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THE k-PSEUDO-SYMMETRIC $\mathfrak{I}$-APPROXIMATE DERIVATIVES

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Abstract. In article [5] the authors introduced the notion of k-pseudo-symmetric approximate derivative of a function

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

which is a generalization of the symmetric derivative. In this paper we give a definition of k-pseudo-symmetric $\mathfrak{I}$-approximate derivative of a function. Next we prove several properties of derivatives of this kind.

1. Introduction

Throughout this paper $\mathfrak{B}$ will denote the family of all subsets of $\mathbb{R}$ (the real line) having Baire property, $\mathfrak{I}$ will denote the σ-ideal of the sets of the first category. For two sets A and B contained in $\mathbb{R}$, the symbol $A \sim B$ will denote that $(A \Delta B) \in \mathfrak{I}$. For $a \in \mathbb{R}$ and $A \subset \mathbb{R}$ we shall use the following denotations

$$a \cdot A = \{ax : x \in A\} \quad \text{and} \quad A - a = \{x - a : x \in A\}.$$

Recall [4] that 0 is an $\mathfrak{I}$-density point of a set $A \in \mathfrak{B}$, if and only if $\chi_{(n \cdot A) \cap [-1,1]} \xrightarrow{\mathfrak{I}} 1$ i.e., if for every increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that $\chi_{(n_{m_p} \cdot A) \cap [-1,1]} \longrightarrow 1$ except on a set belonging to $\mathfrak{I}$ (in abbr. $\mathfrak{I}$-ae). A point $x_0 \in \mathbb{R}$ is an $\mathfrak{I}$-density point of $A \in \mathfrak{B}$ if and only if 0 is an $\mathfrak{I}$-density point of $A - x_0$. The set of all $\mathfrak{I}$-density points of a set A will be denoted by $\Phi(A)$.

Recall that the operation Φ has the following properties:

- $\Phi(A) \sim A$ for each $A \in \mathfrak{B}$;
- if $A \in \mathfrak{B}$, $B \in \mathfrak{B}$ and $A \sim B$, then $\Phi(A) = \Phi(B)$;

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- $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}) = \mathbb{R}$;
- $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for each $A \in \mathfrak{B}$ and $B \in \mathfrak{B}$.

Further the family $\mathcal{T}_{\mathfrak{I}} = \{A \in \mathfrak{B} \ : A \subset \Phi(A)\}$ is a topology which we call $\mathfrak{I}$-density topology (see [4]).

Real functions which are continuous with respect to the topology $\mathcal{T}_{\mathfrak{I}}$ we call the $\mathfrak{I}$-approximately continuous functions.

Throughout this paper $\mathrm{cl}(A)$, $\mathrm{int}(A)$ will denote the closure and the interior of the set A with respect to natural topology. Except where a topology is specifically mentioned, all topological notions are considered with respect to the natural topology.

Definition 1. [1] *For $x \in \mathbb{R}$ the symbol $\mathfrak{P}(x)$ will denote the family of all closed intervals $[a, b]$ such that $x \in (a, b)$ and of all interval sets of the form*

$$\bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{x\},$$

where $b_{n-1} < a_n < b_n < x$ and $x < c_n < d_n < c_{n-1}$ for all positive integers n and

$$x \in \Phi\left(\bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n]\right).$$

Definition 2. [1] *Let τ be the class of all subsets U of $\mathbb{R}$ such that:*

1. $U \in T_{\mathfrak{I}}$;

2. if $U \neq \emptyset$ and $x \in U$, then there exists a set $P \in \mathfrak{P}(x)$ such that $P \subset \mathrm{int}(U) \cup \{x\}$.

Theorem 1. *The class τ is a topology in the set $\mathbb{R}$. Moreover $\tau \subset T_{\mathfrak{I}}$, and if f is an $\mathfrak{I}$-approximately continuous function, then f is continuous with respect to τ.*

The topology τ was introduced in [1] and it is the coarsest topology for which the class of $\mathfrak{I}$-approximately continuous functions is contained in the class of continuous functions with respect to this topology.

In this paper we shall need the following lemmas:

Lemma 1. [4]. *If 0 is an $\mathfrak{I}$-density point of a set A, then for every positive integer n there exists a number $\delta_n > 0$ such that for every h, with $0 < h < \delta_n$, and for every positive integer k fulfilling the inequality $-n \leq k \leq n-1$ we have*

$$A \cap \left[\frac{k}{n} \cdot h, \frac{k+1}{n} \cdot h\right] \neq \emptyset.$$

Lemma 2. *Let $G \subset \mathbb{R}$ be an open set. Then 0 is an $\mathfrak{I}$-dispersion point of G, if and only if for every positive integer n there exist a positive integer k and* *a real number $\delta > 0$ such that for each $h \in (0, \delta)$ and for each $i \in \{1, \ldots, n\}$ there exist two positive integers $j_r \in \{1, \ldots, k\}$ and $j_l \in \{1, \ldots, k\}$ such that*

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{n \cdot k} \right) \cdot h, \left(\frac{i-1}{n} + \frac{j_r}{n \cdot k} \right) \cdot h \right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j_l}{n \cdot k} \right) \cdot h, -\left(\frac{i-1}{n} + \frac{j_l - 1}{n \cdot k} \right) \cdot h \right) = \emptyset.$$

We shall use the above lemmas for $x \in \mathbb{R}$ by translating the set if necessary.

2. Main results

Definition 3. *Let k be a positive integer. Let a function $f : [a, b] \longrightarrow \mathbb{R}$ and $c \in (a, b)$. The greatest lower bound of all the numbers α ($+\infty$ – included) for which the Baire'a cover of the set*

$$\left\{ t : \frac{f(c + kt) - f(c - t)}{(k+1)t} \leq \alpha \right\}$$

has 0 as an $\mathfrak{I}$-density point, is called upper k-pseudo-symmetric $\mathfrak{I}$-approximate derivative of f at the point c, which is denoted by

$$\overline{f}^{[k]}_{s-\mathfrak{I}-ap}(c).$$

Similarly we can define lower k-pseudo-symmetric $\mathfrak{I}$-approximate derivative, which is denoted by $\underline{f}^{[k]}_{s-\mathfrak{I}-ap}(c)$.

If these two derivatives are equal, then their common value is called k-pseudo-symmetric $\mathfrak{I}$-approximate derivative of f at the point c and is denoted by $f^{[k]}_{s-\mathfrak{I}-ap}(c)$.

At the end points a and b, we define

$$\overline{f}^{[k]}_{s-\mathfrak{I}-ap}(a) = \overline{f}_{\mathfrak{I}-ap}(a)$$

and

$$\underline{f}^{[k]}_{s-\mathfrak{I}-ap}(b) = \underline{f}_{\mathfrak{I}-ap}(b),$$

where $\overline{f}_{\mathfrak{I}-ap}$ and $\underline{f}_{\mathfrak{I}-ap}$ are ordinary upper, respectively lower $\mathfrak{I}$-approximate derivatives (see [3]).

Theorem 2. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an $\mathfrak{I}$-approximately continuous function, then $\overline{f}^{[k]}_{s-\mathfrak{I}-ap}$ and $\underline{f}^{[k]}_{s-\mathfrak{I}-ap}$ have the property of Baire.*

Proof. First we observe that

$$\underline{f}^{[k]}_{s-\mathfrak{I}-ap}(x) = -\left(\overline{(-f)}^{[k]}_{s-\mathfrak{I}-ap}(x)\right).$$

Therefore it is sufficient to show that the set

$$A = \left\{x : \overline{f}^{[k]}_{s-\mathfrak{I}-ap}(x) < a\right\}$$

has the property of Baire for each $a \in \mathbb{R}$.

Let

$$H(x,h) = \frac{f(x+kh) - f(x-h)}{(k+1)h},$$

where $x \in \mathbb{R}$ and $h > 0$. Let $a \in \mathbb{R}$ and $\{a_m\}_{m\in\mathbb{N}}$ be an arbitrary sequence such that, $a_m < a_{m+1} < a$ for each $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} a_m = a.$$

For each $m \in \mathbb{N}$, we shall denote

$$B_m(x) = \{h > 0 : H(x,h) > a_m\}$$

and

$$A_m = \{x : B_m(x) \text{ has an } \mathfrak{I}\text{-dispersion point at } 0\}.$$

It is obvious that

$$A = \bigcup_{m=1}^{\infty} A_m.$$

Since for each $x \in \mathbb{R}$ the function $H(x,h) : \mathbb{R}^+ \to \mathbb{R}$ is $\mathfrak{I}$-approximately continuous, then for each $x \in \mathbb{R}$ and $m \in \mathbb{N}$ we have

$$B_m(x) \in \tau, \qquad \operatorname{int}(B_m(x)) \neq \emptyset$$

and

$$A_m = \{x : \operatorname{int}(B_m(x)) \text{ has an } \mathfrak{I}\text{-dispersion point at } 0\}.$$

Let $m \in \mathbb{N}$ and $x \in \mathbb{R}$. From Lemma 5 we conclude that for each $n \in \mathbb{N}$ there exist $l \in \mathbb{N}$ and $p \in \mathbb{N}$ such that for each $\delta \in \left(0, \frac{1}{p}\right)$ and for each $i \in \{1,\dots,n\}$ there exists $j \in \{1,\dots,l\}$ such that

$$\left(\frac{(i-1)\cdot l + j - 1}{n\cdot l}\cdot\delta, \frac{(i-1)\cdot l + j}{n\cdot l}\cdot\delta\right) \cap \operatorname{int}(B_m(x)) = \emptyset.$$

Thus

$$A = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcup_{p=1}^{\infty}\bigcap_{\delta\in\left(0,\frac{1}{p}\right)}\bigcap_{i=1}^{n}\bigcup_{j=1}^{l} D(m,n,l,p,\delta,i,j),$$

where

$$D(m,n,l,p,\delta,i,j) =$$
$$= \left\{x : \left(\frac{(i-1)\cdot l+j-1}{n\cdot l}\delta, \frac{(i-1)\cdot l+j}{n\cdot l}\delta\right) \cap \operatorname{int}(B_m(x)) = \emptyset\right\}.$$

Let $m \in \mathbb{N}$, $n \in \mathbb{N}$, $l \in \mathbb{N}$, $p \in \mathbb{N}$, $\delta \in \left(0, \frac{1}{p}\right)$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, l\}$. We shall show that $D = D(m,n,l,p,\delta,i,j)$ is a closed set with respect to the topology τ.

Let $x_0 \notin D$. Then there exists an open interval (α, β) such that

$$(\alpha,\beta) \subset \operatorname{int}(B_m(x_0)) \cap \left(\frac{(i-1)\cdot l+j-1}{n\cdot l}\cdot\delta, \frac{(i-1)\cdot l+j}{n\cdot l}\cdot\delta\right).$$

From $\mathfrak{I}$-approximately continuity of the function $H(x_0,h)$ in (α,β) (see[4]), we know that there exists $h_0 \in (\alpha,\beta)$ such that $H(x_0,h)$ is continuous at h_0. Since $H(x_0,h_0) > a_m$, then there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that,

$$(*) \qquad \text{if } |h-h_0| < \eta, \text{ then } \frac{f(x_0+kh)-f(x_0-h)}{(k+1)h} > a_m + \varepsilon_0 > a_m.$$

Let $\varepsilon < (k+1)\varepsilon_0 h_0$.

Consider the point $x_0 - h_0$. The function f is $\mathfrak{I}$-approximately continuous at the point $x_0 - h_0$ and therefore $x_0 - h_0$ belongs to the set

$$\left\{t : |f(t) - f(x_0-h_0)| < \frac{\varepsilon}{2}\right\},$$

which is open with respect to the topology τ. If

$$C = \operatorname{int}\left(\left\{t : |f(t) - f(x_0-h_0)| < \frac{\varepsilon}{2}\right\}\right) \cup \{x_0 - h_0\}$$

then $C \in \tau$.

Let

$$E = C \cap (-kC + (k+1)x_0 - (k+1)h_0).$$

We observe that $E \in \tau$, $x_0 - h_0 \in E$ and

$$(**) \qquad |f(t_1) - f(t_2)| < \varepsilon \text{ for each } t_1 \in E,\ t_2 \in E.$$

In a similar way we can state that

$$K = \operatorname{int}\left(\left\{t : |f(t) - f(x_0+h_0)| < \frac{\varepsilon}{2}\right\}\right) \cup \{x_0 + h_0\} \in \tau.$$

We put

$$F = K \cap \left(-\frac{1}{k}\cdot K + \frac{k+1}{k}, x_0 + \frac{k+1}{k}\cdot h_0\right).$$

Then $x_0 + h_0 \in F$ and

$$(***) \qquad |f(t_1) - f(t_2)| < \varepsilon \text{ for each } t_1 \in F,\ \ t_2 \in F.$$

Let

$$E_1 = (E \cap (x_0 - h_0 - \eta, x_0 - h_0]) + h_0$$

and $x \in E_1$. Then $x - x_0 \in [-\eta, 0]$ and $x - h_0 \in E$. Therefore, if $kh = x - x_0$ then $x_0 - h_0 + kh = x - h_0 \in C$. Additionally, since

$$x - h_0 \in -kC + (k+1)x_0 - (k+1)h_0,$$

then $x_0 - h - h_0 \in C$.

Then, by $(*)$ and $(**)$, we infer that

$$\frac{f(x + kh_0) - f(x - h_0)}{(k+1)h_0} = \frac{f(x_0 + kh_0 + kh) - f(x_0 - h_0 + kh)}{(k+1)h_0} >$$

$$> \frac{f(x_0 + kh_0 + kh) - f(x_0 - h_0 - h) - \varepsilon}{(k+1)h_0} >$$

$$> \frac{f(x_0 + kh_0 + kh) - f(x_0 - h_0 - h)}{(k+1)(h_0 + h)} - \frac{\varepsilon}{(k+1)h_0} >$$

$$> a_m + \varepsilon_0 - \frac{\varepsilon}{(k+1)h_0} > a_m.$$

Therefore $x \notin D$.

Now, let $F_1 = (F \cap [x_0 + h_0, x_0 + h_0 + \eta)) - h_0$ and $x \in F_1$. In a similar way, from $(*)$ and $(***)$ we can show that $x \notin D$.

Let $M = F_1 \cup E_1$. Then $M \in \tau$, $x_0 \in M$ and $M \subset \mathbb{R} \setminus D$. Thus D is a closed set with respect to the topology τ, hence D has the property of Baire, which completes the proof. □

In a similar way we can prove the following

Corollary 1. *If $f : R \longrightarrow R$ is a continuous function, then $\overline{f}^{[k]}_{s-\mathfrak{I}-ap}$ and $\underline{f}^{[k]}_{s-\mathfrak{I}-ap}$ belong to the third class of Baire.*

Theorem 3. *Let f be a function defined in an open interval I. If f is monotone, then*

$$\underline{f}^{[k]}_{s-\mathfrak{I}-ap}(x) = \underline{f}^{[k]}_{s}(x)$$

and

$$\overline{f}^{[k]}_{s-\mathfrak{I}-ap}(x) = \overline{f}^{[k]}_{s}(x)$$

for each $x \in I$.

Proof. We shall only prove that

$$\underline{f}^{[k]}_{s-\mathfrak{I}-ap}(x) = \underline{f}^{[k]}_{s}(x).$$

Assume that f is a nondecreasing function. First we observe that

$$\underline{f}^{[k]}_{s}(x) \leq \underline{f}^{[k]}_{s-\mathfrak{I}-ap}(x)$$

for each $x \in I$.

Now suppose that there exists $x_0 \in I$ such that

$$k_1 = \underline{f}^{[k]}_{s}(x_0) < \underline{f}^{[k]}_{s-\mathfrak{I}-ap}(x_0) = k_2.$$

Let $\varepsilon \in \left(0, \frac{k_2-k_1}{2}\right)$ and

$$B = \left\{h > 0 : \frac{f(x_0+kh) - f(x_0-h)}{(k+1)h} \geq k_2 - \frac{\varepsilon}{2}\right\}.$$

Since $\underline{f}^{[k]}_{s-\mathfrak{I}-ap}(x_0) = k_2$, then 0 is a right-hand $\mathfrak{I}$-density point of the set B. Thus in view of Lemma 4

$$\forall_{n\in\mathbb{N}}\exists_{\delta_n>0}\forall_{h\in(0,\delta_n)}\forall_{l\in\mathbb{N}\cap[-n,n-1]}\left(\left[\frac{l}{n}\cdot h, \frac{l+1}{n}\cdot h\right] \cap B \neq \emptyset\right). \tag{1}$$

From assumption that $\underline{f}^{[k]}_{s}(x_0) = k_1 < k_2 - 2\varepsilon$, we infer that there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} h_n = 0$,

$$h_n > 0, \ \ x_0 + kh_n \in I, \ \ x_0 - h_n \in I$$

and

$$\frac{f(x_0+kh_n) - f(x_0-h_n)}{(k+1)h_n} < k_2 - 2\varepsilon \tag{2}$$

for each $n \in \mathbb{N}$.

We shall consider closed intervals

$$J_n = \left[\frac{f(x_0+kh_n) - f(x_0-h_n)}{(k+1)(k_2-\varepsilon)}, h_n\right]$$

for each $n \in \mathbb{N}$ and we shall show that

$$B \cap J_n = \emptyset$$

for each $n \in \mathbb{N}$.

Indeed, if $h \in J_n$ then

$$\frac{f(x_0+kh_n) - f(x_0-h_n)}{(k+1)(k_2-\varepsilon)} \leq h \leq h_n.$$

Thus from the monotonicity of the function f we infer that

$$f(x_0+kh) \leq f(x_0+kh_n) \leq f(x_0-h_n) + (k+1)h(k_2-\varepsilon) \leq$$

$$\leq f\left(x_0 - h\right) + (k+1)h(k_2 - \varepsilon)$$

and consequently

$$\frac{f(x_0 + kh) - f(x_0 - h)}{(k+1)h} \leq k_2 - \varepsilon.$$

Hence $h \notin B$.

Now let $n_0 > \frac{k_2 - \varepsilon}{\varepsilon}$, $\delta_{n_0} \in \mathbb{R}$ be chosen such that δ_{n_0} and n_0 satisfy condition (1). We choose $h_{n_1} \in \{h_n\}_{n \in N}$ in such a way that $h_{n_1} < \delta_{n_0}$. Then from (2) we know that the length of interval J_{n_1} is equal to

$$h_{n_1} - \frac{f(x_0 + kh_{n_1}) - f(x_0 - h_{n_1})}{(k+1)(k_2 - \varepsilon)}$$

and by simple calculations we know that it is not less then $\frac{h_{n_1}}{n_0}$. Thus $\left[\frac{n_0 - 1}{n_0} \cdot h_{n_1}, h_{n_1}\right] \subset J_{n_1}$ and consequently $\left[\frac{n_0 - 1}{n_0} \cdot h_{n_1}, h_{n_1}\right] \cap B = \emptyset$, which gives a contradiction. □

Corollary 2. *If $f : I \longrightarrow \mathbb{R}$ is a monotone and k-pseudo-symmetrically $\mathfrak{I}$-approximately differentiable function, where I is an open interval, then the function f is k-pseudo-symmetrically differentiable.*

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