Folia Mathematica
Vol. 11, No. 1, pp. 15–39

Acta Universitatis Lodziensis


Dedicated to Professor Janusz Matkowski on the occasion of His 60th Birthday.

# LIPSCHITZIAN NEMYTSKII OPERATORS IN THE CONES OF MAPPINGS OF BOUNDED WIENER $\varphi$-VARIATION

VYACHESLAV V. CHISTYAKOV[‡]

**Abstract.** We define and study properties of mappings of bounded generalized Wiener $\varphi$-variation with values in a metric space, metric semigroup or abstract convex cone. Then we present necessary and sufficient conditions in terms of generators for the Nemytskii operators acting between metric semigroups and abstract convex cones of mappings of bounded generalized Wiener $\varphi$-variation to satisfy a Lipschitz condition or be globally bounded (including set-valued Nemytskii operators with closed bounded convex images). Our results develop and generalize recent results by the author [7], Maligranda and Orlicz [23], Matkowski and Miś [31] and Zawadzka [44].

## 1. Introduction

Let $I$, $M$ and $N$ be nonempty sets. Denote by $M^I$ the family of all mappings from $I$ into $M$. Given a mapping $h : I \times N \to M$, the operator $\mathcal{H} : N^I \to M^I$ defined by $(\mathcal{H}g)(x) = h(x, g(x))$ for all $x \in I$ and $g \in N^I$ is called the *Nemytskii* (superposition) *operator* and $h$ is called the *generator* of $\mathcal{H}$.

The Nemytskii operator is the "simplest" (in the terminology of [20, Sec. 17]) classical nonlinear operator acting between functional spaces, and a vast literature is devoted to the investigation of its properties (cf. [1], [20] and references therein). Although the Nemytskii operator is well studied in many classical functional spaces (ideal, Lebesgue, Orlicz, Hölder, Sobolev, etc.), little is known about its properties in spaces of functions and mappings of bounded (and bounded generalized) variation even for the interval

Supported by the Russian Foundation for Basic Research, Grant 03-01-00473.

[‡] *Department of Mathematics, State University Higher School of Economics*, Bol'shaya Pechërskaya 25 St., 603600 Nizny Novogrod, Russia. E-mail: czeslaw@mail.ru.

*Key words and phrases:* Wiener $\varphi$-variation, Nemytskii operator, metric semigroup, convex cone, Lipschitz condition.

*AMS subject classifications:* 26A45, 47H30, 26E25.

$I = [a, b] \subset \mathbb{R}$ and $N = M = \mathbb{R}$ (see [1, Sec. 6.5], [15]). In this respect, more thoroughly is studied an important class of Nemytskii operators satisfying a Lipschitz condition. Let us recall the first result in this direction.

Let $I = [a, b]$, $M = N = \mathbb{R}$ and $\mathrm{Lip}(I) \subset \mathbb{R}^I$ be the Banach algebra of Lipschitz continuous functions on $I$ with the usual norm $\|g\| = |g(a)| + L(g)$, where $L(g) = \sup\{|g(x) - g(y)|/|x - y|\,;\, x, y \in I,\, x \neq y\}$ and $g \in \mathrm{Lip}(I)$. Let $h : I \times \mathbb{R} \to \mathbb{R}$ be the generator of a Nemytskii operator $\mathcal{H} : \mathbb{R}^I \to \mathbb{R}^I$. It was shown by Matkowski [26] that $\mathcal{H}$ maps $\mathrm{Lip}(I)$ into itself and satisfies a Lipschitz condition (i.e., there exists a constant $C \geq 0$ such that $\|\mathcal{H}g_1 - \mathcal{H}g_2\| \leq C\|g_1 - g_2\|$ for all $g_1, g_2 \in \mathrm{Lip}(I)$), if and only if there exist two functions $f$ and $h_0$ in $\mathrm{Lip}(I)$ such that $h(x, u) = f(x)u + h_0(x)$ for all $x \in I$ and $u \in \mathbb{R}$. We note that this result is specific for $\mathrm{Lip}(I)$: it does not hold, e.g., if $\mathrm{Lip}(I)$ is replaced by the space $C(I)$ of continuous functions on $I$ with the sup-norm or by the space $L^p(I)$ of Lebesgue $p$-summable functions on $I$ with the standard norm, $p \geq 1$ (consider, for instance, $h(x, u) = \cos u$, $x \in I$, $u \in \mathbb{R}$).

Subsequently, Matkowski's result has been developed for single-valued and multivalued Lipschitzian Nemytskii operators in various functional spaces: [3]–[10], [25]–[34], [39]–[40] and [44].

The aim of this paper is to present necessary and sufficient conditions in terms of the generators for Lipschitzian Nemytskii operators acting in spaces of mappings of bounded generalized (nonlinear) Wiener $\varphi$-variation. Since we are interested in set-valued Nemytskii operators as well, an abstract setting emerges including mappings with values in metric semigroups and convex cones. Our results (Theorem 1 in Section 3, Theorems 2 and 3 in Section 4 and Theorem 4 in Section 5) are generalizations of the results by the author [7], Maligranda and Orlicz [23], Matkowski and Miś [31] and Zawadzka [44].

The paper is organized as follows. In Sections 2 and 3 we define and study properties of mappings of bounded generalized Wiener $\varphi$-variation with values in a metric space, metric semigroup or abstract convex cone. The sufficient and necessary conditions for the Nemytskii operator acting between metric semigroups and abstract convex cones of mappings of bounded generalized Wiener $\varphi$-variation to be Lipschitzian or globally bounded are presented in Sections 4 and 5, respectively.

The main results of the present paper were presented at the Summer Symposium in Real Analysis XXIV (The University of North Texas, Denton, Texas, USA, May 23–27, 2000), cf. [4].

## 2. Mappings of bounded generalized Wiener $\varphi$-variation

Let $\Phi$ denote the set of all convex functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at zero only (and, hence, continuous on $\mathbb{R}^+$), $\Phi_0 = \{\varphi \in \Phi \mid \lim_{s\to 0} \varphi(s)/s = 0\}$ and $I = [a, b] \subset \mathbb{R}$ be a fixed closed interval (with $a, b \in \mathbb{R}$, $a < b$). Given $\varphi \in \Phi$ and $\lambda > 0$, we set

$$\varphi_\lambda(s) = \varphi(s/\lambda), \qquad s \in \mathbb{R}^+.$$

If $(M, d)$ is a metric space and $\varphi \in \Phi$, we say that a mapping $f : I \to M$ is

(a) *of bounded Wiener $\varphi$-variation* and write $f \in V_\varphi(I; M)$ provided

$$v_\varphi(f) \equiv v_{\varphi,d}(f, I) = \sup_\xi \sum_{i=1}^{m} \varphi\Big(d\big(f(x_i), f(x_{i-1})\big)\Big) < \infty,$$

where the supremum is taken over all partitions $\xi = \{x_i\}_{i=0}^m$ of the interval $I$, i.e., $m \in \mathbb{N}$ and $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$;

(b) *of bounded generalized Wiener $\varphi$-variation* and write $f \in W_\varphi(I; M)$, if there exists a constant $\lambda > 0$ (depending on $f$ in general) such that $v_{\varphi_\lambda}(f) < \infty$; in this case we define the *precise $\varphi$-variation of $f$* by

$$p_\varphi(f) \equiv p_{\varphi,d}(f, I) = \inf\{\lambda > 0 \mid v_{\varphi_\lambda}(f) \le 1\}.$$

For $\varphi(s) = s^q$ ($s \ge 0$, $q \ge 1$), the value $v_\varphi(f)$ with $q = 1$ corresponds to the classical variation in the sense of Jordan [36, Ch. 8], $v_\varphi(f)$ with $q = 2$ was defined by Wiener [41] and with $q > 1$ – by Marcinkiewicz [24] and Young [42]. Definition (a) above with general $\varphi \in \Phi$ is due to Young [43]. The space $W_\varphi(I; M)$ with $M = \mathbb{R}$ has been extensively studied by many authors [12]–[16], [21], [23], [35]. Definition (b) above with $M$ a metric space was given in [4]. The functional $p_\varphi(\cdot)$ is of Luxemburg–Minkowski–Nakano–Orlicz type (cf. [22]).

In the special case $\varphi(s) = s$, corresponding to Jordan's variation, we denote $V_\varphi(I; M) = W_\varphi(I; M)$ by $V(I; M)$, and $v_\varphi(f) = p_\varphi(f)$ – by $v(f) \equiv v_d(f, I)$.

Note that if $\varphi \in \Phi \setminus \Phi_0$, then $\varphi'(0) = \inf_{s>0} \varphi(s)/s > 0$, and so, $V_\varphi(I; M) = V(I; M)$. Indeed, since $s \le \varphi(s)/\varphi'(0)$ for $s \ge 0$, condition $f \in V_\varphi(I; M)$ implies $v(f) \le v_\varphi(f)/\varphi'(0)$. Now if $f \in V(I; M)$, by convexity of $\varphi$ ([20, Sec. I.1.4]) we find that $\varphi(t) + \varphi(s) \le \varphi(t + s)$, $t, s \in \mathbb{R}^+$, whence $v_\varphi(f) \le \varphi(v(f))$.

The following properties of $v_\varphi$ are well known ([11], [35]): $v_\varphi(f, J) \le v_\varphi(f, I)$, if $I$ contains the interval $J$; $v_\varphi(f, [a, x]) + v_\varphi(f, [x, b]) \le v_\varphi(f, [a, b])$, if $a \le x \le b$; and $v_\varphi(f, I) \le \liminf_{k\to\infty} v_{\varphi_k}(f_k, I)$, if $\{f_k\} \subset M^I$ converges pointwise on $I$ to $f \in M^I$ and $\{\varphi_k\} \subset \Phi$ converges pointwise on $\mathbb{R}^+$ to $\varphi \in \Phi$ as $k \to \infty$.

Now we study the embeddings of sets $V_\varphi(I; M)$ corresponding to different functions $\varphi \in \Phi$, which will motivate the definition of spaces $W_\varphi(I; M)$. For the sake of completeness the following two lemmas are presented with proofs (although their proofs resemble those of Propositions 1.14–1.16 from [35]).

**Lemma 1.** *Let $\varphi, \psi \in \Phi$ and $(M, d)$ be a metric space. If*

(1) *there exist $c > 0$ and $s_0 > 0$ such that $\psi(s) \le c\,\varphi(s)$ for all $s \in [0, s_0]$,*

*then $V_\varphi(I; M) \subset V_\psi(I; M)$. Conversely, if $(M, |\cdot|)$ is a normed linear space and $V_\varphi(I; M) \subset V_\psi(I; M)$, then condition* (1) *holds.*

*Proof.* Under condition (1) we show first that for any $s_1 > 0$ there exists a $c_1(s_1) > 0$ such that $\psi(s) \le c_1(s_1)\varphi(s)$ for all $s \in [0, s_1]$: if $s_1 \le s_0$, by virtue of (1), we set $c_1(s_1) = c$, and if $s_1 > s_0$, we may set $c_1(s_1) = c\,\psi(s_1)/\psi(s_0)$, since for all $s \in [s_0, s_1]$ we have:

$$\psi(s) \le \frac{\psi(s_1)}{\psi(s)}\,\frac{\varphi(s)}{\varphi(s_0)}\,\psi(s) = \frac{\psi(s_1)}{\varphi(s_0)}\,\varphi(s) \le c\,\frac{\psi(s_1)}{\psi(s_0)}\,\varphi(s).$$

Given $f \in V_\varphi(I; M)$, it is bounded, i.e., $\omega(f) = \sup_{x,y\in I} d(f(x), f(y))$ is finite, and so, $\psi\big(d(f(x), f(y))\big) \le c_1(\omega(f))\varphi\big(d(f(x), f(y))\big)$ for all $x$, $y \in I$, which gives $v_\psi(f) \le c_1(\omega(f))v_\varphi(f)$ or $f \in V_\psi(I; M)$.

Now we suppose that condition (1) is wrong. Given $n \in \mathbb{N}$, define $s_{0n} \in \mathbb{R}^+$ by $\varphi(s_{0n}) = 1/n^2$. Then there exists $s_n \in [0, s_{0n}]$ such that $\psi(s_n) > n\varphi(s_n)$. We set $k_n = \min\{k \in \mathbb{N} \mid 1/n^2 \le k\varphi(s_n) < 2/n^2\}$ for $n \in \mathbb{N}$, and $k_0 = 0$. If $n \in \mathbb{N}$ and $m \in \mathbb{N}$ are such that $k_1 + \cdots + k_{m-1} < n \le k_1 + \cdots + k_{m-1} + k_m$, we set $s'_n = s_m$ (and $s'_0 = 0$). It follows that the series $\sum_{n=0}^\infty \varphi(s'_n)$ converges, whereas $\sum_{n=0}^\infty \psi(s'_n)$ diverges.

Let us fix an increasing sequence $\{x_n\}_{n=0}^\infty$ on the open interval $(a, b)$ and $u \in M$ with $|u| = 1$. Define $f : [a, b] \to M$ by $f(x_n) = s'_n u$, $n \in \mathbb{N} \cup \{0\}$, and $f(x) = 0$ for $x \ne x_n$, $n = 0, 1, 2, \ldots$ Since for any partition $a = t_0 < t_1 < \cdots < t_m = b$ of $[a, b]$ we have:

$$\sum_{i=1}^m \varphi\big(|f(t_i) - f(t_{i-1})|\big) \le 2\sum_{i=1}^m \varphi(s'_{l_i}) + \sum_{i=1}^m \varphi\big(|s'_{l_i} - s'_{l_{i-1}}|\big) \le 4\sum_{i=0}^\infty \varphi(s'_i),$$

then $v_\varphi(f, [a, b]) < \infty$. On the other hand, for the partition $a = t_0 < t_1 < \cdots < t_{2m} = b$ of $[a, b]$ such that $t_{2i+1} = x_i$, $i = 0, 1, \ldots, m-1$, and $t_{2i} = (x_{i-1} + x_i)/2$, $i = 1, \ldots, m-1$, we find that

$$\sum_{i=1}^{2m} \psi\big(|f(t_i) - f(t_{i-1})|\big) \ge 2\sum_{i=0}^{m-1} \psi(s'_i),$$

whence $v_\psi(f, [a, b]) = \infty$, which proves the lemma. □

In particular, Lemma 1 implies that if $\lim_{s\to+0}\varphi(s)/s>0$, then $V_\varphi(I;M)=V(I;M)$.

Recall ([35, Sec. 1]) that $\varphi\in\Phi$ *satisfies the* $\Delta_2$*-condition near zero* or, in short, the $\Delta_2^0$*-condition* if $\limsup_{\rho\to 0}\varphi(2\rho)/\varphi(\rho)<\infty$, which is equivalent to

$$\exists \text{ constants } c>0 \text{ and } s_0>0 \text{ such that } \varphi(2s)\le c\,\varphi(s)\ \forall\, s\in[0,s_0].$$

Also, the last condition is equivalent to the following:

$$\forall\, s_1>0\ \exists\, c_0(s_1)>0 \text{ such that } \varphi(2s)\le c_0(s_1)\varphi(s)\ \forall\, s\in[0,s_1]; \tag{2}$$

in fact, this is obvious for $s_1\le s_0$, and if $s_1>s_0$ and $s_0\le s\le s_1$, then

$$\varphi(2s)\le\frac{\varphi(2s_1)}{\varphi(2s)}\,\frac{\varphi(s)}{\varphi(s_0)}\,\varphi(2s)=\frac{\varphi(2s_1)}{\varphi(s_0)}\,\varphi(s)\le c\,\frac{\varphi(2s_1)}{\varphi(2s_0)}\,\varphi(s).$$

**Lemma 2.** *Let $M$ be a normed space and $\varphi\in\Phi$. Then $V_\varphi\equiv V_\varphi(I;M)$ is a linear space if and only if $\varphi$ satisfies the $\Delta_2^0$-condition.*

*Proof.* If $V_\varphi$ is linear, then $2f\in V_\varphi$ for all $f\in V_\varphi$, that is, $V_\varphi\subset V_\psi$ with $\psi(s)=\varphi(2s)$, $s\ge 0$. By Lemma 1, there exist constants $c>0$ and $s_0>0$ such that $\varphi(2s)=\psi(s)\le c\,\varphi(s)$ for all $s\in[0,s_0]$.

Now, let $\varphi$ satisfy the $\Delta_2^0$-condition. Let $f,\,g\in V_\varphi$ be such that $\omega(f)\le r_0$, $\omega(g)\le r_0$ (cf. the proof of Lemma 1), $c\in\mathbb{R}$ and $m\in\mathbb{N}$ be the least integer, for which $|c|\le 2^m$. The linearity of $V_\varphi$ follows from the inequalities:

$$v_\varphi(f+g)\le c_0(r_0)\big(v_\varphi(f)+v_\varphi(g)\big),$$
$$v_\varphi(cf)\le\big(c_0(2^{m-1}r_0)\big)^m v_\varphi(f),$$

where the function $c_0(s_1)$ is defined in (2). □

The previous lemma shows that $V_\varphi(I;M)$ is not a linear space, in general, even if $M$ is a normed space. At the same time, the convexity of $\varphi\in\Phi$ implies that the set $V_\varphi(I;M)$ is convex and $f\mapsto v_\varphi(f)$ is a convex functional on it:

$$v_\varphi\big(\theta f+(1-\theta)g\big)\le\theta v_\varphi(f)+(1-\theta)v_\varphi(g),\quad f,\,g\in V_\varphi(I;M),\quad \theta\in[0,1].$$

Also, we see that if $(M,d)$ is a metric space and $\varphi\in\Phi$, then, by Lemma 1, $V_{\varphi_\lambda}(I;M)\subset V_\varphi(I;M)$, if $0<\lambda\le 1$, and $V_\varphi(I;M)\subset V_{\varphi_\lambda}(I;M)$, if $\lambda>1$. In order for the reverse inclusion $V_{\varphi_\lambda}(I;M)\subset V_\varphi(I;M)$ to hold with $\lambda>1$, by Lemma 1 it is sufficient (and if $M$ is a normed space, then it is also necessary) that $\varphi$ satisfy the $\Delta_2^0$-condition. This is the motivation to have introduced the space of mappings of bounded generalized Wiener $\varphi$-variation (for any $\varphi\in\Phi$):

$$W_\varphi(I;M)=\bigcup_{\lambda>0}V_{\varphi_\lambda}(I;M)=\bigcup_{\lambda>1}V_{\varphi_\lambda}(I;M).$$

As we have just seen, if $\varphi$ satisfies the $\Delta_2^0$-condition, then $W_\varphi(I;M) = V_\varphi(I;M)$; conversely, if $M$ is a normed space and $W_\varphi(I;M) = V_\varphi(I;M)$, then $\varphi$ satisfies the $\Delta_2^0$-condition: in fact, the inclusion $V_{\varphi_2}(I;M) \subset V_\varphi(I;M)$ and Lemma 1 yield the existence of $c > 0$ and $s_0 > 0$ such that $\varphi(s) \le c\,\varphi(s/2)$, $s \in [0, s_0]$.

Given $f \in W_\varphi(I;M)$, the precise $\varphi$-variation $p_\varphi(f)$ is well defined, since $v_{\varphi_\lambda}(f) \le v_\varphi(f)/\lambda$, $\lambda \ge 1$. The main properties of $p_\varphi$ are presented in the following lemma (note that any function $\varphi \in \Phi$ is strictly increasing, so it admits the inverse function denoted by $\varphi^{-1}$):

**Lemma 3.** *Given $(M,d)$ a metric space, $\varphi \in \Phi$ and $f \in W_\varphi(I;M)$, we have*:

(a) *$d(f(x), f(y)) \le \varphi^{-1}(1)p_\varphi(f)$ for all $x$, $y \in I$;*
(b) *if $\lambda = p_\varphi(f) > 0$, then $v_{\varphi_\lambda}(f) \le 1$;*
(c) *if $\lambda > 0$, then $p_\varphi(f) \le \lambda$, if and only if $v_{\varphi_\lambda}(f) \le 1$;*
(d) *if $\lambda > 0$ and $v_{\varphi_\lambda}(f) = 1$, then $p_\varphi(f) = \lambda$;*
(e) *if a sequence $\{f_k\} \subset W_\varphi(I;M)$ converges pointwise on $I$ to $f$ as $k \to \infty$, then $p_\varphi(f) \le \liminf_{k\to\infty} p_\varphi(f_k)$;*
(f) *$p_\varphi(f) \le v(f)/\varphi^{-1}(1)$, if $f \in V(I;M)$;*
(g) *if $x \in I = [a,b]$, then $p_\varphi(f, [a,b]) \le p_\varphi(f, [a,x]) + p_\varphi(f, [x,b])$;*
(h) *if $M$ is a normed space, then $W_\varphi(I;M)$ is a linear space and $p_\varphi(\cdot)$ is a seminorm on $W_\varphi(I;M)$.*

*Proof.* (a) By definitions of $v_\varphi(f)$ and $p_\varphi(f)$, we have:

$$\varphi\big(d(f(x), f(y))/\lambda\big) \le v_{\varphi_\lambda}(f) \le 1 \quad \text{for all} \quad \lambda > p_\varphi(f),$$

so that, applying $\varphi^{-1}$ and multiplying by $\lambda$, we obtain the inequality in (a).

(b) Choose numbers $\lambda(k) > \lambda$, $k \in \mathbb{N}$, such that $\lambda(k) \to \lambda$ as $k \to \infty$. By the definition of $p_\varphi(f)$, we have $v_{\varphi_{\lambda(k)}}(f) \le 1$ for all $k \in \mathbb{N}$, and so, $v_{\varphi_\lambda}(f) \le \liminf_{k\to\infty} v_{\varphi_{\lambda(k)}}(f) \le 1$.

(c) The sufficiency is clear from the definition of $p_\varphi(f)$, so we prove the necessity only. Let $p_\varphi(f) > 0$ (otherwise, by (a), $f$ is constant and $v_{\varphi_\lambda}(f) = 0$). If $p_\varphi(f) = \lambda$, then, thanks to item (b), $v_{\varphi_\lambda}(f) \le 1$. It remains to show that

$$\text{if } \ p_\varphi(f) < \lambda, \ \text{ then } \ v_{\varphi_\lambda}(f) < 1. \tag{3}$$

In fact, setting $\mu = p_\varphi(f)$ and taking into account the convexity of $\varphi$ and the conclusion of (b), we have: $v_{\varphi_\lambda}(f) \le (\mu/\lambda)v_{\varphi_\mu}(f) \le \mu/\lambda < 1$.

(d) By (c), $p_\varphi(f) \le \lambda$, and by (3) we have $p_\varphi(f) = \lambda$.

(e) It suffices to suppose that $\lambda = \liminf_{k\to\infty} p_\varphi(f_k)$ is finite. Then $p_\varphi(f_{k_l})$ tends to $\lambda$ as $l \to \infty$ for some subsequence $\{f_{k_l}\}_{l=1}^\infty$ of $\{f_k\}_{k=1}^\infty$, so that for any $\varepsilon > 0$ there exists $l_0(\varepsilon) \in \mathbb{N}$ such that $p_\varphi(f_{k_l}) < \lambda + \varepsilon$ for all $l \geq l_0(\varepsilon)$. The definition of $p_\varphi(f_{k_l})$ implies $v_{\varphi_{\lambda+\varepsilon}}(f_{k_l}) \leq 1$, if $l \geq l_0(\varepsilon)$, and since $f_{k_l}$ converges to $f$ pointwise on $I$ as $l \to \infty$, we get: $v_{\varphi_{\lambda+\varepsilon}}(f) \leq \liminf_{l\to\infty} v_{\varphi_{\lambda+\varepsilon}}(f_{k_l}) \leq 1$, whence $p_\varphi(f) \leq \lambda + \varepsilon$ for all $\varepsilon > 0$.

(f) Set $\lambda = v(f)/\varphi^{-1}(1)$, suppose $v(f) > 0$, note that $v_{\varphi_\lambda}(f) \leq \varphi_\lambda(v(f)) = 1$, and apply (c).

(g) Let $a < x < b$, and set $\lambda = p_\varphi(f, [a, x])$ and $\mu = p_\varphi(f, [x, b])$. If $\lambda \cdot \mu = 0$, then, by (a), the inequality (actually, the equality) is obvious. Let $\lambda \cdot \mu > 0$. By (b), $v_{\varphi_\lambda}(f, [a, x]) \leq 1$ and $v_{\varphi_\mu}(f, [x, b]) \leq 1$. Since, by (c), $p_\varphi(f, I) \leq \lambda + \mu$ is equivalent to $v_{\varphi_{\lambda+\mu}}(f, I) \leq 1$, we prove the latter inequality. Let $\xi = \{x_i\}_{i=0}^m$ be a partition of $I$ such that $x_{k-1} \leq x \leq x_k$ for some $k \in \{1, \ldots, m\}$. By virtue of monotonicity and convexity of $\varphi$, we get:

$$\begin{aligned}
\varphi_{\lambda+\mu}\Big(d\big(f(x_i), f(x_{i-1})\big)\Big) &\leq \frac{\lambda}{\lambda+\mu}\varphi_\lambda\Big(d\big(f(x_i), f(x_{i-1})\big)\Big),\ i = 1, \ldots, k-1, \\
\varphi_{\lambda+\mu}\Big(d\big(f(x_k), f(x_{k-1})\big)\Big) &\leq \frac{\lambda}{\lambda+\mu}\varphi_\lambda\Big(d\big(f(x), f(x_{k-1})\big)\Big) + \\
&+ \frac{\mu}{\lambda+\mu}\varphi_\mu\Big(d\big(f(x_k), f(x)\big)\Big), \\
\varphi_{\lambda+\mu}\Big(d\big(f(x_i), f(x_{i-1})\big)\Big) &\leq \frac{\mu}{\lambda+\mu}\varphi_\mu\Big(d\big(f(x_i), f(x_{i-1})\big)\Big),\ i = k+1, \ldots, m.
\end{aligned}$$

It follows that

$$\sum_{i=1}^m \varphi_{\lambda+\mu}\Big(d\big(f(x_i), f(x_{i-1})\big)\Big) \leq \frac{\lambda}{\lambda+\mu} v_{\varphi_\lambda}(f, [a, x]) + \frac{\mu}{\lambda+\mu} v_{\varphi_\mu}(f, [x, b]) \leq 1,$$

and so, $v_{\varphi_{\lambda+\mu}}(f, I) \leq 1$.

(h) Suppose $M$ is a normed space. If $f, g \in W_\varphi(I; M)$, then there exist $\lambda > 0$ and $\mu > 0$ such that $f \in V_{\varphi_\lambda}(I; M)$ and $g \in V_{\varphi_\mu}(I; M)$, and so, by the convexity of $v_\varphi$, we find that

$$v_{\varphi_{\lambda+\mu}}(f + g) \leq \frac{\lambda}{\lambda+\mu} v_{\varphi_\lambda}(f) + \frac{\mu}{\lambda+\mu} v_{\varphi_\mu}(g), \tag{4}$$

whence $f + g \in W_\varphi(I; M)$. It is also clear that $cf \in W_\varphi(I; M)$, if $c$ is a scalar, and $p_\varphi(cf) = |c| p_\varphi(f)$. Now set $\lambda = p_\varphi(f)$ and $\mu = p_\varphi(g)$. If $\lambda \cdot \mu = 0$, then clearly $p_\varphi(f + g) \leq \lambda + \mu$, and if $\lambda \cdot \mu > 0$, then by virtue of (4) and (b) we get $v_\varphi\big((f + g)/(\lambda + \mu)\big) \leq 1$, and so, by the definition of $p_\varphi$, $p_\varphi(f + g) \leq \lambda + \mu$, which was to be proved. □

One of the advantages to define the space $W_\varphi(I; M)$ is that it is invariant with respect to equivalent metrics on $M$: if $d$ and $\rho$ are metrics on $M$ such that $c\,d(u, v) \leq \rho(u, v) \leq C d(u, v)$ for some constants $c > 0$, and

$C > 0$ and all $u, v \in M$, and $f$ belongs to $W_\varphi(I;M)$ with respect to $d$, then $f \in W_\varphi(I;M)$ with respect to $\rho$ and $c\, p_{\varphi,d}(f,I) \le p_{\varphi,\rho}(f,I) \le C p_{\varphi,d}(f,I)$.

In order to establish the relations between spaces $W_\varphi(I;M)$, generated by different functions $\varphi \in \Phi$, let us recall certain definitions from [13]. Given $\varphi, \psi \in \Phi$, we write $\psi \preccurlyeq \varphi$ and say that $\varphi$ *dominates* $\psi$ *near zero* if there exist constants $r > 0$, $c > 0$ and $s_0 > 0$ such that $\psi(s) \le r\varphi(c\,s)$ for all $s \in [0, s_0]$. Functions $\varphi$ and $\psi$ are said to be *equivalent near zero*, in symbols $\varphi \sim \psi$, provided $\psi \preccurlyeq \varphi$ and $\varphi \preccurlyeq \psi$; in other words, $\varphi \sim \psi$, if and only if there exist positive constants $r_1$, $r_2$, $c_1$, $c_2$ and $s_0$ such that $r_1\varphi(c_1 s) \le \psi(s) \le r_2\varphi(c_2 s)$ for all $s \in [0, s_0]$.

**Lemma 4.** *Let $\varphi, \psi \in \Phi$ and $(M, d)$ be a metric space. If $\psi \preccurlyeq \varphi$, then $W_\varphi(I;M) \subset W_\psi(I;M)$ and there exists a number $\kappa = \kappa(\varphi, \psi) > 0$, depending only on $\varphi$ and $\psi$, such that $p_\psi(f) \le \kappa p_\varphi(f)$ for all $f \in W_\varphi(I;M)$. Conversely, if $(M, |\cdot|)$ is a normed linear space and $W_\varphi(I;M) \subset W_\psi(I;M)$, then $\psi \preccurlyeq \varphi$. Thus, the spaces $W_\varphi(I;M)$ and $W_\psi(I;M)$ consist of the same mappings, if and only if $\varphi \sim \psi$, in which case $p_\varphi$ and $p_\psi$ are equivalent functionals.*

*Proof.* Let $f \in W_\varphi(I;M)$ and $\lambda = p_\varphi(f)$. If $\lambda = 0$, then $f$ is constant according to Lemma 3(a), and so, $p_\psi(f) = 0$. Suppose $\lambda > 0$. Since $\psi \preccurlyeq \varphi$, there exist positive constants $r$, $c$ and $s_0$ such that $\psi(s) \le r\varphi(c\,s)$ for all $s \in [0, s_0]$. Let us show that

$$\forall\, s_1 > 0 \ \exists\, r_1(s_1) > 0 \ \text{such that} \ \psi(s) \le r_1(s_1)\varphi(c\,s) \ \ \forall\, s \in [0, s_1]. \tag{5}$$

In fact, this is clear if $s_1 \le s_0$, and if $s_1 > s_0$ and $s_0 \le s \le s_1$, we have:

$$\psi(s) \le \frac{\psi(s_1)}{\psi(s)}\,\frac{\varphi(c\,s)}{\varphi(c\,s_0)}\,\psi(s) = \frac{\psi(s_1)}{\varphi(c\,s_0)}\,\varphi(c\,s) \le r\frac{\psi(s_1)}{\psi(s_0)}\,\varphi(c\,s) \equiv r_1(s_1)\varphi(c\,s).$$

We set $s_1 = \varphi^{-1}(1)/c$, $r_2 = \max\{1, r_1(s_1)\}$ and $\mu = r_2 c\,\lambda$. By Lemma 3(a), we have $d\big(f(x), f(y)\big)/\mu \le s_1$ for all $x, y \in I$, and so, (5) and the convexity of $\varphi$ give

$$\psi\Big(d\big(f(x), f(y)\big)/\mu\Big) \le \frac{r_1(s_1)}{r_2}\,\varphi\Big(d\big(f(x), f(y)\big)/\lambda\Big).$$

Let $\xi = \{x_i\}_{i=0}^m$ be a partition of $I$. Since $r_1(s_1)/r_2 \le 1$, by Lemma 3(b), we have:

$$\sum_{i=1}^m \psi_\mu\Big(d\big(f(x_i), f(x_{i-1})\big)\Big) \le \sum_{i=1}^m \varphi_\lambda\Big(d\big(f(x_i), f(x_{i-1})\big)\Big) \le v_{\varphi_\lambda}(f) \le 1.$$

It follows that $v_{\psi_\mu}(f) \le 1$, which implies $p_\psi(f) \le \mu = r_2 c\, p_\varphi(f) \equiv \kappa p_\varphi(f)$.

The reverse assertion can be established similarly to the second part of the proof of Lemma 1 (or see [13, Sec. 4.1.1, 2.2.2 and 2.1] in the case $M = \mathbb{R}$). □

## 3. Semigroups and cones of mappings

A *metric semigroup* is a triple $(N, \rho, +)$ where $(N, \rho)$ is a metric space with the metric $\rho$, $(N, +)$ is an Abelian semigroup with the addition operation $+$ and $\rho$ is translation invariant: $\rho(u + w, v + w) = \rho(u, v)$ for all $u, v, w \in N$. The semigroup $(N, \rho, +)$ is said to be *complete* provided $(N, \rho)$ is a complete metric space. If $N$ contains a *neutral* element (called *zero*) $0 \in N$, so that $u + 0 = 0 + u = u$ for all $u \in N$, then given $u \in N$, we set $|u|_\rho = \rho(u, 0)$.

For any elements $u, v, \overline{u}, \overline{v} \in N$ of a metric semigroup $(N, \rho, +)$ we have:

$$\rho(u, v) \le \rho(u + \overline{u}, v + \overline{v}) + \rho(\overline{u}, \overline{v}), \tag{6}$$

$$\rho(u + \overline{u}, v + \overline{v}) \le \rho(u, v) + \rho(\overline{u}, \overline{v}). \tag{7}$$

If sequences $\{u_k\} = \{u_k\}_{k\in\mathbb{N}}$, $\{v_k\}$, $\{\overline{u}_k\}$ and $\{\overline{v}_k\}$ of elements of $N$ converge in $N$ to elements $u$, $v$, $\overline{u}$ and $\overline{v}$, respectively, as $k \to \infty$, then, by (7),

$$\lim_{k\to\infty} \rho(u_k + \overline{u}_k, v_k + \overline{v}_k) = \rho(u + \overline{u}, v + \overline{v}), \tag{8}$$

and, in particular, the addition operation $(u, v) \mapsto u + v$ is a continuous mapping from $N \times N$ into $N$.

The quadruple $(N, \rho, +, \cdot)$ is called an *abstract convex cone*, if $(N, \rho, +)$ is a metric semigroup with zero $0 \in N$ and the operation $\cdot : \mathbb{R}^+ \times N \to N$ of multiplication of elements of $N$ by nonnegative numbers, given by $(\lambda, u) \mapsto \lambda u$, satisfies the following properties: $\lambda(u+v) = \lambda u+\lambda v$, $(\lambda+\mu)u = \lambda u+\mu u$, $\lambda(\mu u) = (\lambda\mu)u$, $1 \cdot u = u$ and $\rho(\lambda u, \lambda v) = \lambda\rho(u, v)$ for all $u, v \in N$ and $\lambda, \mu \in \mathbb{R}^+$. If $(N, \rho)$ is complete, the cone is said to be *complete*, and given $u \in N$, we set, as above, $|u|_\rho = \rho(u, 0)$.

Note that the following equality holds in an abstract convex cone $(N,\rho,+,\cdot)$:

$$\rho(\lambda u + \mu v, \lambda v + \mu u) = |\lambda - \mu|\rho(u, v), \qquad \lambda, \mu \in \mathbb{R}^+, \quad u, v \in N. \tag{9}$$

It follows that $\rho(\lambda u, \mu v) \le \lambda\rho(u, v) + |\lambda - \mu| \cdot |v|_\rho$, and so, the operation of multiplication of elements of $N$ by nonnegative numbers is a continuous mapping from $\mathbb{R}^+ \times N$ into $N$.

Since we are interested also in set-valued superposition operators, the following provides the appropriate setting.

Let $(Y, |\cdot|)$ be a real normed space. Denote by $\mathrm{cbc}(Y)$ the family of all nonempty closed bounded convex subsets of $Y$ equipped with the Hausdorff metric $D$ generated by the norm in $Y$:

$$D(P,Q)=\max\Big\{\sup_{p\in P}\inf_{q\in Q}|p-q|,\sup_{q\in Q}\inf_{p\in P}|p-q|\Big\},\qquad P,\ Q\in \mathrm{cbc}(Y).$$

Given $P,Q\in\mathrm{cbc}(Y)$, we set $P+Q=\{p+q\mid p\in P,q\in Q\}$, $\lambda P=\{\lambda p\mid p\in P\}$, $\lambda\in\mathbb{R}^+$, and $P\overset{*}{+}Q=\mathrm{cl}(P+Q)$, where cl means the closure in $Y$. The following equalities hold in $\mathrm{cbc}(Y)$ ([19]): $P\overset{*}{+}Q=\mathrm{cl}(\mathrm{cl}P+\mathrm{cl}Q)$, $\lambda(P\overset{*}{+}Q)=\lambda P\overset{*}{+}\lambda Q$, $(\lambda+\mu)P=\lambda P\overset{*}{+}\mu P$, $\lambda(\mu P)=(\lambda\mu)P$ and $D(\lambda P,\lambda Q)=\lambda D(P,Q)$ for all $\lambda,\mu\in\mathbb{R}^+$. Moreover, since (see [38, Lemma 3], [17, Lemma 2.2])

$$D(P\overset{*}{+}R,Q\overset{*}{+}R)=D(P+R,Q+R)=D(P,Q),\qquad P,Q,R\in\mathrm{cbc}(Y),$$

then $(\mathrm{cbc}(Y),D,\overset{*}{+},\cdot)$ is an abstract convex cone; this cone is complete provided $Y$ is a Banach space (this is a consequence of properties of $D$, see Theorems II-9 and II-14 in [2]).

More examples of metric semigroups and abstract convex cones relevant for our purposes are to follow below in this Section.

Let $(N,\rho)$ and $(M,d)$ be two metric spaces. Recall that $T:N\to M$ is called a *Lipschitzian* operator if its (*least*) *Lipschitz constant* given by

$$L(T)=\sup\big\{d(Tu,Tv)/\rho(u,v)\ \big|\ u,v\in N,\ u\neq v\big\}$$

is finite, and the set of all these operators is denoted by $\mathrm{Lip}(N;M)$.

If $(M,d,+)$ is a metric semigroup (abstract convex cone), the set $\mathrm{Lip}(N;M)$ is closed with respect to the pointwise addition operation (and multiplication by a number $\lambda\in\mathbb{R}^+$), since, by (7), $L(T+S)\le L(T)+L(S)$ (and $L(\lambda T)=\lambda L(T)$) for all $T,S\in\mathrm{Lip}(N;M)$. For a fixed $u_0\in N$ the translation invariant metric $d_L$ on $\mathrm{Lip}(N;M)$ is defined by ([39]):

$$d_L(T,S)=d(Tu_0,Su_0)+d_\ell(T,S),\qquad T,\ S\in\mathrm{Lip}(N;M),\tag{10}$$

where

$$d_\ell(T,S)=\sup\big\{d(Tu+Sv,Su+Tv)/\rho(u,v)\ \big|\ u,\ v\in N,\ u\neq v\big\}.$$

The straightforward properties of the translation invariant semimetric $d_\ell$ are gathered in the following

**Lemma 5.** *Given $T,S\in\mathrm{Lip}(N;M)$, we have:*

*(a) $|d(Tu,Su)-d(Tv,Sv)|\le d(Tu+Sv,Su+Tv)\le d_\ell(T,S)\rho(u,v)$ for $u,v\in N$;*

*(b) $|L(T)-L(S)|\le d_\ell(T,S)\le L(T)+L(S)$;*

*(c) if $\{T_k,S_k\}\subset\mathrm{Lip}(N;M)$, $d(T_ku,Tu)\to 0$ and $d(S_ku,Su)\to 0$ as $k\to\infty$ for all $u\in N$, then $d_\ell(T,S)\le\liminf_{k\to\infty}d_\ell(T_k,S_k)$.*

In this way $(\mathrm{Lip}(N;M), d_L, +)$ becomes a metric semigroup (abstract convex cone), which is complete provided the metric semigroup $(M, d, +)$ is complete.

Let $(N, +)$ and $(M, +)$ be two Abelian semigroups. Recall that $T : N \to M$ is an *additive* operator if it satisfies the Cauchy equation: $T(u + v) = Tu + Tv$ in $M$ for all $u, v \in N$.

If $(N, \rho, +)$ and $(M, d, +)$ are two metric semigroups, we designate by $\mathrm{L}(N;M)$ the set of all Lipschitzian additive operators from $N$ into $M$. If, moreover, $N$ and $M$ are metric semigroups with zeros (denoted by the same symbol 0) and $T : N \to M$ is additive, then $T(0) = 0$; in fact, $T(0) = T(0 + 0) = T(0) + T(0)$ and $d(0, T(0)) = d(T(0), T(0) + T(0)) = 0$. In this case $d_L = d_\ell$ (cf. (10) with $u_0 = 0$) is a metric on $\mathrm{L}(N;M)$, and we have $L(T) = d_L(T, 0) = |T|_{d_L}$.

Let $(M, d, +)$ be a metric semigroup (or an abstract convex cone) and $\varphi \in \Phi$. We are going to endow the set $W_\varphi(I;M)$ with the structure of a metric semigroup (or an abstract convex cone) as follows (cf. [4]). Let $f, g \in W_\varphi(I;M)$. The operation of *addition* (*multiplication by a number* $c \in \mathbb{R}^+$) is defined pointwise: $(f + g)(x) = f(x) + g(x)$ $((cf)(x) = cf(x)$, respectively), $x \in I$, and the translation invariant *metric* $d_\varphi$ on $W_\varphi(I;M)$ is defined by

$$d_\varphi(f, g) = d\big(f(a), g(a)\big) + \Delta_\varphi(f, g), \qquad f,\, g \in W_\varphi(I;M),$$

where

$$\Delta_\varphi(f, g) \equiv \Delta_{\varphi,d}(f, g, I) = \inf\big\{\lambda > 0 \,\big|\, w_{\varphi_\lambda}(f, g) \le 1\big\}$$

and

$$w_\varphi(f, g) \equiv w_{\varphi,d}(f, g, I) = \sup_\xi \sum_{i=1}^{m} \varphi\Big(d\big(f(x_i) + g(x_{i-1}), g(x_i) + f(x_{i-1})\big)\Big),$$

the supremum being taken over all partitions $\xi = \{x_i\}_{i=0}^m$ $(m \in \mathbb{N})$ of $I = [a, b]$. In the special case $\varphi(s) = s$, when $W_\varphi(I;M) = V(I;M)$, we denote $d_\varphi$ by $d_1$ and $w_\varphi = \Delta_\varphi$ — by $\Delta_1$. The metric $d_1$ was employed by Zawadzka [44] (when $d$ is the Hausdorff metric on the family of all compact convex subsets of a real normed space).

The addition operation in $W_\varphi(I;M)$ is well defined, for if $f, g \in W_\varphi(I;M)$, then, by (7) and the monotonicity and convexity of $\varphi$, we have (4), so that $f + g \in W_\varphi(I;M)$ and, thanks to Lemma 3(b), $p_\varphi(f + g) \le p_\varphi(f) + p_\varphi(g)$. (It is also clear that $cf \in W_\varphi(I;M)$ and $p_\varphi(cf) = cp_\varphi(f)$, if $c \ge 0$.)

It will be shown below (in Theorem 1) that $\Delta_\varphi$ is a semimetric and $d_\varphi$ is a metric on $W_\varphi(I;M)$, which are translation invariant. Now let us verify that the value $\Delta_\varphi(f, g)$ is finite. In fact, since $v_{\varphi_\lambda}(f) < \infty$ and $v_{\varphi_\mu}(g) < \infty$

for some constants $\lambda > 0$ and $\mu > 0$, given $x, y \in I$, inequality (7) implies

$$\frac{d\big(f(x)+g(y),g(x)+f(y)\big)}{\lambda+\mu} \le \frac{\lambda}{\lambda+\mu}\,\frac{d\big(f(x),f(y)\big)}{\lambda} + \frac{\mu}{\lambda+\mu}\,\frac{d\big(g(x),g(y)\big)}{\mu}\,,$$

hence (again by applying the monotonicity and convexity of $\varphi$)

$$w_{\varphi_{\lambda+\mu}}(f,g) \le \frac{\lambda}{\lambda+\mu}\,v_{\varphi_\lambda}(f) + \frac{\mu}{\lambda+\mu}\,v_{\varphi_\mu}(g). \tag{11}$$

Again by the convexity of $\varphi$, for any $\eta \ge \lambda+\mu$ we have

$$w_{\varphi_\eta}(f,g) \le \frac{\lambda+\mu}{\eta}\,w_{\varphi_{\lambda+\mu}}(f,g) \to 0 \quad \text{as} \quad \eta\to\infty,$$

and so, the number $\Delta_\varphi(f,g)$ is well defined.

The main properties of $\Delta_\varphi$ and $w_\varphi$ are gathered in the following

**Lemma 6.** *If $(M,d,+)$ is a metric semigroup, $\varphi \in \Phi$ and $f, g \in W_\varphi(I;M)$, then*

*(a)* $|d(f(x),g(x)) - d(f(y),g(y))| \le d\big(f(x)+g(y),g(x)+f(y)\big) \le$ $\le \varphi^{-1}(1)\Delta_\varphi(f,g)$ *for all* $x, y \in I$;

*(b) if* $\lambda = \Delta_\varphi(f,g) > 0$, *then* $w_{\varphi_\lambda}(f,g) \le 1$;

*(c) given* $\lambda>0$, $\Delta_\varphi(f,g)\le\lambda$, *if and only if* $w_{\varphi_\lambda}(f,g)\le 1$;

*(d) if* $\lambda > 0$ *and* $w_{\varphi_\lambda}(f,g) = 1$, *then* $\Delta_\varphi(f,g) = \lambda$;

*(e) if sequences* $\{f_k\}, \{g_k\} \subset W_\varphi(I;M)$ *converge pointwise on* $I$ *to* $f$ *and* $g$, *respectively, as* $k\to\infty$, *then* $\Delta_\varphi(f,g) \le \liminf_{k\to\infty} \Delta_\varphi(f_k,g_k)$;

*(f)* $|p_\varphi(f) - p_\varphi(g)| \le \Delta_\varphi(f,g) \le p_\varphi(f) + p_\varphi(g)$.

*Proof.* (a) The first inequality is a consequence of (6). In order to obtain the second inequality, we note that by definitions of $\Delta_\varphi$ and $w_\varphi$, we have

$$\varphi\biggl(\frac{d\big(f(x)+g(y),g(x)+f(y)\big)}{\lambda}\biggr) \le w_{\varphi_\lambda}(f,g) \le 1, \quad \text{if} \quad \lambda > \Delta_\varphi(f,g),$$

and it suffices to take the inverse function $\varphi^{-1}$ and multiply by $\lambda$.

(b) First, let us show that if conditions of (e) are satisfied and $\lambda(k)\to\lambda$ as $k\to\infty$, where $\lambda(k) > 0$, $k\in\mathbb{N}$, and $\lambda > 0$, then

$$w_{\varphi_\lambda}(f,g) \le \liminf_{k\to\infty} w_{\varphi_{\lambda(k)}}(f_k,g_k). \tag{12}$$

The pointwise convergence of $f_k$ to $f$ and $g_k$ to $g$ and property (8) yield

$$\lim_{k\to\infty} d\big(f_k(x)+g_k(y),g_k(x)+f_k(y)\big) = d\big(f(x)+g(y),g(x)+f(y)\big), \qquad x,y\in I.$$

For any partition $\{x_i\}_{i=0}^m$ of $I$ and any $k\in\mathbb{N}$ the definition of $w_\varphi$ gives

$$\sum_{i=1}^m \varphi\biggl(\frac{d\big(f_k(x_i)+g_k(x_{i-1}),g_k(x_i)+f_k(x_{i-1})\big)}{\lambda(k)}\biggr) \le w_{\varphi_{\lambda(k)}}(f_k,g_k).$$

Now (12) follows if we pass to the limit inferior as $k \to \infty$, make use of the continuity of $\varphi$ and then take the supremum over all partitions of $I$.

To prove (b), let $\lambda(k) > \lambda$, $k \in \mathbb{N}$, be such that $\lambda(k) \to \lambda$ as $k \to \infty$. Since $w_{\varphi_{\lambda(k)}}(f,g) \le 1$ for $k \in \mathbb{N}$ by definition of $\Delta_\varphi$, (12) implies $w_{\varphi_\lambda}(f,g) \le 1$.

(c) Similarly to the proof of Lemma 3(c), by virtue of (a) and (b), it suffices to show that, if $0 < \Delta_\varphi(f,g) < \lambda$, then $w_{\varphi_\lambda}(f,g) < 1$: set $\mu = \Delta_\varphi(f,g)$ and note that $w_{\varphi_\lambda}(f,g) \le (\mu/\lambda) w_{\varphi_\mu}(f,g) \le \mu/\lambda < 1$.

(d) By (c), the cases $\Delta_\varphi(f,g) < \lambda$ or $\Delta_\varphi(f,g) > \lambda$ cannot hold.

(e) Replace $p_\varphi(f_k)$ by $\Delta_\varphi(f_k, g_k)$ in the proof of Lemma 3(e) and apply (12).

(f) To prove the first inequality, set $\lambda = \Delta_\varphi(f,g)$ and $\mu = p_\varphi(g)$ and assume first that $\lambda > 0$ and $\mu > 0$. Since, by (6),

$$(13)\quad d\big(f(x), f(y)\big) \le d\big(f(x)+g(y), g(x)+f(y)\big) + d\big(g(x), g(y)\big), \qquad x, y \in I,$$

the convexity of $\varphi$ and Lemmas 3(b) and 6(b) imply

$$v_{\varphi_{\lambda+\mu}} \le \frac{\lambda}{\lambda+\mu} w_{\varphi_\lambda}(f,g) + \frac{\mu}{\lambda+\mu} v_{\varphi_\mu}(g) \le 1,$$

whence $p_\varphi(f) \le \lambda + \mu = \Delta_\varphi(f,g) + p_\varphi(g)$. Similarly, $p_\varphi(g) \le \Delta_\varphi(f,g) + p_\varphi(f)$. Now if $\lambda = 0$, by Lemma 6(a), (13) and the symmetry in $f$ and $g$, $d\big(f(x), f(y)\big) = d\big(g(x), g(y)\big)$ for all $x, y \in I$, and so, $p_\varphi(f) = p_\varphi(g)$. If $\mu = 0$, Lemma 3(a) implies that $g$ is constant and, hence, $d\big(f(x), f(y)\big) = d\big(f(x)+g(y), g(x)+f(y)\big)$, $x, y \in I$, so that $\Delta_\varphi(f,g) = p_\varphi(f)$.

To prove the second inequality, set $\lambda = p_\varphi(f)$ and $\mu = p_\varphi(g)$. If $\lambda \cdot \mu = 0$, the (in)equality is obvious by virtue of Lemma 3(a). In the case $\lambda \cdot \mu > 0$ Lemma 3(b) implies $v_{\varphi_\lambda}(f) \le 1$ and $v_{\varphi_\mu}(g) \le 1$, and so, $w_{\varphi_{\lambda+\mu}}(f,g) \le 1$ thanks to (11). From the definition of $\Delta_\varphi$ we conclude that $\Delta_\varphi(f,g) \le \lambda + \mu = p_\varphi(f) + p_\varphi(g)$. □

**Theorem 1.** *If $\varphi \in \Phi$ and $(M, d, +)$ is a (complete) metric semigroup (or an abstract convex cone), then the triple $(W_\varphi(I;M), d_\varphi, +)$ is also a (complete) metric semigroup (or an abstract convex cone, respectively).*

*Proof.* Let $f, g, h \in W_\varphi(I;M)$. The translation invariance of $d_\varphi$ follows from equality $\Delta_\varphi(f+h, g+h) = \Delta_\varphi(f,g)$, which is a straightforward consequence of the translation invariance of $d$ and the following equality for $x, y \in I$:

$$d\big((f+h)(x)+(g+h)(y), (g+h)(x)+(f+h)(y)\big) = d\big(f(x)+g(y), g(x)+f(y)\big).$$

Now let us show that $d_\varphi$ is a metric on $W_\varphi(I;M)$. If $d_\varphi(f,g) = 0$, Lemma 6(a) implies $d\big(f(x), g(x)\big) = 0$, $x \in I$, i.e., $f = g$. Clearly, $d_\varphi(f,g) = d_\varphi(g,f)$. The triangle inequality for $d_\varphi$ will follow if we prove that $\Delta_\varphi(f,g) \le$

$\Delta_\varphi(f,h)+\Delta_\varphi(g,h)$. Given $x,y\in I$, from (6) and the translation invariance of $d$ we have:

$$\begin{aligned} d\big(f(x)+g(y),g(x)+f(y)\big) &\le d\big(f(x)+h(y),h(x)+f(y)\big)+\\ &+d\big(g(x)+h(y),h(x)+g(y)\big). \end{aligned} \tag{14}$$

First assume that $\Delta_\varphi(f,h)=0$. By Lemma 6(a), the first term at the right hand side of (14) is equal to zero, and so, by (14), $w_{\varphi_\lambda}(f,g)\le w_{\varphi_\lambda}(g,h)$ for all $\lambda>0$. This gives $\Delta_\varphi(f,g)\le\Delta_\varphi(g,h)$ and, by symmetry in $f$ and $g$, $\Delta_\varphi(f,g)=\Delta_\varphi(g,h)$. In a similar manner, if $\Delta_\varphi(g,h)=0$, then $\Delta_\varphi(f,g)=\Delta_\varphi(f,h)$. Now, let $\lambda=\Delta_\varphi(f,h)$ and $\mu=\Delta_\varphi(g,h)$ be positive, so that, by Lemma 6(b), $w_{\varphi_\lambda}(f,h)\le 1$ and $w_{\varphi_\mu}(g,h)\le 1$. From (14), definition of $w_\varphi$ and the monotonicity and convexity of $\varphi$ we have:

$$w_{\varphi_{\lambda+\mu}}(f,g)\le\frac{\lambda}{\lambda+\mu}\,w_{\varphi_\lambda}(f,h)+\frac{\mu}{\lambda+\mu}\,w_{\varphi_\mu}(g,h)\le 1,$$

which proves that $\Delta_\varphi(f,g)\le\lambda+\mu=\Delta_\varphi(f,h)+\Delta_\varphi(g,h)$.

Let $(M,d,+)$ be complete and $\{f_k\}\subset W_\varphi(I;M)$ be a Cauchy sequence:

$$d_\varphi(f_k,f_j)=d\big(f_k(a),f_j(a)\big)+\Delta_\varphi(f_k,f_j)\to 0\quad\text{as}\quad k,j\to\infty. \tag{15}$$

Lemma 6(a) implies that $\{f_k(x)\}$ is a Cauchy sequence in $M$ for all $x\in I$, and so, there exists a mapping $f:I\to M$ such that $d\big(f_k(x),f(x)\big)\to 0$ as $k\to\infty$ for all $x\in I$. By Lemma 6(e),

$$\Delta_\varphi(f_k,f)\le\liminf_{j\to\infty}\Delta_\varphi(f_k,f_j)\le\lim_{j\to\infty}d_\varphi(f_k,f_j)\in\mathbb{R}^+,\qquad k\in\mathbb{N}.$$

Since $\{f_k\}$ is a Cauchy sequence in $W_\varphi(I;M)$,

$$\limsup_{k\to\infty}\Delta_\varphi(f_k,f)\le\lim_{k\to\infty}\lim_{j\to\infty}d_\varphi(f_k,f_j)=0,$$

whence $d_\varphi(f_k,f)\to 0$ as $k\to\infty$. It remains to note that $f\in W_\varphi(I;M)$: (15) and Lemma 6(f) imply that $\{p_\varphi(f_k)\}$ is a Cauchy sequence in $\mathbb{R}$, and so, it is convergent and $p_\varphi(f)\le\lim_{k\to\infty}p_\varphi(f_k)<\infty$. □

Now we study the embeddings of metric semigroups $W_\varphi(I;M)$ corresponding to different functions $\varphi\in\Phi$.

**Lemma 7.** *If $(M,d,+)$ is a metric semigroup and $\varphi,\psi\in\Phi$, then*
- *(a) $V(I;M)\subset W_\varphi(I;M)$, and $\Delta_\varphi(f,g)\le\Delta_1(f,g)/\varphi^{-1}(1)$ for $f,g\in V(I;M)$;*
- *(b) if $\psi\preccurlyeq\varphi$, then $W_\varphi(I;M)\subset W_\psi(I;M)$, and there exist constants $\kappa>0$ and $\kappa_0>0$, depending on $\varphi$ and $\psi$ only, such that $\Delta_\psi(f,g)\le\kappa\Delta_\varphi(f,g)$ and $d_\psi(f,g)\le\kappa_0 d_\varphi(f,g)$ for all $f,g\in W_\varphi(I;M)$.*

*Proof.* (a) The embedding is a consequence of Lemma 3(f). To prove the inequality, we note that $w_{\varphi_\lambda}(f,g) \le \varphi_\lambda\big(\Delta_1(f,g)\big) = 1$, if $\lambda = \Delta_1(f,g)/\varphi^{-1}(1)$, and so, $\Delta_\varphi(f,g) \le \lambda$.

(b) We employ the notation and idea from Lemma 4. Let $f, g \in W_\varphi(I;M)$ and $\lambda = \Delta_\varphi(f,g)$. If $\lambda = 0$, then, by Lemma 6(a), $d\big(f(x)+g(y), g(x)+f(y)\big) = 0$ for all $x, y \in I$, and so, $\Delta_\psi(f,g) = 0$. If $\lambda > 0$, we take into account (5) and define $s_1$, $r_2$ and $\mu$ as in the proof of Lemma 4. Then by Lemma 6(a),

$$d\big(f(x)+g(y), g(x)+f(y)\big)/\mu \le s_1 \quad \text{for all} \quad x, y \in I,$$

and so, (5) and the convexity of $\varphi$ provide the inequality:

$$\psi\bigg(\frac{d\big(f(x)+g(y), g(x)+f(y)\big)}{\mu}\bigg) \le \frac{r_1(s_1)}{r_2}\,\varphi\bigg(\frac{d\big(f(x)+g(y), g(x)+f(y)\big)}{\lambda}\bigg).$$

By Lemma 6(b), given a partition $\xi = \{x_i\}_{i=1}^m$ of $I$, we have:

$$\begin{aligned}\sum_{i=1}^m \psi_\mu\Big(d\big(f(x_i)+f(x_{i-1}), g(x_i)+f(x_{i-1})\big)\Big) \le \\ \le \sum_{i=1}^m \varphi_\lambda\Big(d\big(f(x_i)+g(x_{i-1}), g(x_i)+f(x_{i-1})\big)\Big) \le \\ \le w_{\varphi_\lambda}(f,g) \le 1.\end{aligned}$$

Thus, $w_{\psi_\mu}(f,g) \le 1$, and so, $\Delta_\psi(f,g) \le \mu = r_2 c\,\Delta_\varphi(f,g) \equiv \kappa\Delta_\varphi(f,g)$.

To prove the second inequality, it suffices to set $\kappa_0 = \max\{1, r_2 c\}$. □

## 4. Lipschitzian Nemytskii operators. Sufficient condition

If a metric semigroup $(M,d,+)$ contains zero, $\varphi \in \Phi$ and $f \in W_\varphi(I;M)$, we set

$$\|f\|_{\varphi,d} = |f(a)|_d + p_{\varphi,d}(f), \quad \text{where} \quad |f(a)|_d = d\big(f(a), 0\big).$$

Recall that $d_L$ is a metric on $\mathrm{L}(N;M)$ with properties considered in Section 3.

**Theorem 2.** *Suppose that $(N,\rho,+)$ and $(M,d,+)$ are two metric semigroups with zeros, $\mathrm{Lip}_0(N;M) = \{T \in \mathrm{Lip}(N;M) \mid T(0) = 0\}$ and $\psi \in \Phi$. If $f \in W_\psi(I;\mathrm{Lip}_0(N;M))$ and $g \in W_\psi(I;N)$, then the mapping $fg : I \to M$, defined by $(fg)(x) = f(x)g(x)$ for all $x \in I$, belongs to $W_\psi(I;M)$ and the following inequality holds $\|fg\|_{\psi,d} \le \gamma(\psi)\|f\|_{\psi,d_L}\|g\|_{\psi,\rho}$, where $\gamma(\psi) = \max\{1, 2\psi^{-1}(1)\}$.*

*Proof.* The value $\|fg\|_{\psi,d}$ is given by $\|fg\|_{\psi,d} = |(fg)(a)|_d + p_{\psi,d}(fg)$. By the definition of the Lipschitz constant of $f(a)$, we have:

$$|(fg)(a)|_d = d\big((fg)(a), 0\big) = d\big(f(a)g(a), f(a)(0)\big) \leq$$
$$\leq L(f(a))\rho(g(a), 0) = |f(a)|_{d_L} \cdot |g(a)|_\rho.$$

We are going to show that the second term $p_{\psi,d}(fg)$ is estimated by

$$p_{\psi,d}(fg) \leq |||f|||\, p_{\psi,\rho}(g) + p_{\psi,d_L}(f)\, \|g\| \equiv \eta,$$

where, taking into account Lemmas 5(b) and 3(a),

$$|||f||| \equiv \sup_{x\in I} L(f(x)) \leq L(f(a)) + d_L\big(f(x), f(a)\big) \leq |f(a)|_{d_L} + \psi^{-1}(1)p_{\psi,d_L}(f),$$
$$\|g\| \equiv \sup_{y\in I} \rho\big(g(y), 0\big) \leq \rho\big(g(a), 0\big) + \rho\big(g(y), g(a)\big) \leq |g(a)|_\rho + \psi^{-1}(1)p_{\psi,\rho}(g).$$

By definitions of Lipschitz constant $L(\cdot)$ and $d_L$, for any $x, y \in I$ we have:

$$d\big((fg)(x), (fg)(y)\big) \leq d\big(f(x)g(x), f(x)g(y)\big) + d\big(f(x)g(y), f(y)g(y)\big) \leq$$
$$\leq L(f(x))\rho\big(g(x), g(y)\big) + d_L\big(f(x), f(y)\big)\rho\big(g(y), 0\big) \leq$$
$$\leq |||f|||\, \rho\big(g(x), g(y)\big) + d_L\big(f(x), f(y)\big)\, \|g\|.$$

Without loss of generality we assume that $\lambda = p_{\psi,d_L}(f)$, $\mu = p_{\psi,\rho}(g)$, $|||f|||$ and $\|g\|$ are strictly positive. By the monotonicity and convexity of $\psi$ and Lemma 3(b), for any partition $\xi = \{x_i\}_{i=0}^m$ of $I$ we get

$$\begin{aligned}\sum_{i=1}^m \psi_\eta\Big(d\big((fg)(x_i), (fg)(x_{i-1})\big)\Big) &\leq \frac{|||f|||\,\mu}{\eta}\sum_{i=1}^m \psi_\mu\Big(\rho\big(g(x_i), g(x_{i-1})\big)\Big) + \\ &+ \frac{\lambda\,\|g\|}{\eta}\sum_{i=1}^m \psi_\lambda\Big(d_L\big(f(x_i), f(x_{i-1})\big)\Big) \leq \\ &\leq \frac{|||f|||\,\mu}{\eta}\, v_{\psi_\mu,\rho}(g) + \frac{\lambda\,\|g\|}{\eta}\, v_{\psi_\lambda,d_L}(f) \leq 1.\end{aligned}$$

It follows that $v_{\psi_\eta,d}(fg) \leq 1$, and so, $p_{\psi,d}(fg) \leq \eta$. It remains to take into account the above estimates on $p_{\psi,d}(fg)$, $|||f|||$ and $\|g\|$. □

**Remark 1.** *Theorem 2 contains as particular cases the results of* [7, Theorem 4] *(when $N$ and $M$ are normed spaces) and* [23, Theorem 2] *(when $N = M = \mathbb{R}$). Set-valued versions of Theorem 2, as well as the following Theorem 3, hold if we replace $(M, d, +)$ there by $(\mathrm{cbc}(Y), D, \overset{*}{+})$ with $Y$ a normed space (see page 24).*

**Theorem 3.** *Let $(N, \rho, +)$ and $(M, d, +)$ be two metric semigroups with zeros. Let $\varphi, \psi \in \Phi$, $\psi \preccurlyeq \varphi$, and the generator $h : I \times N \to M$ of the Nemytskii operator $\mathcal{H} : N^I \to M^I$ be defined by $h(x, u) = f(x)u + h_0(x)$, $x \in I$, $u \in N$, where $f \in W_\psi\big(I; \mathrm{L}(N; M)\big)$ and $h_0 \in W_\psi(I; M)$. Then $\mathcal{H} \in \mathrm{Lip}\big(W_\varphi(I; N); W_\psi(I; M)\big)$ and the following inequality holds: $L(\mathcal{H}) \le \gamma(\psi)\kappa_0(\varphi, \psi)\|f\|_{d_L,\psi}$.*

*Proof.* First, we show that $\mathcal{H}$ maps $W_\varphi(I; N)$ into $W_\psi(I; M)$. Let $g \in W_\varphi(I; N)$. The Nemytskii operator $\mathcal{H}$ is given by $(\mathcal{H}g)(x) = f(x)g(x) + h_0(x)$, $x \in I$. By Lemma 4, we have $g \in W_\psi(I; N)$ and $p_{\psi,\rho}(g) \le \kappa p_{\varphi,\rho}(g)$. Theorem 2 implies $fg \in W_\psi(I; M)$, and so, $p_{\psi,d}(\mathcal{H}g) \le p_{\psi,d}(fg) + p_{\psi,d}(h_0)$. In this way we have shown that $\mathcal{H}g \in W_\psi(I; M)$.

Let us prove that $\mathcal{H}$ is Lipschitzian. Suppose that $g_1, g_2 \in W_\varphi(I; N)$. From Lemma 7 we know that $\rho_\psi(g_1, g_2) \le \kappa_0(\varphi, \psi)\rho_\varphi(g_1, g_2)$. Now we have to estimate the following distance:

$$d_\psi(\mathcal{H}g_1, \mathcal{H}g_2) = d\big((\mathcal{H}g_1)(a), (\mathcal{H}g_2)(a)\big) + \Delta_{\psi,d}(\mathcal{H}g_1, \mathcal{H}g_2).$$

In what follows we make use of the translation invariance of $d$ several times. We have:

$$d\big((\mathcal{H}g_1)(a), (\mathcal{H}g_2)(a)\big) = d\big(f(a)g_1(a), f(a)g_2(a)\big) \le |f(a)|_{d_L}\, \rho\big(g_1(a), g_2(a)\big).$$

We will prove that the second term is estimated as follows:

$$\Delta_{\psi,d}(\mathcal{H}g_1, \mathcal{H}g_2) \le |||f|||\Delta_{\psi,\rho}(g_1, g_2) + p_{\psi,d_L}(f)\|g_1, g_2\| \equiv \eta,$$

where $|||f|||$ is defined and estimated in the proof of Theorem 2 and the value $\|g_1, g_2\|$ is defined and estimated, in view of Lemma 6(a), by

$$\|g_1, g_2\| \equiv \sup_{y \in I} \rho\big(g_1(y), g_2(y)\big) \le \rho\big(g_1(a), g_2(a)\big) + \psi^{-1}(1)\Delta_{\psi,\rho}(g_1, g_2).$$

Given $x$, $y \in I$, by virtue of the additivity of operators $f(x)$, the following equality holds:

$$\begin{aligned}&\big[(fg_1)(x) + (fg_2)(y)\big] + \big[f(x)\big(g_2(x) + g_1(y)\big)\big] + \big[f(y)g_1(y) + f(x)g_2(y)\big] = \\ &= \big[(fg_2)(x) + (fg_1)(y)\big] + \big[f(x)\big(g_1(x) + g_2(y)\big)\big] + \big[f(x)g_1(y) + f(y)g_2(y)\big].\end{aligned}$$

Denote by $l_k$ (respectively, $r_k$) the $k$-th term in square brackets at the left (respectively, right) hand side of this equality, $k = 1, 2, 3$. Then, by (7),

$$\begin{aligned}d(l_1, r_1) &= d(l_1 + l_2 + l_3, r_1 + l_2 + l_3) = d(r_1 + r_2 + r_3, r_1 + l_2 + l_3) = \\ &= d(r_2 + r_3, l_2 + l_3) \le d(r_2, l_2) + d(r_3, l_3),\end{aligned}$$

that is,

$$\begin{aligned}
d\big((\mathcal{H}g_1)(x)+(\mathcal{H}g_2)(y),(\mathcal{H}g_2)(x)+(\mathcal{H}g_1)(y)\big) &= \\
&= d\big((fg_1)(x)+(fg_2)(y),(fg_2)(x)+(fg_1)(y)\big) \le \\
&\le d\big(f(x)(g_1(x)+g_2(y)),f(x)(g_2(x)+g_1(y))\big)+ \\
&\qquad + d\big(f(x)g_1(y)+f(y)g_2(y),f(y)g_1(y)+f(x)g_2(y)\big) \le \\
&\le L(f(x))\,\rho\big(g_1(x)+g_2(y),g_2(x)+g_1(y)\big)+ \\
&\qquad + d_L\big(f(x),f(y)\big)\,\rho\big(g_1(y),g_2(y)\big) \le \\
&\le |||f|||\,\rho\big(g_1(x)+g_2(y),g_2(x)+g_1(y)\big)+ \\
&\qquad + d_L\big(f(x),f(y)\big)\,\|g_1,g_2\|.
\end{aligned}$$

Assume, with no loss of generality, that the quantities $\lambda = p_{\psi,d_L}(f)$, $\mu = \Delta_{\psi,\rho}(g_1,g_2)$, $|||f|||$ and $\|g_1,g_2\|$ are positive. By the monotonicity and convexity of $\psi$ and Lemmas 6(b) and 3(b), for any partition $\xi=\{x_i\}_{i=0}^m$ of $I$ we get:

$$\begin{aligned}
\sum_{i=1}^m \psi_\eta\Big(d\big((\mathcal{H}g_1)(x_i)+(\mathcal{H}g_2)(x_{i-1}),(\mathcal{H}g_2)(x_i)+(\mathcal{H}g_1)(x_{i-1})\big)\Big) &\le \\
&\le \frac{|||f|||\,\mu}{\eta}\sum_{i=1}^m \psi_\mu\Big(\rho\big(g_1(x_i)+g_2(x_{i-1}),g_2(x_i)+g_1(x_{i-1})\big)\Big)+ \\
&\qquad + \frac{\lambda\,\|g_1,g_2\|}{\eta}\sum_{i=1}^m \psi_\lambda\Big(d_L\big(f(x_i),f(x_{i-1})\big)\Big) \le \\
&\le \frac{|||f|||\,\mu}{\eta}\,w_{\psi_\mu,\rho}(g_1,g_2)+\frac{\lambda\,\|g_1,g_2\|}{\eta}\,v_{\psi_\lambda,d_L}(f) \le 1.
\end{aligned}$$

It follows that $w_{\psi_\eta,d}(\mathcal{H}g_1,\mathcal{H}g_2)\le 1$, and so, $\Delta_{\psi,d}(\mathcal{H}g_1,\mathcal{H}g_2)\le\eta$. It remains to note that

$$\begin{aligned}
d_\psi(\mathcal{H}g_1,\mathcal{H}g_2) &\le |f(a)|_{d_L}\,\rho\big(g_1(a),g_2(a)\big)+ \\
&\qquad + \Big(|f(a)|_{d_L}+\psi^{-1}(1)p_{\psi,d_L}(f)\Big)\,\Delta_{\psi,\rho}(g_1,g_2)+ \\
&\qquad + p_{\psi,d_L}(f)\Big(\rho\big(g_1(a),g_2(a)\big)+\psi^{-1}(1)\Delta_{\psi,\rho}(g_1,g_2)\Big) \le \\
&\le \gamma(\psi)\,\|f\|_{\psi,d_L}\,\rho_\psi(g_1,g_2) \le \\
&\le \gamma(\psi)\,\kappa_0(\varphi,\psi)\,\|f\|_{\psi,d_L}\,\rho_\varphi(g_1,g_2). \qquad \square
\end{aligned}$$

## 5. Lipschitzian Nemytskii operators. Necessary condition

It turns out that Theorem 3 almost characterizes Lipschitzian Nemytskii operators between $W_\varphi(I;M)$. In order to establish the reverse theorem (Theorem 4), we need four more Lemmas (Lemmas 8–11).

**Lemma 8** ([35, Proposition 1.03]). *If $\zeta : I = [a,b] \to \mathbb{R}$ is a monotone function and $\varphi \in \Phi$, then $v_\varphi(\zeta) = \varphi\big(|\zeta(b) - \zeta(a)|\big)$.*

**Lemma 9** ([21, 2.13–2.14], if $M = \mathbb{R}$ and [11, 4.1–4.2] in the general case). *Let $(M,d)$ be a complete metric space and $\varphi \in \Phi$. Then any mapping $f \in W_\varphi(I;M)$ admits the limit from the left $f(x-0) \in M$ at each point $x \in (a,b]$ and the limit from the right $f(x+0) \in M$ at each point $x \in [a,b)$, and the set of discontinuity points of $f$ on $I$ is at most countable.*

Let $(M,d)$ be a complete metric space, $\varphi \in \Phi$ and $f \in W_\varphi(I;M)$. We define the *left regularization* $f^- : I \to M$ of $f$ as follows:

$$f^-(x) = \lim_{y\to x-0} f(y) \text{ if } x \in (a,b] \text{ and } f^-(a) = \lim_{x\to a+0} f^-(x) = \lim_{x\to a+0} f(x) \text{ in } M.$$

Denote by $W_\varphi^-(I;M)$ the set of those $f \in W_\varphi(I;M)$, which are left-continuous on $(a,b]$ (i.e., $f^-(x) = f(x)$ for all $x \in (a,b]$). We have:

**Lemma 10** ([7, Lemma 6]). *If $(M,d)$ is a complete metric space, $\varphi \in \Phi$ and $f \in W_\varphi(I;M)$, then $f^- \in W_\varphi^-(I;M)$ and $p_\varphi(f^-) \le p_\varphi(f)$.*

We remark that the proof of Lemma 6 from [7], formulated for a Banach space $M$, is valid for the case of a complete metric space $M$.

**Lemma 11** ([40, Theorem 1 and Corollary 2]). *Let $(N,+)$ be an Abelian semigroup with zero and division by 2 and $(M,d,+,\cdot)$ be a complete abstract convex cone. Then the operator $T : N \to M$ satisfies the Jensen functional equation*

$$2\,T\Big(\frac{u+v}{2}\Big) = Tu + Tv \quad \text{for all} \quad u,v \in N,$$

*if and only if there exist a unique additive operator $A : N \to M$ and a constant $h_0 \in M$ such that $Tu = Au + h_0$ for all $u \in N$.*

Particular cases of Lemma 11 for operators $T$ with compact convex values were obtained earlier in [18, Theorem 2] and [37, Theorem 5.6].

The main result of this Section is the following

**Theorem 4.** *Let $(N,\rho,+,\cdot)$ and $(M,d,+,\cdot)$ be two abstract convex cones with $M$ complete, $h : I \times N \to M$ be the generator of the Nemytskii operator $\mathcal{H}$ and $\varphi,\psi \in \Phi$. If $\mathcal{H} \in \mathrm{Lip}\big(W_\varphi(I;N); W_\psi(I;M)\big)$, then $h(x,\cdot) \in \mathrm{Lip}(N;M)$ for all $x \in I$ and there exist two mappings $f : I \to \mathrm{L}(N;M)$ and $h_0 : I \to M$ such that $f(\cdot)u$, $h_0 \in W_\psi^-(I;M)$ for all $u \in N$ and the following* *representation holds*: $h^-(x,u)=f(x)u+h_0(x)$ *for all* $x\in I$ *and* $u\in N$, *where* $f(\cdot)u$ *is given by* $x\mapsto f(x)u$ *and* $h^-(\cdot,u)$ *is the left regularization of* $h(\cdot,u)$ *for each fixed* $u\in N$.

*Proof.* We divide the proof into three steps for clarity. Given $g_1,g_2\in W_\varphi(I;N)$, set $\lambda=L(\mathcal{H})\rho_\varphi(g_1,g_2)$, and since $\mathcal{H}$ is Lipschitzian, we have $d_\psi(\mathcal{H}g_1,\mathcal{H}g_2)\le\lambda$ or even $\Delta_{\psi,d}(\mathcal{H}g_1,\mathcal{H}g_2)\le\lambda$. By Lemma 6(c), if $\lambda>0$, then the last inequality is equivalent to $w_{\psi_\lambda,d}(\mathcal{H}g_1,\mathcal{H}g_2)\le 1$. By definitions of $w_{\psi_\lambda,d}$ and $\mathcal{H}$, we find that, if $n\in\mathbb{N}$ and $a\le\alpha_1<\beta_1<\alpha_2<\beta_2<\dots<\alpha_n<\beta_n\le b$, then

(16)
$$\sum_{i=1}^n\psi\biggl(\frac{d\bigl(h(\beta_i,g_1(\beta_i))+h(\alpha_i,g_2(\alpha_i)),h(\beta_i,g_2(\beta_i))+h(\alpha_i,g_1(\alpha_i))\bigr)}{L(\mathcal{H})\rho_\varphi(g_1,g_2)}\biggr)\le 1.$$

If $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta$, we define $\zeta_{\alpha,\beta}\in\mathrm{Lip}(\mathbb{R};[0,1])$ by

$$\zeta_{\alpha,\beta}(y)=\begin{cases}0, & \text{if}\quad y\le\alpha,\\ (y-\alpha)/(\beta-\alpha), & \text{if}\quad \alpha\le y\le\beta,\\ 1, & \text{if}\quad y\ge\beta.\end{cases}$$

1. Let us show that $h(x,\cdot)\in\mathrm{Lip}(N;M)$ for all $x\in I$. (We note that in this step $M$ may be any metric semigroup). Let $u_1,u_2\in N$ be arbitrary, $u_1\ne u_2$. First, suppose that $x\in(a,b]$. We set in (16): $n=1$, $\beta_1=x$, $\alpha_1=a$ and $g_k(y)=\zeta_{a,x}(y)u_k$ for $y\in I$ and $k=1,2$, so that $g_k(a)=0$ and $g_k(x)=u_k$, $k=1,2$. By (9) and Lemma 8,

$$w_{\varphi_\lambda,\rho}(g_1,g_2)=v_{\varphi_\lambda}\bigl(\rho(u_1,u_2)\zeta_{a,x}\bigr)=\varphi_\lambda\bigl(\rho(u_1,u_2)\bigr)=1,$$

if and only if $\lambda=\rho(u_1,u_2)/\varphi^{-1}(1)$, and so, $\Delta_{\varphi,\rho}(g_1,g_2)=\rho(u_1,u_2)/\varphi^{-1}(1)$ according to Lemma 6(d). Since $\rho_\varphi(g_1,g_2)=\Delta_{\varphi,\rho}(g_1,g_2)$, inequality (16) gives

$$d\bigl(h(x,u_1),h(x,u_2)\bigr)\le L(\mathcal{H})\frac{\psi^{-1}(1)}{\varphi^{-1}(1)}\,\rho(u_1,u_2).$$

Now, let $x=a$. We set in (16): $n=1$, $\beta_1=b$, $\alpha_1=a$ and $g_k(y)=\bigl(1-\zeta_{a,b}(y)\bigr)u_k$, $y\in I$, $k=1,2$, so that $g_k(a)=u_k$ and $g_k(b)=0$, $k=1,2$. In a similar manner as above, $\Delta_{\varphi,\rho}(g_1,g_2)=\rho(u_1,u_2)/\varphi^{-1}(1)$, and so,

$$\rho_\varphi(g_1,g_2)=\rho\bigl(g_1(a),g_2(a)\bigr)+\Delta_{\varphi,\rho}(g_1,g_2)=\Bigl(1+\frac{1}{\varphi^{-1}(1)}\Bigr)\rho(u_1,u_2).$$

Thus, inequality (16) implies

$$d\bigl(h(a,u_2),h(a,u_1)\bigr)\le L(\mathcal{H})\psi^{-1}(1)\Bigl(1+\frac{1}{\varphi^{-1}(1)}\Bigr)\rho(u_1,u_2)\equiv L_0\rho(u_1,u_2).$$

By the definition of $h^-$ and Lemma 9, we also find that

$$d\big(h^-(x,u_1),h^-(x,u_2)\big)\le L_0\rho(u_1,u_2),\qquad x\in I,\quad u_1,u_2\in N. \tag{17}$$

2. Let us establish the representation for $h^-(x,u)$. To start with, let $x\in(a,b]$ and $a<\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\alpha_n<\beta_n<x$. Define $\zeta_n\in\mathrm{Lip}(I;[0,1])$ by

$$\zeta_n(y)=\begin{cases}0, & \text{if } a\le y\le\alpha_1,\\ \zeta_{\alpha_i,\beta_i}(y), & \text{if } \alpha_i\le y\le\beta_i,\ i=1,\dots,n,\\ 1-\zeta_{\beta_i,\alpha_{i+1}}(y), & \text{if } \beta_i\le y\le\alpha_{i+1},\ i=1,\dots,n-1,\\ 1, & \text{if } \beta_n\le y\le b.\end{cases}$$

Given $u_1$, $u_2\in N$, $u_1\ne u_2$, we set

$$g_k(y)=\frac12\,\zeta_n(y)u_1+\frac12\big(1-\zeta_n(y)\big)u_2+\frac12\,u_k,\qquad y\in I,\quad k=1,\,2.$$

We note that, since $w_{\varphi_\lambda,\rho}(g_1,g_2)=0$ for all $\lambda>0$, then $\Delta_{\varphi,\rho}(g_1,g_2)=0$, and by virtue of (9), $\rho_\varphi(g_1,g_2)=\rho\big(g_1(a),g_2(a)\big)=\rho(u_1,u_2)/2$. Also, $g_1(\beta_i)=u_1$, $g_2(\beta_i)=(u_1+u_2)/2$, $g_1(\alpha_i)=(u_1+u_2)/2$ and $g_2(\alpha_i)=u_2$. Therefore, inequality (16) implies

$$\sum_{i=1}^{n}\psi\left(\frac{d\big(h(\beta_i,u_1)+h(\alpha_i,u_2),h(\beta_i,(u_1+u_2)/2)+h(\alpha_i,(u_1+u_2)/2)\big)}{L(\mathcal H)\big(\rho(u_1,u_2)/2\big)}\right)\le 1. \tag{18}$$

Since $\mathcal H$ maps $W_\varphi(I;N)$ into $W_\psi(I;M)$ and constant mappings belong to $W_\varphi(I;N)$, we have $h(\cdot,u)=\mathcal Hu\in W_\psi(I;M)$, and so, by Lemma 10, $h^-(\cdot,u)$ is in $W_\psi^-(I;M)$ for all $u\in N$. Taking into account the completeness of $M$, the definition of the left regularization $h^-(\cdot,u)$, the continuity of the addition operation $+$ in $M$ and the continuity of $\psi$ and passing to the limit as $\alpha_1\to x-0$ in (18) we find that

$$d\Big(h^-(x,u_1)+h^-(x,u_2),h^-\Big(x,\frac{u_1+u_2}{2}\Big)+h^-\Big(x,\frac{u_1+u_2}{2}\Big)\Big)\le$$
$$\le L(\mathcal H)\big(\rho(u_1,u_2)/2\big)\psi^{-1}(1/n).$$

As $n\to\infty$, it follows that

$$d\Big(h^-(x,u_1)+h^-(x,u_2),h^-\Big(x,\frac{u_1+u_2}{2}\Big)+h^-\Big(x,\frac{u_1+u_2}{2}\Big)\Big)=0$$

for all $x\in(a,b]$ and, by definition of $h^-(a,u)$, the last equality holds also at $x=a$. Since $d$ is a metric on $M$ and $M$ is an abstract convex cone, we have:

$$h^-(x,u_1)+h^-(x,u_2)=h^-\Big(x,\frac{u_1+u_2}{2}\Big)+h^-\Big(x,\frac{u_1+u_2}{2}\Big)=2h^-\Big(x,\frac{u_1+u_2}{2}\Big).$$

Thus, for each $x \in I$ the operator $h^-(x,\cdot) : N \to M$ satisfies the following Jensen functional equation:

$$2h^-\Big(x, \frac{u_1+u_2}{2}\Big) = h^-(x,u_1) + h^-(x,u_2), \qquad u_1,\, u_2 \in N.$$

By Lemma 11, for each $x \in I$ there exist an additive operator $f(x)(\cdot) : N \to M$ and a constant $h_0(x) \in M$ such that

$$h^-(x,u) = f(x)u + h_0(x), \qquad u \in N. \tag{19}$$

Since $f(x)(0) = 0$, (19) yields $h^-(x,0) = h_0(x)$ for all $x \in I$, and so, $h_0$ belongs to $W^-_\psi(I;M)$. From (19) and (17) we get, for all $u_1, u_2 \in N$ and $x \in I$,

$$\begin{aligned} d\big(f(x)u_1, f(x)u_2\big) &= d\big(f(x)u_1 + h_0(x), f(x)u_2 + h_0(x)\big) = \\ &= d\big(h^-(x,u_1), h^-(x,u_2)\big) \le \\ &\le L_0\rho(u_1,u_2), \end{aligned}$$

and so, $f(x) \in \mathrm{L}(N;M)$, that is, $f : I \to \mathrm{L}(N;M)$.

3. It remains to show that, given $u \in N$, we have $f(\cdot)u \in W^-_\psi(I;M)$. By virtue of (6) and (19), for any $x,\, y \in I$ we have

$$\begin{aligned} d\big(f(x)u, f(y)u\big) &\le d\big(f(x)u + h_0(x), f(y)u + h_0(y)\big) + d\big(h_0(x), h_0(y)\big) = \\ &= d\big(h^-(x,u), h^-(y,u)\big) + d\big(h_0(x), h_0(y)\big). \end{aligned} \tag{20}$$

As $h^-(\cdot,u), h_0 \in W^-_\psi(I;M)$, we set $\lambda = p_{\psi,d}\big(h^-(\cdot,u)\big)$ and $\mu = p_{\psi,d}(h_0)$ and suppose, with no loss of generality, that $\lambda \cdot \mu > 0$. It follows that

$$\frac{d\big(f(x)u, f(y)u\big)}{\lambda+\mu} \le \frac{\lambda}{\lambda+\mu}\,\frac{d\big(h^-(x,u), h^-(y,u)\big)}{\lambda} + \frac{\mu}{\lambda+\mu}\,\frac{d\big(h_0(x), h_0(y)\big)}{\mu},$$

and by the convexity of $\psi$ and Lemma 3(b),

$$v_{\psi_{\lambda+\mu},d}\big(f(\cdot)u\big) \le \frac{\lambda}{\lambda+\mu}\, v_{\psi_\lambda,d}\big(h^-(\cdot,u)\big) + \frac{\mu}{\lambda+\mu}\, v_{\psi_\mu,d}(h_0) \le 1,$$

and so, $p_{\psi,d}\big(f(\cdot)u\big) \le \lambda+\mu$ and $f(\cdot)u \in W_\psi(I;M)$. That $f(\cdot)u$ is left-continuous on $(a,b]$ is a consequence of (20): if $x \in (a,b]$, we pass to the limit as $y \to x-0$ in (20) and note, that both terms at the right hand side of (20) tend to zero. □

**Remark 2.** *Theorem 4 contains as particular cases the results from* [6, Theorem 5.5]*,* [7, Theorem 7]*,* [29, Theorem 2]*,* [31, Theorems 1] *and* [44, Theorem 1]*. The representation of the form* $h^-(x,u) = f(x)u + h_0(x)$ *for generators of Lipschitzian Nemytskii operators was found by Matkowski* [26, 28] *(in the space of Lipschitz functions and mappings) and Matkowski and Miś* [31] *(in the case* $\varphi(s) = \psi(s) = s$ *and* $N = M = \mathbb{R}$*). The idea* *to employ the (set-valued) Jensen functional equation belongs to A. and W. Smajdors* [39].

**Remark 3.** *A theorem similar to Theorem 4 is valid for the right regularization of* $h(\cdot, u)$. *However, the regularized mapping* $h^-(x,u)$ *in the representation of Theorem 4 cannot be replaced by* $h(x,u)$ *in general; the corresponding example (for the space* $V(I;\mathbb{R})$*) is constructed in* [31, p. 157].

**Remark 4.** *Let* $\mathrm{P}(I;N) \subset N^I$ *be the set of mappings satisfying the condition: for all* $u_1, u_2 \in N$, $n \in \mathbb{N}$ *and* $a < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < b$ *the mapping given by* $I \ni y \mapsto \zeta_n(y)u_1 + u_2 \in N$ *belongs to* $\mathrm{P}(I;N)$, *the function* $\zeta_n$ *being defined on p. 35. We endow* $\mathrm{P}(I;N)$ *with the metric* $\rho_\varphi$. *Then the conclusion of Theorem 4 holds if we replace the Lipschitz condition on* $\mathcal{H}$ *in it by* $\mathcal{H} \in \mathrm{Lip}\big(\mathrm{P}(I;N); W_\psi(I;M)\big)$.

**Remark 5.** *Suppose that the conditions of Theorem 4 are satisfied. Denote by* $\mathrm{B}(N;M)$ *the set of all bounded additive operators from* $N$ *into* $M$. *The proof of Theorem 4 shows that the following result is valid: if the Nemytskii operator* $\mathcal{H}$ *maps* $W_\varphi(I;N)$ *into* $W_\psi(I;M)$ *and is (*globally*)* bounded, *i. e. there exists a constant* $C \geq 0$ *such that* $d_\psi(\mathcal{H}g_1, \mathcal{H}g_2) \leq C$ *for all* $g_1, g_2 \in W_\varphi(I;N)$ *(cf. also Remark 4), then* $h(x,\cdot) \in \mathrm{B}(N;M)$ *for all* $x \in I$ *and there exists* $h_0 \in W_\psi^-(I;M)$ *such that* $h^-(x,u) = h_0(x)$ *for all* $x \in I$ *and* $u \in N$. *In fact, there exist* $f : I \to \mathrm{B}(N;M)$ *and* $h_0 \in W_\psi^-(I;M)$, *for which* $h^-(x,u) = f(x)u + h_0(x)$, $x \in I$, $u \in N$. *Since* $d\big(h(x,u_1), h(x,u_2)\big) \leq C$ *for all* $x \in I$ *and* $u_1, u_2 \in N$, *then* $d\big(f(x)u_1, f(x)u_2\big) = d\big(h^-(x,u_1), h^-(x,u_2)\big) \leq C$. *Thus, for any rational* $\lambda > 0$ *and any* $u \in N$ *we have:*

$$\lambda d\big(f(x)u, 0\big) = d\big(\lambda f(x)u, 0\big) = d\big(f(x)(\lambda u), f(x)(0)\big) \leq C,$$

*and so,* $f(x)u = 0$, *that is,* $f(x) = 0$ *for all* $x \in I$.

**Remark 6.** *A set-valued version of Theorem 4 holds if we replace* $(M,d,+,\cdot)$ *in it by* $(\mathrm{cbc}(Y), D, \overset{*}{+}, \cdot)$ *with* $Y$ *a Banach space (see p. 24).*

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