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REGULARITY FROM THE STANDPOINT OF SMALL SYSTEMS

OĹGA KULCSÁROVÁ‡

Abstract. The abstract formulation of regularity was studied e.g. in papers [8], [10] and [11] ([6]). In this note there will be shown that the regularity of a Baire small system is a consequence of the regularity of every subadditive Baire measure.

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Let X be a locally compact topological space, $\mathbb{C}$ the system of all compact G_δ subsets of X, $\mathcal{S}$ the σ-ring of subsets of X generated by $\mathbb{C}$ (Baire sets), $\mathbb{U}$ the system of all open sets, belonging to $\mathcal{S}$, $N = \{0, 1, 2, \dots\}$. The set of all reals is denote by $\mathcal{R}$.

A sequence $\{\mathcal{N}_n\}_{n=0}^{\infty}$ of subsets of $\mathcal{S}$ is called a small system if it fulfils the following conditions:

(1) $\forall n \in N\ \phi \in \mathcal{N}_n$; $A, B \in \mathcal{N}_0 \implies A \cup B \in \mathcal{N}_0$;
(2) $E_i \in \mathcal{S}$, $E_{i+1} \subset E_i$, $i = 1, 2, \dots$,
$\bigcap_{i=1}^{\infty} E_i = \phi \Rightarrow \forall n \in N\ \exists m \in N\ E_m \in \mathcal{N}_n$;
(3) $\forall n \in N\ \mathcal{N}_{n+1} \subset \mathcal{N}_n$;
(4) $\forall n \in N\ E_i \in \mathcal{N}_{n+i}$, $i = 1, 2, \dots \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n$.

A small system is called an aproximative small system if it fulfils moreover the condition

(5) $E_i \in \mathcal{N}_{r_i}$, $i = 1, 2, \dots, k$, $\sum_{i=1}^{k} \frac{1}{2^{r_i}} \leqq \frac{1}{2^n} \implies \bigcup_{i=1}^{k} E_i \in \mathcal{N}_n$.

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‡ *Faculty of Sciences, Šafárik University*, SK–041 54 Košice, Slovakia. E-mail: kulcsarova@science.upjs.sk.
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Example 1. *Let* $X = \langle 0,1\rangle$, $\mathcal{S}$ *be a set of all Baire subsets of* X, $\mu\ \mathcal{S} \to \langle 0,1\rangle$ *is the Lebesgue measure,* $\mathcal{N}_n = \left\{E \in \mathcal{S}\ \mu(E) < \frac{1}{2^{n+1}}\right\}$, $n = 3,4,\dots,$ $\mathcal{N}_2 = \left\{E \in \mathcal{S}\ \mu(E) < \frac{1}{3}\right\}$, $\mathcal{N}_1 = \left\{E \in \mathcal{S}\ \mu(E) < \frac{1}{2}\right\}$, $\mathcal{N}_0 = \mathcal{S}$. *The sequence* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *is a small system, but it isn't an aproximative small system (*[9]*).*

If $\{\mathcal{N}_n\}_{n=0}^{\infty}$ is an aproximative small system of subsets of $\mathcal{S}$, then there holds ([9]):

(a) $E \subset F,\ F \in \mathcal{N}_n,\ E \in \mathcal{S} \implies E \in \mathcal{N}_n$;
(b) $\forall n \in N\ \ \exists p,q \in N\ A \in \mathcal{N}_p,\ B \in \mathcal{N}_q \implies A \cup B \in \mathcal{N}_n$;
(c) $\forall n \in N\ \ \exists m \in N\ A,B \in \mathcal{N}_m \implies A \cup B \in \mathcal{N}_n$;
(d) $\forall n \in N\ \ \exists \{k_i\}_{i=1}^{\infty}\ E_i \in \mathcal{N}_{k_i},\ \ i = 1,2,\dots \implies \bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{N}_n$.

The sequence $\{\mathcal{N}_n\}_{n=0}^{\infty}$ of subsets of $\mathcal{S}$ will be called regular if it fulfils the following condition:

(r) $\forall n \in N\ \ \forall E \in \mathcal{S}\ \ \exists U \in \mathbb{U},\ \ \exists C \in \mathbb{C}\ C \subset E \subset U$ and $U \setminus C \in \mathcal{N}_n$.

The condition (r) includes both of properties $U \setminus E \in \mathcal{N}_n$, $E \setminus C \in \mathcal{N}_n$ from Theorem 2, [8] (see [11]).

A function $\mu\mathcal{S} \to \langle 0,\infty\rangle$ is called a subadditive measure if it is monotone, subadditive, $\mu(\phi) = 0$ and upper continuous in ϕ.

A subadditive measure μ defined on $\mathcal{S}$ is called regular if

$$\forall \varepsilon \in R,\ \varepsilon > 0\ \ \forall E \in \mathcal{S}\ \ \exists U \in \mathbb{U},\ \exists C \in \mathbb{C}\ C \subset E \subset U \text{ and } \mu(U \setminus C) < \varepsilon.$$

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Let μ be a subadditive measure defined on $\mathcal{S}$ and $\{\mathcal{N}_n\}_{n=0}^{\infty}$ be a small system on $\mathcal{S}$. We say that μ and $\{\mathcal{N}_n\}_{n=0}^{\infty}$ are equivalent (see [9], [5]) on $\mathcal{S}$ if the conditions are fulfiled:

(i) $\forall \varepsilon \in R,\ \varepsilon > 0\ \ \exists n \in N\, \forall E \in \mathcal{S},\ E \in \mathcal{N}_n \implies \mu(E) < \varepsilon$;
(ii) $\forall n \in N\ \ \exists \varepsilon \in R,\ \varepsilon > 0\, \forall E \in \mathcal{S},\ \mu(E) < \varepsilon \implies E \in \mathcal{N}_n$.

The following theorem is known ([5]).

Theorem 1. *For every subadditive measure* μ*, defined on* $\mathcal{S}$*, there exists a small system* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *of subsets of* $\mathcal{S}$ *such that* μ *and* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *are equivalent on* $\mathcal{S}$*. For every aproximative small system* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *of subsets of* $\mathcal{S}$ *there exists a subadditive measure* ν*, defined on* $\mathcal{S}$*, such that* ν *and* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *are equivalent on* $\mathcal{S}$*.*

Theorem 2. *Let there be given a subadditive measure* μ*, defined on* $\mathcal{S}$*, and a small system* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *of subsets of* $\mathcal{S}$*. Let* μ *and* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *are equivalent on* $\mathcal{S}$*. Then* μ *is regular iff* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *is regular.*

Proof. Let μ be regular and $n \in N$. There exists $\varepsilon > 0$ (since μ and $\{\mathcal{N}_n\}_{n=0}^{\infty}$ are equaivalent on $\mathcal{S}$) such, that if $\mu(A) < \varepsilon$, $A \in \mathcal{S}$, then $A \in \mathcal{N}_n$. For this ε and any $E \in \mathcal{S}$ there exist $U \in \mathbb{U}$, $C \in \mathbb{C}$, for which $C \subset E \subset U$ and $\mu(U \setminus C) < \varepsilon$, i.e. $U \setminus C \in \mathcal{N}_n$. Now let $\{\mathcal{N}_n\}_{n=0}^{\infty}$ be regular and $\varepsilon > 0$. Then there exists $n \in N$ such, that if $A \in \mathcal{N}_n$, $A \in \mathcal{S}$, then $\mu(A) < \varepsilon$. For this n and any $E \in \mathcal{S}$ there exist $U \in \mathbb{U}$, $C \in \mathbb{C}$, such that $C \subset E \subset U$ and $U \setminus C \in \mathcal{N}_n$, i.e. $\mu(U \setminus C) < \varepsilon$. □

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Theorem 3. *Let μ be a subadditive Baire measure. Then it is regular.*

Proof. Denote

$$\mathcal{K} = \{E \in \mathcal{S}\, \forall \varepsilon \in R,\ \varepsilon > 0\ \exists C \in \mathbb{C},\ \exists U \in \mathbb{U}\ C \subset E \subset U \text{ and } \mu(U \setminus C) < \varepsilon\}.$$

We are going to prove $\mathbb{C} \subset \mathcal{K}$.

Let $\varepsilon > 0$, $D \in \mathbb{C}$. Then we can choose a monotone sequence $\{U_n\}_{n=0}^{\infty} \subset \mathbb{U}$, $U_{n+1} \subset U_n$, $n = 1, 2, \ldots$ such that $D = \bigcap_{n=1}^{\infty} U_n$. There is $\bigcap_{n=1}^{\infty} (U_n \setminus D) = \phi$ and $U_{n+1} \setminus D \subset U_n \setminus D$, $n = 1, 2, \ldots$ Since μ is upper continuous in ϕ, for $\varepsilon > 0$ there exists a number $m \in N$, for which $\mu(U_m \setminus D) < \varepsilon$. Now if $C = D$ and $U = U_m$, then $D \in \mathcal{K}$ since $D \subset D \subset U_m$ and $\mu(U_m \setminus D) < \varepsilon$.

Next we will prove that $\mathcal{K}$ is a σ-ring.

If $A, B \in \mathcal{K}$, then $A \setminus B \in \mathcal{K}$, too. Namely, if $A, B \in \mathcal{K}$ and $\varepsilon > 0$, then there exist $C_A, C_B \in \mathbb{C}_m$, $U_A, U_B \in \mathbb{U}$, such that $C_A \subset A \subset U_A$, $C_B \subset B \subset U_B$, $\mu(U_A \setminus C_A) < \frac{\varepsilon}{2}$, $\mu(U_B \setminus C_B) < \frac{\varepsilon}{2}$. Take $C_A \setminus U_B \in \mathbb{C}$, $U_A \setminus C_B \in \mathbb{U}$. It is easy to verify, that $C_A \setminus U_B \subset A \setminus B \subset U_A \setminus C_B$ and $\mu((U_A \setminus C_B) \setminus (C_A \setminus U_B)) < \varepsilon$. Now let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{K}$, $A = \bigcup_{n=1}^{\infty} A_n$. Put $B_n = \bigcup_{i=1}^{n} A_i$, $n = 1, 2, \ldots$, then $B_n \subset B_{n+1}$, $n = 1, 2, \ldots$, $A = \bigcup_{n=1}^{\infty} B_n$. Using Theorem 3 of [1] (X, §52), for A and $\varepsilon > 0$, there exist $U \in \mathbb{U}$ and $C \in \mathbb{C}$, such that $\mu(U \setminus A) < \frac{\varepsilon}{2}$ and $\mu(A \setminus C) < \frac{\varepsilon}{2}$. It follows

$$\mu(U \setminus C) = \mu((U \setminus A) \cup (A \setminus C)) \le \mu(U \setminus A) + \mu(A \setminus C) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\mathcal{K} = \mathcal{S}$ follows from the facts, that $\mathcal{K}$ is a σ-ring, $\mathbb{C} \subset \mathcal{K}$ and $\mathcal{K} \subset \mathcal{S}$. This completes the proof. □

Claim 1. *Let $\{\mathcal{N}_n\}_{n=0}^{\infty}$ be an aproximative small system of subsets of $\mathcal{S}$. Then it is regular.*

Proof. If $\{\mathcal{N}_n\}_{n=0}^{\infty}$ is an aproximative small system of subsets of $\mathcal{S}$, then by theorem 1 there exists a subadditive Baire measure, which is equivalent to $\{\mathcal{N}_n\}_{n=0}^{\infty}$ on $\mathcal{S}$. This subadditive Baire measure is regular and then the sequence $\{\mathcal{N}_n\}_{n=0}^{\infty}$ is also regular too, by Theorem 2. □

Theorem 3 generalizes the well know fact from measure theory (see [6]). Many problems of (not only) measure theory can be formulated and proved by the help of the small systems (see for instance [3], [12], [13], [2], [6], [7]).

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