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STATISTICAL DENSITY POINTS

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Abstract. In this note we shall introduce a new kind of density point of the measurable subset of the real line.

Throughout the paper **N** will denote the set of positive integers, **R** – the set of real numbers, χ_A – the characteristic function of $A \subset \mathbf{N}$ or $A \subset [-1,1]$, $n \cdot A = \{nx : x \in A\}$ for $n \in \mathbf{N}$ and $A \subset \mathbf{R}$, $A - x_0 = \{x - x_0 : x \in A\}$, λ – a Lebesgue measure on the real line.

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Let $A \subset \mathbf{R}$ be a measurable set. Put $f_n = \chi_{n\cdot(A-x_0))\cap[-1,1]}$. It is well known that x_0 is a density point of A, if and only if $f_n \underset{n\to\infty}{\longrightarrow} \chi_{[-1,1]}$ in measure, which means that $\lim_{n\to\infty} \lambda(\{x \in [-1,1] : |f_n(x) - 1| \geq \epsilon\}) = 0$ for each $\epsilon > 0$, and taking into account that f_n are characteristic functions, simply that $\lim_{n\to\infty} \lambda(\{x \in [-1,1] : f_n(x) = 0\}) = 0$ (compare [3]). The set of all density points of A will be denoted by $\Phi(A)$, as usual.

Take $N \subset \mathbf{N}$. Put $d_n(N) = \frac{1}{n}\sum_{k=1}^{n} \chi_N(k)$ and $d(N) = \limsup_{n\to\infty} d_n(N)$. We shall say that a sequence $\{x_n\}_{n\in\mathbf{N}}$ of real numbers converges statistically to $g \in \mathbf{R}$ ($g = n \to \infty_{\lim\text{stat}} x_n$), if and only if for each $\epsilon > 0$ we have $d(N(\epsilon)) = 0$, where $N(\epsilon) = \{n \in \mathbf{N} : |x_n - g| \geq \epsilon\}$ (compare [2]). Using this notion we shall introduce the following definition.

Definition 1. *We shall say that $x_0 \in \mathbf{R}$ is a point of of statistical density of a measurable set $A \subset \mathbf{R}$, if and only if $n \to \infty_{\lim\text{stat}} \lambda(\{x \in [-1,1] : f_n(x) = 0\}) = 0$ (where $f_n = \chi_{(n(A-x_0))\cap[-1,1]}$, as before).*

Remark 1. *According to* [1] *the condition in the definition is equivalent to* $\lim\text{stat}_{n\to\infty} f_n(x) = 0$ *a.e. in* $[-1,1]$.

Let $\Phi_s(A) = \{x \in \mathbf{R} : \mathbf{x}$ is a point of statistical density of $A\}$ for measurable set A. Obviously $\Phi(A) \subset \Phi_s(A)$. We shall show that the equality holds.

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Theorem 1. *For each measurable set A we have $\Phi(A) = \Phi_s(A)$.*

Proof. By virtue of the above observation it is sufficient to show that $\Phi_s(A) \subset \Phi(A)$. Suppose that x_0 is not a density point of A. For simplicity of denotations assume that $x_0 = 0$. Then there exist a number $a > 0$ and a sequence $\{x_k\}_{k\in\mathbf{N}}$ convergent decreasingly to 0, such that for each $k \in \mathbf{N}$ we have $\frac{1}{2x_k} \cdot \lambda(A' \cap [-x_k, x_k]) > a$. A moment of reflection shows, that we can assume (taking, if necessary, smaller a) that $x_k = (2n_k)^{-1}$, where $n_k \in \mathbf{N}$. Of course, $n_k \underset{k\to\infty}{\longrightarrow} \infty$. Let $j \in \{n_k, n_k+1, \dots, 2n_k\}$. We have $(j \cdot A') \cap [-1,1] \supset (j \cdot A') \cap [-j \cdot (2n_k)^{-1}, j \cdot (2n_k)^{-1}] = j \cdot (A' \cap [-(2n_k)^{-1}, (2n_k)^{-1}]$. Hence $\lambda((j \cdot A') \cap [-1,1]) \geq j \cdot \lambda(A' \cap [-(2n_k)^{-1}, (2n_k)^{-1}]) > j \cdot a \cdot (n_k)^{-1} \geq a > 0$.

Hence for $\epsilon \in (0,a)$ we have $N(\epsilon) = \{n \in N : \lambda(\{x \in [-1,1] : f_n(x) = 0\}) > a\} \supset \{n_k, n_k+1, \dots, 2n_k\}$ for each $k \in N$, so $d(N(\epsilon)) \geq \frac{1}{2}$ and 0 is not a point of statistical density of A. □

2

Recall now that the original definition of a density point looks as follows: x_0 is a density point of a measurable set $A \subset \mathbf{R}$, if and only if $\lim_{h\to 0^+} \frac{\lambda(A\cap[x_0-h,x_0+h])}{2h} = 1$, which means that for each sequence $\{h_n\}_{n\in N}$ of positive numbers convergent to 0 we have $\lim_{n\to\infty} \frac{\lambda(A\cap[x_0-h_n,x_0+h_n])}{2h_n} = 1$, so for each such sequence the sequence of characteristic functions

$$\{\chi_{(\frac{1}{h_n}\cdot(A-x_0))\cap[-1,1]}\}_{n\in N}$$

converges to $\chi_{[-1,1]}$ in measure. In 1. we have used the sequence $h_n = n^{-1}$, because this is, in some sense, a ,,universal" sequence, i.e. the convergence of the above described sequence of characteristic functions for $\{n^{-1}\}_{n\in N}$ implies the convergence for each $\{h_n\}_{n\in N}$. A sequence $\{h_n\}_{n\in N}$ is a ,,universal" sequence, if and only if

$$\liminf_{n\to\infty} \frac{h_{n+1}}{h_n} > 0$$

(compare, for example [4]). For points of statistical density the situation is entirely different. Put $f_n = \chi_{(\frac{1}{h_n}(A-x_0))\cap[-1,1]}$ for $n \in \mathbf{N}$.

Definition 2. *We shall say, that $x_0 \in \mathbf{R}$ is a point of statistical density of a measurable set $A \subset \mathbf{R}$ with respect to a sequence $\{h_n\}_{n\in N}$ of positive numbers convergent decreasingly to 0, if and only if $\lim \mathrm{stat}_{n\to\infty} \lambda(\{x \in [-1,1] : f_n(x) = 0\}) = 0$.*

Theorem 2. *If $h_n = 2^{-n}$ for $n \in \mathbf{N}$, then there exists a measurable set $A \subset \mathbf{R}$ such that 0 is a point of statistical density of A with respect to $\{h_n\}_{n\in N}$, but 0 is not a point of density of A.*

Proof. Choose an increasing sequence $\{n_k\}_{k\in N}$ of positive integers such that $k : (n_k - n_{k-1}) \underset{k\to\infty}{\longrightarrow} 0$. Assume that $n_0 = 1$.

Put $A = \mathbf{R} \setminus \bigcup_{\mathbf{k=1}}^{\infty} [\mathbf{2^{-n_k}}, \mathbf{2^{-n_k+1}}]$. Obviously 0 is not a density point of A. If $\epsilon_m \in [2^{-m-1}, 2^{-m})$, then $\lambda((2^j \cdot A') \cap [-1,1]) > \epsilon_m$ for $j \in \bigcup_{k=1}^{\infty}\{n_k - m, n_k - m + 1, n_k - m + 2, \ldots, n_k - 1, n_k\} = N(\epsilon_m)$. It is not difficult to prove that $d(N(\epsilon_m)) = 0$ for each $m \in N$, so 0 is a point of statistical density of A with respect to $\{2^{-n}\}_{n\in N}$. □

It would be interesting to characterize ,,universal" sequences for points of statistical density.

3

Let now $\{h_n\}_{n\in N}$ be a sequence of positive numbers convergent decreasing to 0. Denote for measurable set by $\Phi_s(A, \{h_n\})$ the set of all points of statistical density of A with respect to $\{h_n\}_{n\in N}$.

Theorem 3. *The operator Φ_s for fixed $\{h_n\}$ has the following properties:*

(1) $\Phi_s(A, \{h_n\})$ *is a measurable set for measurable A,*
(2) $\lambda(\Phi_s(A, \{h_n\}) \setminus A) = 0$,
(3) if $\lambda(A \triangle B) = 0$*, then* $\Phi_s(A, \{h_n\}) = \Phi_s(B, \{h_n\})$,
(4) $\Phi_s(\emptyset, \{h_n\}) = \emptyset$*,* $\Phi_s(\mathbf{R}, \{\mathbf{h_n}\}) = \mathbf{R}$,
(5) $\Phi_s(A \cap B, \{h_n\}) = \Phi_s(A, \{h_n\}) \cap \Phi_s(B, \{h_n\})$ *for measurable A and B.*

Proof.

Ad 2). Suppose that A is a measurable set and $x_0 \in \Phi_s(A, \{h_n\})$. Then there exists an increasing sequence $\{n_k\}_{k\in N}$ of positive integers such that $\lim_{k\to\infty} \lambda(\{s \in [-1,1] : f_{n_k}(x) = 0\}) = 0$. Hence $x_0 \notin \Phi(\mathbf{R} \setminus \mathbf{A})$, so $\Phi_s(A, \{h_n\}) \subset \mathbf{R} \setminus \Phi(\mathbf{R} \setminus A)$ and $\Phi_s(A, \{h_n\}) \setminus A \subset (\mathbf{R} \setminus A) \setminus \Phi(\mathbf{R} \setminus A)$, and the conclusion follows from the Lebesgue Density Theorem.

Ad 1). We have $\Phi(A) \subset \Phi_s(A, \{h_n\}) \subset \mathbf{R} \setminus \Phi(\mathbf{R} \setminus A)$, so $\Phi_s(A, \{h_n\})$ is included between two measurable sets which differ by the null set (again from LDT). The proofs of 3), 4) and 5) are standard. □

In this situation for arbitrary sequence $\mathbf{h} = \{h_n\}$ decreasingly convergent to 0 we can define the family $\mathcal{T}_{\mathbf{h}} = \{A \subset \mathbf{R} : \mathbf{A}$ is measurable and $A \subset \Phi_s(A, \{h_n\})\}$. In a standard way one can prove that each $\mathcal{T}_h$ is a topology stronger than the natural topology on the real line.

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