Folia Mathematica
Vol. 11, No. 1, pp. 53–57
Acta Universitatis Lodziensis
© 2004 for University of Łódź Press

SOME PROPERTIES OF THE SET OF DISCONTINUITY POINTS OF A MONOTONIC FUNCTION OF SEVERAL VARIABLES

WANDA LINDNER[‡]

Abstract. In the note [1] the neccesary and sufficient condition for a set $E \subset \mathbf{R}^2$ to be the set of discontinuity points for some nondecreasing function $f : \mathbf{R}^2 \to \mathbf{R}$ has been proved. The problem of the similar characterization for the function of more than two variables is still open.

This paper provides the neccesary condition for the set $E \subset \mathbf{R}^n$ to be the set of discontinuity points of some nondecreasing function.

1. Introduction

Let $\mathbf{x} = (x_1, x_2, \dots, x_n),\ \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbf{R}^n$.

Definition 1. *We say, that* $\mathbf{x} \prec \mathbf{y}$ *iff* $(x_i < y_i)$ *for every* $i \in \{1, 2, \dots, n\}$.

Definition 2. *We say, that* $\mathbf{x} \preceq \mathbf{y}$ *iff* $(x_i \le y_i)$ *for every* $i \in \{1, 2, \dots, n\}$. *If one of the following relations occurs:*

$$\mathbf{x} \preceq \mathbf{y} \quad or \quad \mathbf{y} \preceq \mathbf{x}$$

we say, that this points are comparable.

Definition 3. *The function* $f : \mathbf{R}^n \to \mathbf{R}$ *is called nondecreasing iff for every* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$

$$\mathbf{x} \preceq \mathbf{y} \quad \Rightarrow \quad f(\mathbf{x}) \le f(\mathbf{y}).$$

Definition 4. *The function* $f : \mathbf{R}^n \to \mathbf{R}$ *is called increasing iff for every* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$

$$(\mathbf{x} \neq \mathbf{y}) \wedge (\mathbf{x} \preceq \mathbf{y}) \quad \Rightarrow \quad f(\mathbf{x}) < f(\mathbf{y}).$$

Proposition 1. *The function* $f : \mathbf{R}^n \to \mathbf{R}$ *is nondecreasing (increasing) iff it is nondecreasing (increasing) as the function of one variable, with the others fixed.*

[‡] *Technical University of Łódź, Institute of Mathematics*, Al. Politechniki 11, 90–924 Łódź. Poland.. E-mail: lindner@imul.uni.lodz.pl.

Key words and phrases: monotonic function, discontinuity points.

AMS subject classifications: 26A48.

We will consider the space $\mathbf{R}^n$ equipped with the metric

$$d(\mathbf{x},\mathbf{y}) = \max\{|x_i - y_i| : i = 1, 2, \dots, n\}.$$

2. A dislocation of the set of discontinuity points

For every function $g : R^k \to R$ we will denote by $g^{(i)}$ the function defined as follows:

$$g^{(i)}(u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_k) = g(u_1, u_2, \dots, u_{i-1}, -u_i, u_{i+1}, \dots, u_k).$$

Definition 5. *The hypersurface*

$$\Gamma = \left\{\varphi(\mathbf{u}) = (g_1(\mathbf{u}), g_2(\mathbf{u}), \dots, g_n(\mathbf{u})) : \mathbf{u} \in \mathbf{R}^{n-1}\right\}$$

is called semidecreasing iff all the functions

$$g_1(\mathbf{u}),\ g_2^{(1)}(\mathbf{u}),\ g_3^{(2)}(\mathbf{u}), \dots,\ g_n^{(n-1)}(\mathbf{u})$$

are continuous and nondecreasing.

Definition 6. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a nondecreasing function. Let $t \in \mathbf{R}$. The set*

$$P_t(f) = \{\mathbf{x}\ :\ \mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{y}) \geq t \quad \wedge \quad \mathbf{y} \prec \mathbf{x} \Rightarrow f(\mathbf{y}) < t\}$$

is called the level line of the function f on the level t.

Lemma 1. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a nondecreasing function. If $\mathbf{x} \prec \mathbf{y}$ then for every number t the conjunction $\mathbf{x} \in P_t(f)\ \wedge\ \mathbf{y} \in P_t(f)$ is false.*

Proof. In the oposite case we have that

$$\mathbf{x} \prec \frac{\mathbf{x}+\mathbf{y}}{2}, \quad \text{so} \quad f(\frac{\mathbf{x}+\mathbf{y}}{2}) \geq t$$

and

$$\frac{\mathbf{x}+\mathbf{y}}{2} \prec \mathbf{y} \quad \text{so} \quad f(\frac{\mathbf{x}+\mathbf{y}}{2}) < t,$$

the contradiction. □

Lemma 2. *Each nonempty level line of the nondecreasing function $f : \mathbf{R}^n \to \mathbf{R}$ is a semidecreasing hypersurface.*

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in P_t(f)$. Fix $\mathbf{u} = (u_1, u_2, \dots, u_{n-1}) \in \mathbf{R}^{n-1}$. Let $a = \min\{x_1, x_2 + u_1, x_3 + u_2, \dots, x_n + u_{n-1}\}\ - 1$, and $b = \max\{x_1, x_2 + u_1, x_3 + u_2, \dots, x_n + u_{n-1}\}\ + 1$. Then

$$f(a, a - u_1, \dots, a - u_{n-1}) < t \leq f(b, b - u_1, \dots, b - u_{n-1}).$$

Hence the number $x(\mathbf{u}) = \inf\{x : t \leq f(x, x - u_1, \dots, x - u_{n-1})\}$ is finite.

Let us define:

$$g_1(\mathbf{u}) = x(\mathbf{u}),\ g_2(\mathbf{u}) = x(\mathbf{u}) - u_1,\ \dots,\ g_n(\mathbf{u}) = x(\mathbf{u}) - u_{n-1}.$$

This is easy to see, that for every $\mathbf{u} \in \mathbf{R}^{n-1}$ the point

$$(g_1(\mathbf{u}), g_2(\mathbf{u}), \ldots, g_n(\mathbf{u})) \in P_t(f).$$

Now we show, that $g_1, g_2^{(1)}, \ldots, g_n^{(n-1)}$ are nondecreasing functions of $(n-1)$ variables.

Let $\mathbf{u} \preceq \mathbf{v}$. Then for every $x \in \mathbf{R}$

$$(x, x-u_1, \ldots, x-u_{n-1}) \succeq (x, x-v_1, \ldots, x-v_{n-1}),$$

so

$$f(x, x-u_1, \ldots, x-u_{n-1}) \geq f(x, x-v_1, \ldots, x-v_{n-1}),$$

and, consequently

$$\{x : t \leq f(x, x-u_1, \ldots, x-u_{n-1})\} \supset \{x : t \leq f(x, x-v_1, \ldots, x-v_{n-1})\}$$

hence

$$\inf\{x : t \leq f(x, x-u_1, \ldots, x-u_{n-1})\} \leq \inf\{x : t \leq f(x, x-v_1, \ldots, x-v_{n-1})\}.$$

As a result $g_1(\mathbf{u}) \leq g_1(\mathbf{v})$.

Let us fix $i \in \{2, 3, \ldots, n\}$. We will prove that the function $g_i^{(i-1)}$ is nondecreasing. By virtue of proposition 1 it suffices to check, that the function g_i is *nonincreasing* with respect to the $(i-1)$-th variable, and is nondecreasing with respect to others.

At first we show, that $g_i(u_1, u_2, \ldots, u_{n-1}) = g_1(u_1, u_2, \ldots, u_{n-1}) - u_{i-1}$ is nonincreasig with respect to the $(i-1)$-th variable. In order to do this consider $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{u}^* = (u_1, u_2, \ldots, u_{i-1}+h, \ldots, u_n)$, where $h > 0$. We have

$$\begin{aligned}
g_1(\mathbf{u}^*) &= \inf\{x : t \leq f(x, x-u_1, \ldots, x-u_{i-1}-h, \ldots, x-u_{n-1})\} = \\
&= \inf\{x+h : t \leq f(x+h, x-u_1+h, \ldots, x-u_{i-1}, \ldots, x-u_{n-1}+h)\} = \\
&= \inf\{x : t \leq f(x+h, x-u_1+h, \ldots, x-u_{i-1}, \ldots, x-u_{n-1}+h)\} + h \leq \\
&\leq \inf\{x : t \leq f(x, x-u_1, \ldots, x-u_{i-1}, \ldots, x-u_{n-1})\} + h = g_1(\mathbf{u}) + h.
\end{aligned}$$

So

$$\begin{aligned}
g_1(\mathbf{u}^*) - h &\leq g_1(\mathbf{u}), \\
g_1(\mathbf{u}^*) - u_{i-1} - h &\leq g_1(\mathbf{u}), \\
g_i(\mathbf{u}^*) &\leq g_i(\mathbf{u}).
\end{aligned}$$

The last inequality shows, that g_i is nonincreasing with respect to the $(i-1)$-th variable.

Of course $g_i(\mathbf{u}) = x(\mathbf{u}) - u_{i-1}$ is nondecreasing with respect to any other variable, as g_1 is.

Now we check, that the function g_1 is continuous. In order to do this take $\mathbf{u}, \mathbf{v} \in \mathbf{R}^{n-1}$. Let

$$\mathbf{u}^* = (\min(u_1, v_1), \min(u_2, v_2), \dots, \min(u_n, v_n))$$

and

$$\mathbf{u}^{**} = (\max(u_1, v_1), \max(u_2, v_2), \dots, \max(u_n, v_n)).$$

Note, that $d[\mathbf{u}^*, \mathbf{u}^{**}] = d[\mathbf{u}, \mathbf{v}]$.

According to the previous part of the proof, by induction we have, that

$$g_1(\mathbf{u}^*) \le g_1(\mathbf{u}^{**}) \le g_1(\mathbf{u}^*) + \sum_{i=1}^{n-1} |u_i^{**} - u_i^*|,$$

hence

$$|g_1(\mathbf{u}^*) - g_1(\mathbf{u}^{**})| \le \sum_{i=1}^{n-1} |u_i^{**} - u_i^*| \le (n-1) * d[\mathbf{u}^*, \mathbf{u}^{**}].$$

and, further

$$|g_1(\mathbf{u}) - g_1(\mathbf{v})| \le (n-1) * d[\mathbf{u}, \mathbf{v}].$$

From this we obtain, that the function g_1 satisfies the Lipschitz condition, and, as a result, is continuous. Of course it implies the continuity of functions $g_2, \ g_3, \dots, \ g_n$. □

Theorem 1. *The set of discontinuity points of a given nondecreasing function $f : \mathbf{R}^n \to \mathbf{R}$ is contained in a countable union of semidecreasing hypersurfaces.*

Proof. The set E of discontinuity points of a given nondecreasing function $f : \mathbf{R}^n \to \mathbf{R}$ can be expressed as $E = \bigcup_{k \in \mathbf{N}} E_k$, where

$$E_k = \left\{ \mathbf{x} : \inf_{\mathbf{x} \prec \mathbf{y}} f(\mathbf{y}) - \sup_{\mathbf{y} \prec \mathbf{x}} f(\mathbf{y}) \ge \frac{1}{k} \right\}.$$

Let W be a given countable and dense subset of $\mathbf{R}$. We show, that

$$E_k \subset \bigcup_{j \in W} P_j(f).$$

Let $\mathbf{x} \in E_k$. There exists such a number $j \in W$, that

$$\sup_{\mathbf{y} \prec \mathbf{x}} f(\mathbf{y}) < j \le \inf_{\mathbf{x} \prec \mathbf{y}} f(\mathbf{y}),$$

so $\mathbf{x} \in P_j(f)$. □

Lemma 3. *If $\mathbf{x}, \mathbf{y} \in P_t(f)$ and $\mathbf{x} \preceq \mathbf{y}$, then the segment $\overline{\mathbf{xy}} \subset P_t(f)$.*

The proof is obvious and hence ommited.

Theorem 2. *Let* $f : \mathbf{R}^n \to \mathbf{R}$ *be a nondecreasing function. If* $\mathbf{x}, \mathbf{y} \in P_t(f)$ *are comparable, and the function* f *is continuous in those points, then* f *is continuous in every point of the segment* $\overline{\mathbf{xy}}$.

Proof. Suppose, contrary to our claim, that there exists such a point $\mathbf{z} \in \overline{\mathbf{xy}}$ that the function f is not continuous in $\mathbf{z}$. Hence there exists such $\epsilon > 0$ that for any $\delta > 0$ there exists a point $\mathbf{z}^*$ such that $d(\mathbf{z}, \mathbf{z}^*) < \delta$ and $|f(\mathbf{z}) - f(\mathbf{z}^*)| > \epsilon$.

Since f is continuous in $\mathbf{x}$, there exists such $\delta_{\mathbf{x}} > 0$ that the condition $d(\mathbf{x}, \mathbf{x}^*) < \delta_{\mathbf{x}}$ implies $|t - f(\mathbf{x}^*)| < \frac{\epsilon}{2}$. Analogously, there exists such $\delta_{\mathbf{y}} > 0$ that the condition $d(\mathbf{y}, \mathbf{y}^*) < \delta_{\mathbf{y}}$ implies $|t - f(\mathbf{y}^*)| < \frac{\epsilon}{2}$.

Assume, that $\mathbf{x} \preceq \mathbf{y}$. Let $\delta = \frac{1}{2}\min(\delta_{\mathbf{x}}, \delta_{\mathbf{y}})$, and let $d(\mathbf{z}, \mathbf{z}^*) < \delta$ and $|f(\mathbf{z}) - f(\mathbf{z}^*)| > \epsilon$. Then

$$(x_1 - \delta, x_2 - \delta, \ldots, x_n - \delta) \prec \mathbf{z}^* \prec (y_1 + \delta, y_2 + \delta, \ldots, y_n + \delta),$$

but

$$f(\mathbf{z}^*) < f(x_1 - \delta, x_2 - \delta, \ldots, x_n - \delta) \text{ or } f(\mathbf{z}^*) > f(y_1 + \delta, y_2 + \delta, \ldots, y_n + \delta).$$

This contradicts the monotonicity of the function f. □

References

[1] W. Lindner, *A characterization of the set of discontinuity points of a two variables monotone function,* Scientific Bulletin of Łódź Technical University **466** (1986).

[2] S. Mazurkiewicz, *Podstawy rachunku prawdopodobieństwa,* Monografie Matematyczne **32** (1956), PWN, Warszawa.