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FIXED POINT INDICES OF ITERATED PLANAR MAPS

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Abstract. In this expository paper we survey recent results on the form of indices of iterated planar maps, formulate some open questions and give new proofs for theorems concerning C^1-case.

1. Introduction

Let $f : U \to \mathbb{R}^2$, where U is an open subset of $\mathbb{R}^2$, be a continuous map such that for each $n > 0$ ($n \neq 0$ for a homeomorphism) 0 is an isolated fixed point for f^n. In this case the fixed point index $\mathrm{ind}(f^n, 0)$ is well defined for f^n restricted to a small neighborhood of 0. In this paper we present recent results concerning the form of $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ (or $\{\mathrm{ind}(f^n, 0)\}_{n\neq 0}$, if f is a homeomorphism).

Definition 1. *For each natural n we define integers $i_n(f, 0)$ by the following equality:*

$$i_n(f,0) = \sum_{k|n} \mu(n/k)\mathrm{ind}(f^k, 0),$$

where μ is classical Möbius function [9].

It was proved by A. Dold in 1983 [11] that there are some congruences, called Dold relations, that are satisfied by any sequence of indices of iterations, namely:

Theorem 1. *For each natural n there is: $i_n(f, 0) \equiv 0 \pmod{n}$.*

Let $f : U \to X$, where U is an open subset of X be a map which belongs to a given class of maps G. Restrictions on X or G lead to bounds on the form of local indices of iterations of f at a fixed-point. For example, continuous differentiability of a map $g : \mathbb{R}^k \to \mathbb{R}^k$ implies periodicity of the

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sequence of indices of its iterations (cf. [10]). What is more, the less is the dimension k, the simpler is the form of the sequence. The natural question is what are the all constrains, except for Dold relations, for $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$, where x_0 is a fixed point, for a given X and G. The aim of this paper is to discuss various latest results concerning this problem for different classes of two-dimensional maps.

2. k-periodic expansion and continuous maps

Definition 2. *For a given k we define the sequence (called regular representation):*

$$\mathrm{reg}_k(n) = \begin{cases} k, & \text{if } k|n, \\ 0, & \text{if } k \not| n. \end{cases}$$

Note that reg_k is an elementary periodic sequence $\{0, \ldots, 0, k, 0, \ldots, 0, k, \ldots, \ldots\}$, where the non-zero entries appear for indices divisible by k.

Theorem 2. ([19]) *Any sequence $\psi : \mathbb{N} \to \mathbb{C}$ can be written in the following form of so-called k-periodic expansion: $\psi = \sum_1^\infty a_k(\psi)\mathrm{reg}_k$, where $a_k(\psi) = i_k(\psi)/k = \frac{1}{k}\sum_{d|k}\mu(d)\psi(k/d)$.*

Moreover, ψ takes integer values and satisfies Dold relations iff $a_k(\psi) \in \mathbb{Z}$ for every $k \in \mathbb{N}$.

The notion of k-adic expansion is a very useful device in studying the possible form of indices of iterations. We are interested in bounds on coefficients a_k in the expansion from Theorem 2. Another important information gives the determination of the set $\{k \in \mathbb{N} : a_k \neq 0\}$.

Now, let f represent an element from the class of planar continuous map, we want to know whether there are any additional constrains except for Dold relations for the sequence of indices of its iterations. The following theorem states that, if we assume only continuity of f, the answer to that question is negative.

Theorem 3. ([3], [16]) *For any sequence of integers $\{\psi_n\}_{n=1}^\infty$ which satisfies Dold relations, i.e. $a_k(\psi) \in \mathbb{Z}$, there exists a self-map f of the unit disk D^2 in $\mathbb{R}^2$, such that $\psi_n = \mathrm{ind}(f^n, 0)$ for each n.*

3. Smooth maps

In this section we reprove a part of the classical theorem of Chow, Mallet-Paret and Yorke (Theorem 5). An application of this result for planar C^1-maps enables us to give a new proof of the description of indices of iterations formulated by S. A. Babenko and I. K. Bogatyi (Theorem 6).

Let us introduce the following notation: let Δ be the set of all degrees of primitive roots of unity contained in $\sigma(Df(0))$-the spectrum of derivative at 0. By σ_+ we denote the number of eigenvalues of $Df(0)$ greater then 1, σ_- the number of eigenvalues of $Df(0)$ less then -1, in both cases counting with multiplicities. Let

$$\bar{O} = \{\mathrm{LCM}(K) : K \subset \Delta\},$$

where $\mathrm{LCM}(K)$ denotes the lowest common multiple of all elements in K and $\mathrm{LCM}(\emptyset) = 1$. Let $\bar{O}_{odd} = \{k \in \bar{O} : k \textit{ is odd}\}$.

The following theorem was proved in 1983 by S. N. Chow, J. Mallet-Paret and J. A. Yorke (cf. [10]). For the sake of our considerations we present it in the reformulated form, using the language of k-periodic expansion (cf. [19]).

Theorem 4. *Let U be an open neighborhood of 0, $f : U \to \mathbb{R}^m$ be C^1-map with 0 an isolated fixed point for f^n, $n \geq 1$. Then there exist integers a_k, such that:* $\mathrm{ind}(f^n, 0) = \sum_{k \in O} a_k \mathrm{reg}_k(n)$, *where*

$$O = \begin{cases} \bar{O}, & \textit{if } \sigma_- \textit{ is even,} \\ \bar{O} \cup 2\bar{O}_{odd}, & \textit{if } \sigma_- \textit{is odd.} \end{cases}$$

What is more, there exist integers c_k such that: for σ_- even there is: $a_k = c_k$, for σ_- odd there is

$$a_k = \begin{cases} c_k, & \textit{if } k \in \bar{O} \setminus 2\bar{O}_{odd}, \\ c_k - c_{k/2}, & \textit{if } k \in 2\bar{O}_{odd} \cap \bar{O}, \\ -c_{k/2}, & \textit{if } k \in 2\bar{O}_{odd} \setminus \bar{O}. \end{cases}$$

Theorem 5. ([10]) *The coefficients c_1, c_2 from Theorem 4 satisfy the following bounds:*

(A) $c_1 = (-1)^{\sigma_+}$, *if* $1 \notin \sigma(Df(0))$,
(B) $c_1 \in \{-1, 0, 1\}$, *if* 1 *is the eigenvalue of* $Df(0)$ *with multiplicity* 1.
(C) $c_2 \in \{0, (-1)^{\sigma_+ + \sigma_- + 1}\}$, *if* $1 \notin \sigma(Df(0))$ *and* -1 *is the eigenvalue of* $Df(0)$ *with multiplicity* 1.

Proof. The item (A) is a direct consequence of the Hopf formula. To prove (B) it is enough to show that $\mathrm{ind}(f, 0) \in \{-1, 0, 1\}$, with this assumption. Let $E_1 := \mathrm{Ker}\,(\mathrm{Id} - Df(0))$ and $E_2 := \mathrm{Im}\,(\mathrm{Id} - Df(0))$ be a complementing linear subspace, i.e. $E_1 \oplus E_2 = \mathbb{R}^m$ and $x = (x_1, x_2)$ coordinates corresponding to this decomposition. Let $\Phi(x_1, x_2) = (\Phi_1(x_1, x_2), \Phi_2(x_1, x_2)) := (x_1, x_2) - f(x_1, x_2) = (x_1 - f_1(x_1, x_2), x_2 - f_2(x_1, x_2))$. Then $\mathrm{ind}(f, 0) = \deg(\Phi, 0)$ and $\frac{\partial \Phi_2(0,0)}{\partial x_2}$ is nonsingular.

Near 0 we introduce the following local change of coordinates $s : U \to \mathbb{R}^m$, $s(x_1, x_2) = (\widetilde{x}_1(x_1, x_2), \widetilde{x}_2(x_1, x_2))$, where

$$\widetilde{x}_1 = x_1,$$
$$\widetilde{x}_2 = x_2 - f_2(x_1, x_2).$$

Then the Jacobi matrix in $(0,0)$ of this change of coordinates has the shape

$$\frac{D(\widetilde{x}_1, \widetilde{x}_2)}{D(x_1, x_2)} = \begin{bmatrix} 1 & 0 \\ \square & \frac{\partial \Phi_2(0,0)}{\partial x_2} \end{bmatrix}$$

with the determinant equal to $T = \det(\mathrm{Id} - \frac{\partial f_2(0,0)}{\partial x_2}) = \pm 1$. Thus $\deg(s, 0) = \operatorname{sgn} T$.

We have from the multiplicativity property of degree

$$\deg(\Phi, 0) = \deg(s\Phi s^{-1}, 0) = \deg(s, 0)\deg(\Phi s^{-1}, 0) = \operatorname{sgn} T \deg(\Phi s^{-1}, 0).$$

Notice now that the following equality holds

$$\Phi s^{-1}(\widetilde{x}_1, \widetilde{x}_2) = (\Phi_1 s^{-1}(\widetilde{x}_1, \widetilde{x}_2), \widetilde{x}_2)$$

and for each fixed $\widetilde{x}_2$ $\Phi_1 s^{-1}(\cdot, \widetilde{x}_2)$ is a self-map of real line.

By the assumption 0 is isolated in the set $\{x : \Phi(x) = 0\}$. On a small enough neighborhood V of 0, we define homotopy h_t by the formula

$$h_t(\widetilde{x}_1, \widetilde{x}_2) = ((1-t)\Phi_1 s^{-1}(\widetilde{x}_1, \widetilde{x}_2) + t\Phi_1 s^{-1}(\widetilde{x}_1, 0), \widetilde{x}_2).$$

There is: $h_0 = \Phi s^{-1}$ and $h_1(\widetilde{x}_1, \widetilde{x}_2) = (\Phi_1 s^{-1}(\widetilde{x}_1, 0), \widetilde{x}_2)$. As the zeros of h_t lie only on the line $\widetilde{x}_2 = 0$, we see that $0 \notin h_t(\partial V)$. Thus, by homotopy invariance of degree, we get: $\deg(\Phi s^{-1}, 0) = \deg(h_1, 0) = \deg((\Phi_1 s^{-1}(\cdot, 0), \widetilde{x}_2), 0)$. On the other hand the product formula for degree gives: $\deg(\Phi s^{-1}, 0) = \deg(\Phi_1 s^{-1}(\cdot, 0), 0) \in \{-1, 0, 1\}$, because $\Phi_1 s^{-1}(\cdot, 0)$ is one-dimensional map. Finally we obtain

$$\deg(\Phi, 0) = \operatorname{sgn} T \deg(\Phi_1 s^{-1}(\cdot, 0), 0) \in \{-1, 0, 1\},$$

which ends the proof of part (B).

To prove part (C) assume that $1 \notin \sigma(Df(0))$ and -1 is the eigenvalue of $Df(0)$ with multiplicity 1. Then $c_1 = \operatorname{ind}(f, 0) = (-1)^{\sigma_+}$ by (A), and $\operatorname{ind}(f^2, 0) \in \{-1, 0, 1\}$ by (B). On the other hand, if σ_- is even then: $\operatorname{ind}(f^2, 0) = c_1 \operatorname{reg}_1(2) + c_2 \operatorname{reg}_2(2)$. Consequently, $c_2 = \frac{1}{2}(\operatorname{ind}(f^2, 0) - \operatorname{ind}(f, 0))$ and by Dold relations c_2 is equal either 0 or $-\operatorname{ind}(f, 0) = (-1)^{1+\sigma_+} = (-1)^{1+\sigma_+ + \sigma_-}$. If σ_- is odd then: $\operatorname{ind}(f^2, 0) = c_1 \operatorname{reg}_1(2) + (c_2 - c_1)\operatorname{reg}_2(2)$, thus $c_2 = \frac{1}{2}(\operatorname{ind}(f^2, 0) + \operatorname{ind}(f, 0))$ hence $c_2 = 0$ or $c_2 = \operatorname{ind}(f, 0) = (-1)^{\sigma_+} = (-1)^{1+\sigma_+ + \sigma_-}$. □

If we consider planar maps, then there are strong restrictions on the set O in the theorem above. This enables to describe all sequences of iterations that may appear for planar C^1-maps.

Theorem 6. ([3]) *Let f be a planar C^1-map, then there are only four possible forms of indices of iterations:*
(a) $\operatorname{ind}(f^n,0) = a_1 \operatorname{reg}_1(n)$,
(b) $\operatorname{ind}(f^n,0) = a_2 \operatorname{reg}_2(n)$,
(c) $\operatorname{ind}(f^n,0) = \operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$,
(d) $\operatorname{ind}(f^n,0) = -\operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$,
where $a_i \in \mathbb{Z}$, $d \geq 2$.

Proof. Using the language of k-periodic expansion, Theorem 2, Theorems 4 and 5 we give an alternative proof to the one given in [3].

First remark that the set Δ is one of the following:
(1) $\Delta = \emptyset$,
(2) $\Delta = \{1\}$,
(3) $\Delta = \{2\}$,
(4) $\Delta = \{1, 2\}$,
(5) $\Delta = \{d\}$, where $d \geq 3$.

In the first two cases $\bar{O} = \{1\}$ and Theorem 4 leads to the two cases: either $O = \{1\}$ and $\operatorname{ind}(f^n,0) = c_1 \operatorname{reg}_1(n)$, if σ_- is even, or $O = \{1,2\}$ and $\operatorname{ind}(f^n,0) = c_1 \operatorname{reg}_1(n) - c_1 \operatorname{reg}_2(n)$, if σ_- is odd. The first case is covered by (a). If σ_- is odd, thus equal to 1 here, we have $c_1 = \operatorname{ind}(f,0) \in \{-1,0,1\}$ by Theorem 5 part (B). Consequently we get $\operatorname{ind}(f^n,0) = c_1(\operatorname{reg}_1(n) - \operatorname{reg}_2(n))$, where $c_1 \in \{-1,0,1\}$. If $c_1 = 0$ then we have case (a) with $a_1 = 0$, if $c_1 = 1$ then (c) with $d = 2$ and $a_2 = -1$, if $c_1 = -1$ then (d) with $a_2 = 1$.

In cases (3) and (4) we have $O = \{1,2\}$. There is: $\operatorname{ind}(f^n,0) = c_1 \operatorname{reg}_1(n) + c_2 \operatorname{reg}_2(n)$ for σ_- even. Since $-1 \in \sigma(Df(0))$, 1 can be at most simple eigenvalue. Using once more Theorem 5 parts (A) and (B) we get $c_1 \in \{-1,0,1\}$. Then the expansion is one of the cases (d), (b), or (c) correspondingly, with $d = 2$ in (c).

If σ_- is odd, thus equal to 1, then $\operatorname{ind}(f^n,0) = c_1(\operatorname{reg}_1(n) - \operatorname{reg}_2(n)) + c_2 \operatorname{reg}_2(n)$. On the other hand $1 \notin \sigma(Df(0))$ and $\sigma_+ = 0$ then, by the dimension argument, and consequently $c_1 = \operatorname{ind}(f,0) = (-1)^{\sigma_+} = 1$. Furthermore, -1 is the eigenvalue of $Df(0)$ with multiplicity 1, so by Theorem 5 part (C) we get $c_2 \in \{0, (-1)^2\} = \{0,1\}$.

If $c_2 = 0$, then we have the case (c) with $d = 2$, $a_2 = -1$. If $c_2 = 1$, then we have the case (a) with $a_1 = 1$.

Finally, in case (5) the set $\sigma(Df(0))$ consists of only one not real root of unity of degree $d \geq 3$. Consequently $c_1 = \operatorname{ind}(f,0) = 1$ and $\sigma_- = 0$ then. By Theorem 4 $O = \{1,d\}$ and $\operatorname{ind}(f^n,0) = \operatorname{reg}_1(n) + c_d \operatorname{reg}_d(n)$ which is the case (c).

We get to the conclusion that the bounds on the indices of iterations formulated in the thesis are necessary conditions (note that we do not have to check whether Dold relations are satisfied, because by Theorem 2 for any

k-periodic expansion these congruences hold). On the other hand all types are realized for some C^1-map. I. K. Babenko and S. A. Bogatyi showed the examples of each type being polynomial local diffeomorphisms [3]. □

The important special case appears when f is a holomorphic map. Repeating the same reasoning as above with additional remark that any real eigenvalue of real equivalent of complex matrix has even multiplicity one gets that two forms of indices of iterations are possible [1], [2]:

Theorem 7. *Let f be a holomorphic map, then there are two possible forms of indices of iterations:*
(1) $\mathrm{ind}(f^n,0) = a_1\mathrm{reg}_1(n),$ where $a_1 \geq 1$,
(2) $\mathrm{ind}(f^n,0) = \mathrm{reg}_1(n) + a_q\mathrm{reg}_q(n),$ where $a_q \geq 1$,
what is more both types may be realized by polynomial local diffeomorphisms.

At the end of this section let us point out that a very similar description may be obtained for the simplicial maps of smooth type, with the set O expressed in terms of the simplicial structure instead of derivative $Df(0)$ [14].

4. Homeomorphisms

The impact of the dimension of the space on the form of indices of iterations clearly visible for homeomorphisms. In 3-dimensional space the Theorem 3 is still true if we replace f by g, where $g : D^3 \to D^3$ is a homeomorphism, D^3 is a unit disk in $\mathbb{R}^3$ [3], but is not true for planar homeomorphisms, for which there are strong restrictions on $\{\mathrm{ind}(f^n,0)\}_{n\neq 0}$ which are a consequence of topological properties of the plane.

We present below the current state of knowledge on indices of iterations for three important classes of homeomorphisms.

4.1. Orientation preserving homeomorphisms

Let H_n be the space of all orientation preserving planar homeomorphisms h such that the origin is the only fixed point and $\mathrm{ind}(h,0) = n$. It was stated by B. Schmitt [23] and reproved by M. Bonino [6] that H_n is path connected in compact-open topology. In 1990 M. Brown used this fact to describe the behaviour of the sequence of indices of iterations $\{\mathrm{ind}(f^n,0)\}_{n\neq 0}$:

Theorem 8. ([4]) *Let f be an orientation preserving local homeomorphism of the plane. Then there is an integer $p \neq 1$ such that for each $n \neq 0$*

$$\mathrm{ind}(f^n,0) = \begin{cases} p, & \text{if } \mathrm{ind}(f,0) = p, \\ 1 \text{ or } p, & \text{if } \mathrm{ind}(f,0) = 1. \end{cases}$$

Corollary 1. *If* $\mathrm{ind}(f^n,0) = p$*, then for every $k \in \mathbb{Z}\setminus\{0\}$* $\mathrm{ind}(f^{kn},0) = p$.

We will consider only $n > 0$ since for orientation preserving planar homeomorphisms $\mathrm{ind}(f^n, 0) = \mathrm{ind}(f^{-n}, 0)$.

Definition 3. *Let us define A, the set of generators for $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$, in the following way:*

$$A = \{a \in \mathbb{N} : \mathrm{ind}(f^a, 0) = p \text{ and } \forall_{b|a,\, b\neq a}\ \mathrm{ind}(f^b, 0) = 1\}.$$

From Theorem 8 we have: if $A = \emptyset$, then for each n $\mathrm{ind}(f^n, 0) = 1$.

Theorem 9. ([15]) *Let f be an orientation preserving local homeomorphism of the plane, $A \neq \emptyset$. Then A is finite and*

$$\mathrm{LCM}(A)|(p-1),$$

where $\mathrm{LCM}(A)$ *denotes the lowest common multiplicity of all elements in the set A.*

Problem 1. ([15]) The open question is whether there are further restrictions on the set of generators A. It is likely that there is no room in $\mathbb{R}^2$ for more then one element in A. This statement is an equivalent of the Babenko and Bogatyi hypothesis that indices of an iterated planar homeomorphism behave in the same way as indices of a planar C^1-map [3].

4.2. Orientation reversing homeomorphisms

Considering the indices of iterations of orientation reversing homeomorphisms we take weaker assumptions on f due to the following theorem proved by M. Bonino.

Theorem 10. ([5]) *Let $f : U \to \mathbb{R}^2$, where $U \subset \mathbb{R}^2$ is open, be a an orientation reversing homeomorphism with* 0 *as an isolated fixed point. If there exists an integer $k \geq 1$ such that any neighborhood of* 0 *contains a k-periodic point of f, then there is also a* 2*-periodic point in every neighborhood of the point* 0.

By Theorem 10 the whole sequence $\mathrm{ind}(f^n, 0)$ is defined if and only if the second term of it, i.e. $\mathrm{ind}(f^2, 0)$ is defined, thus in this section we assume only that 0 is an isolated fixed point for f^2.

Recent result of M. Bonino about the possible values of fixed point index in orientation-reversing case (announced earlier by M. Brown in [4]) provides also information about indices of iterations for this class of homeomorphisms.

Theorem 11. ([7]) *Let f be an orientation reversing local homeomorphism of the plane. Then:* $\mathrm{ind}(f, 0) \in \{-1, 0, 1\}$.

The theorem above and Dold relations determine the form of indices of odd iterations of an orientation reversing homeomorphism.

Theorem 12. ([15]) *Let f be an orientation reversing local homeomorphism of the plane. Then for every n odd* $\mathrm{ind}(f^n, 0) \in \{-1, 0, 1\}$ *and*

$$\mathrm{ind}(f^n, 0) = \begin{cases} \mathrm{ind}(f, 0), & \text{if } n > 0, \\ -\mathrm{ind}(f, 0), & \text{if } n < 0. \end{cases}$$

What is more, by Dold relations one may easily deduce the form of indices of even iterations when $\mathrm{ind}(f, 0) = 0$, namely $\mathrm{ind}(f^{2n}, 0) = 2l$, where l is an integer.

Problem 2. ([15]) It is obvious that f^2 is always a homeomorphism which preserves the orientation. Is it true that for a homeomorphism f which reverses the orientation $\{\mathrm{ind}(f^{2n}, 0)\}_{n=1}^{\infty}$ must be a constant sequence, i.e. the set of generators A for this sequence is either empty or equal to $\{1\}$.

4.3. Calvez–Yoccoz homeomorphisms

We say that a local planar homeomorphism f belongs to the class of Calvez–Yoccoz if the following two conditions are satisfied:

(A) There is no V – a neighborhood of 0, such that

$$f(V) \subset V \quad \text{or} \quad V \subset f(V).$$

(B) There is W – a neighborhood of 0, such that

$$\bigcap_{k \in \mathbb{Z}} f^k(W) = \{0\}.$$

Theorem 13. ([8]) *Let f be an orientation preserving Calvez–Yoccoz homeomorphism. Then there exist positive integers r, q, such that for each $n \neq 0$*

$$\mathrm{ind}(f^n, 0) = \begin{cases} 1 - rq, & \text{if } q | n, \\ 1, & \text{if } q \not| n. \end{cases}$$

Theorem 13 may be expressed in terms used in Theorem 8 and Definition 3 as: $A = \{q\}$ and $p \leq 0$, (namely: $p = 1 - rq$) or in the terms of q-periodic expansion as $\mathrm{ind}(f^n, 0) = \mathrm{reg}_1(n) + a_q \mathrm{reg}_q(n)$, where $a_q = -r < 0$. This information was used by P. Calvez and J.-C. Yoccoz to show non-existence of minimal homeomorphisms of a punctered sphere [8], which was a classical problem of S. Ulam from *Scottish Book* ([17], Problem 115).

It is easy to see that (B) is equivalent to the condition that $\{0\}$ is an isolated invariant set, which enables one to use the Conley index theory to study Calvez–Yoccoz homeomorphisms. Following this line J. Franks obtained in a simpler way a part of Theorem 13: he showed that in the sequence $\{\mathrm{ind}(f^n, 0)\}_{n \neq 0}$ there are infinitely many negative terms [12].

The formula of Calvez and Yoccoz from Theorem 13 was reproved by an application of discrete Conley index by F. Ruiz del Portal and J. Salazar [22].

These authors used the same methods to study the dynamics of Calvez–Yoccoz homeomorphisms in more general setting [20], [21]. In [22] they described the indices of iterations in orientation reversing case.

Theorem 14. *Let f be an orientation reversing Calvez–Yoccoz homeomorphism. Then there exist integers $\gamma \in \{-1,0,1\}$ and $r \geq 1$ such that for each $n > 0$*

$$\operatorname{ind}(f^n,0) = \begin{cases} \gamma, & \text{if } 2 \nmid n, \\ \gamma - 2r, & \text{if } 2|n. \end{cases}$$

By Theorems 12, $\gamma = \operatorname{ind}(f,0)$. Notice also that in this case the hypothesis formulated in Problem 2 is true.

We refer now an elegant geometric framework of the method used by Ruiz del Portal and Salazar in the proof of Calvez–Yoccoz formula and the formula from Theorem 14.

Theorem 15. ([22]) *Assume that conditions (A) and (B) are satisfied for a local planar homeomorphism f. Then there is an absolute retract D containing a neighborhood $\bar{V}$ of 0 and a finite set $\{q_1,\ldots,q_m\} \subset D$ and a map $\bar{f} : D \to D$ such that $\bar{f}_{|\bar{V}} = f_{|\bar{V}}$ and for every natural n $\mathrm{Fix}\,(\bar{f}^n) \subset \{0,q_1,\ldots,q_m\}$. What is more:*

1. *if f is orientation preserving, then all the periodic orbits of $\bar{f}$ in $\{q_1,\ldots,q_m\}$ have the same period,*
2. *if f is orientation reversing, then $\bar{f}$ has no more then two fixed points in $\{q_1,\ldots,q_m\}$ and the period of its periodic points is less or equal 2.*

By part (1) of Theorem 15 the set $\{q_1,\ldots,q_m\}$ consists of q periodic orbits (inequality $q \geq 1$ follows from the condition (A)) all of them having the same period $r \geq 1$. Using the fact that D is an absolute retract and Lefschetz–Hopf Theorem we get

$$1 = \operatorname{ind}(\bar{f}^n,D) = \operatorname{ind}(\bar{f}^n,0) + \sum_{q_i \in \mathrm{Fix}\,(\bar{f}^n)} \operatorname{ind}(\bar{f}^n,q_i).$$

The further part of the proof is based on the existence of so-called generalized filtration pair (cf. also [13]), which implies that in a small enough neighborhood of $q_i \in \mathrm{Fix}\,(\bar{f}^n)$ the map $\bar{f}^n$ is constant. Thus, for each i there is: $\operatorname{ind}(\bar{f}^n,q_i) = 1$. Finally $\operatorname{ind}(\bar{f}^n,0) = 1 - rq$ for $r|n$ and $\operatorname{ind}(\bar{f}^n,0) = 1$ for $r \nmid n$, which ends the sketch of the main idea of the proof of Theorem 13. In an analogous way, using part (2) of 15, one obtains the proof of Theorem 14. Note that the methods of Conley index theory work also in the reversing-orientation case.

The simplicity of the form of indices of iterations of Calvez–Yoccoz homeomorphisms encourages us to formulate the hypothesis that a similar regularity appears for continuous maps of Calvez–Yoccoz class.

Problem 3. Let us consider a continuous map f which satisfies conditions (A) and (B). Is it true that $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ is a periodic sequence.

Another problem is the question what could be the form of indices of iterations for maps of "elliptic" type, i.e. maps which behave in a similar way as discretizations of elliptic continuous dynamical systems. In such case the Conley index methods are not applicable. Nevertheless, it is likely that there is a kind of symmetry in the shape of indices of iterations to Calvez–Yoccoz homeomorphisms, which are "hyperbolic" in their nature. Let us consider the class of homeomorphisms which satisfy condition (A) and the following condition of "ellipticity": (B') there is W – a neighborhood of 0, such that for each point $x \in W$ either $\lim_{n\to\infty} f^n(x) = 0$ or $\lim_{n\to\infty} f^{-n}(x) = 0$.

Problem 4. Is it true that for homeomorphisms which satisfy the conditions (A) and (B') the following formula holds:

$$\mathrm{ind}(f^n, 0) = \mathrm{reg}_1(n) + a_q \mathrm{reg}_q(n),$$

and $a_q > 0$, $q \geq 2$.

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