). We can assume that, for \( i = 1, \ldots, l \), \( A_i \in \mathcal{F}_{m_0} \). Hence for \( n > m_0 \) considering that \( E_M(X_{\varepsilon}|\mathcal{F}_{m_0}) = X_{\varepsilon} \) and that the conditional expectation is a contraction
\[
|X - X_n|_M \leq |X - X_{\varepsilon}|_M + |X_{\varepsilon} - X_n|_M \leq \frac{\varepsilon}{2} + |E_M((X_{\varepsilon} - X)|\mathcal{F}_n)|M \leq \frac{\varepsilon}{2} + |X_{\varepsilon} - X|_M < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Thus \( X_n \) is \(| \cdot |_M\) convergent to \( X \), then (i) holds. Trivially (ii) implies (iii) and the proposition follows. \( \Box \)

The condition (ii) \( \Rightarrow \) (i) in the previous proposition says that a closed martingale is \(| \cdot |_M\) convergent. We have the following:

**Proposition 2.** Let \((X_n, \mathcal{F}_n)\) be a martingale of \( M \)-integrable functions. Then, for all \( A \in \bigcup_n \mathcal{F}_n \), the set function \( \mu(A) = \lim_n M \int_A X_n \) is absolutely continuous and has norm relatively compact range, if and only if the martingale \((X_n, \mathcal{F}_n)\) is \(| \cdot |_M\) Cauchy.

**Proof.** The proof follows as in [8], Proposition 2 with small changes. \( \Box \)

Proposition 1 and 2 hold also for \( M \)-integrable martingales indexed by a directed set.

We prove a convergence theorem for \( M \)-integrable martingales.

**Theorem 7.** Let \((X_n, \mathcal{F}_n)\) be a uniformly integrable martingale of \( M \)-integrable functions and suppose that there exists a weakly measurable function \( X : \Omega \to E \) such that \( X_n \) converges to \( X \) a.s. in the weak topology. Then \( X_n \) is \(| \cdot |_M\) convergent to \( X \).
Proof. Since \((X_n)_n\) is uniformly integrable the set function \(\nu : \cup_n F_n \to E\) defined as
\[
\nu(A) = \lim_n M \int_A X_n
\]
is an absolutely continuous measure of bounded variation and it can be extended to the whole \(\mathcal{F}\) to an absolutely continuous measure of bounded variation. Also as in Theorem 1 for each \(A \in \mathcal{F}\)
\[
\nu(A) = \lim_n M \int_A X_n.
\]
Moreover for each \(\omega \notin N\) with \(P(N) = 0\), \(f(X_n(\omega))\) converges to \(f(X(\omega))\) for each \(f \in E^*\). Hence it follows from [3, Theorem 4A] that \(X\) is \(M\)-integrable and \(\nu(\Omega) = M \int_{\Omega} X\). Then for each \(A \in \cup_n F_n\)
\[
\lim_n M \int_A X_n = M \int_A X
\]
and the assertion follows from Proposition 1.

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References

