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# $B$-$(p, r)$-PRE-INVEX FUNCTIONS

TADEUSZ ANTCZAK‡

**Abstract.** Notions of pre-invexity is generalized. A class of functions called $B$-$(p, r)$-pre-invex is introduced by relaxing of $(p, r)$-pre-invex and $B$-pre-invex functions. Examples are given to show relations with other generalizations of pre-invex functions. Some (geometric) properties of this class of functions are derived.

## 1. Introduction

The notion of convexity undoubtedly plays a dominant role in almost all aspect of mathematical programming. Ben-Israel and Mond [5] considered a class of nondifferentiable functions were called pre-invex by Weir and Jeyakumar [11] as a generalization of convexity. Using the definition of a weighted $r$-mean (where $r$ is a real number) for a sequence of positive numbers, Antczak [1] introduced a new class of (nonconvex) functions called them $(p, r)$-pre-invex with respect to $\eta$. The class of $(p, r)$-pre-invex functions with respect to $\eta$ is an extension of the class of pre-invex functions with respect to $\eta$ introduced by Ben-Israel and Mond [5]. The concept of pre-invexity of functions was also generalized to $B$-pre-invex functions by Suneja, Singh and Bector [10].

In this paper, we introduce a new class of nonconvex functions, called $B$-$(p, r)$-pre-invex functions. Thus, we extend a notion of pre-invexity since the defined class of functions contains the classes of $(p, r)$-pre-invex and $B$-pre-invex functions. We also give examples to show that there exist $B$-$(p, r)$-pre-invex functions with respect to $\eta$, $b_1$, $b_2$, which are not either $(p, r)$-pre-invex with respect to the same function $\eta$ or $B$-pre-invex with respect to the same function $\eta$, $b_1$, $b_2$,. Making use of the definition of a level set, an epigraph of a function and an introduced definition of a $B$-$(p, r)$-invex set with respect to $\eta$, $b_1$, $b_2$, we shall give a characterization of geometric

‡*Faculty of Mathematics, University of Łódź*, Banacha 22 St., 90–238 Łódź, Poland. E-mail: antczak@math.uni.lodz.pl.

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properties of $B$-$(p,r)$-pre-invexity. Further, a characterization of the fundamental properties (not only geometric) of the introduced class of functions is dealt with. We show that the class of functions which are characterized by $B$-$(p,r)$-invexity possesses a principal property which has been the base of invexity theory, i.e. any local minimum of these functions is a global minimum. Some of their properties are obtained on the lines of $(p,r)$-pre-invex and $B$-pre-invex functions.

## 2. Definitions and properties of $B$-$(p,r)$-pre-invex functions

Before introducing the definition of a $B$-$(p,r)$-pre-invex function with respect to $\eta$, we recall the definition of a weighted $r$-mean (where $r$ is a real number) for a sequence of positive numbers [7], which will be useful in our further considerations.

**Definition 1.** *Let $a \in R^m$, $q \in R^m$ be vectors whose coordinates are positive and nonnegative numbers, respectively, and let $r$ be an arbitrary real number. If we assume that $\sum_{i=1}^{m} q_i = 1$ and $r$ is any finite real number, then the weighted $r$-mean is defined by*

$$M_r(a\,;q) := M_r(a_1,\,\dots\,,\,a_m;q) = \begin{cases} \left(\sum_{i=1}^{m} q_i a_i^r\right)^{1/r} & \text{for } r \neq 0, \\ \prod_{i=1}^{m} a_i^{q_i} & \text{for } r = 0. \end{cases} \tag{1}$$

We shall also use a definition of an $p$-invex set with respect to $\eta$. The definition of a set of this type was given by Antczak [1] who considered (not necessarily differentiable) functions called $(p,r)$-pre-invex with respect to $\eta$ which were defined on such a set.

**Definition 2.** *Let $X$ be a nonempty subset of $R^n$, $\eta : X \times X \to R^n$ and let $u$ be an arbitrary point of $X$. Then the set $X$ is said to be $p$-invex at $u$ with respect to $\eta$ if, for each $x \in X$, any $q_1 \geq 0$, $q_2 \geq 0$, $q_1 + q_2 = 1$, the following relation*

$$\left(\log\left[M_p\left(e^{\eta_1(x,u)+u_1}, e^{u_1}; q\right)\right], \dots, \log\left[M_p\left(e^{\eta_n(x,u)+u_n}, e^{u_n}; q\right)\right]\right) \in X \tag{2}$$

*is satisfied. Using the definition of a weighted $p$-mean, we may write down the above relation as follows:*

$$\begin{array}{ll} \log\left(\lambda e^{p(\eta(x,u)+u)} + (1-\lambda)e^{pu}\right)^{1/p} \in X & \text{for } \; p \neq 0, \\ u + \lambda\,\eta(x,u) \in X & \text{for } \; p = 0, \end{array}$$

*where the logarithm and the exponentials appearing in the relation are understood to be taken componentwise.*
*$X$ is said to be an $p$-invex set with respect to $\eta$ if $X$ is $p$-invex at each $u \in X$ with respect to the same $\eta$.*

**Remark 1.** *In particular, invex sets, were introduced by Ben-Israel and Mond* [5] *and subsequently studied by many authors including Mohan and Neogy* [8], *Pini* [9]. *The above definition of a $p$-invex set is a generalization of the definition of an invex set. Indeed, in the case when $p = 0$ we obtain the definition of this type set.*

Now we give some example of a $p$-invex set with $p \neq 0$.

**Example 1.** *Let $X = (a,b) \cup (b,c)$ be a nonempty subset in $R$. Note that $X$ is not a convex subset in $R$. We set*

$$\eta(x,u) = \begin{cases} x-u, & \text{if } x,u \in (a,b) \vee x,u \in (b,c), \\ 0, & \text{in other cases.} \end{cases}$$

*It is not difficult to prove by Definition 2 that $X$ is an 1-invex set with respect to $\eta$ defined above.*

Throughout the paper we shall assume that $X$ is a nonempty $p$-invex set with respect to $\eta$, where $\eta$ is a vector-valued function $\eta : X \times X \to R^n$. Further, let $f : X \to R$, $b_1 : X \times X \times [0,1] \to R_+$, and $b_2 : X \times X \times [0,1] \to R_+$.

For the shake of brevity we shall omit the argument of $b_1$, $b_2$ unless needed for specification.

Now we give the definition of a $B$-$(p,r)$-pre-invex function with respect $\eta$, $b_1$, $b_2$ by taking

$$b_1(x,u,\lambda) + b_2(x,u,\lambda) = 1, \quad b_1(x,u,1) = 1 = b_2(x,u,0). \tag{3}$$

**Definition 3.** *Let $\eta : X \times X \to R^n$ be a vector-valued function. A function $f : X \to R$ defined on a $p$-invex set $X \subset R^n$ with respect to $\eta$ is called $B$-$(p,r)$-pre-invex with respect to $\eta$, $b_1$, $b_2$ at $u \in X$ on $X$ if for any $x \in X$, any $q_1 \geq 0$, $q_2 \geq 0$, $q_1 + q_2 = 1$, the following inequality is satisfied:*

$$f\left(\log M_p\left(\mathrm{e}^{\eta(x,u)+u}, \mathrm{e}^{u}; q\right)\right) \leq \log\left(M_r\left(\mathrm{e}^{f(x)}, \mathrm{e}^{f(u)}; b\right)\right), \tag{4}$$

*where $b_1(x,u,\lambda) + b_2(x,u,\lambda) = 1$, $b_1(x,u,1) = 1 = b_2(x,u,0)$, and the logarithm and the exponentials appearing on left-hand side of the inequality are understood to be taken componentwise.*

*If inequality (4) is satisfied at any point $u \in X$, then $f$ is said to be $B$-$(p,r)$-pre-invex with respect to $\eta$ on $X$.*

Using the definition of a weighted $r$-mean, we may write down inequality (4) as follows:

$$(5)\quad f\left(\log\left(\lambda e^{p(\eta(x,u)+u)}+(1-\lambda)e^{pu}\right)^{\frac{1}{p}}\right)\leq$$
$$\leq \log\left(b_1(x,u,\lambda)e^{rf(x)}+b_2(x,u,\lambda)e^{rf(u)}\right)^{\frac{1}{r}}, \qquad \text{if } p\neq 0, r\neq 0,$$
$$f\left(\log\left(\lambda e^{p(\eta(x,u)+u)}+(1-\lambda)e^{pu}\right)^{\frac{1}{p}}\right)\leq$$
$$\leq b_1(x,u,\lambda)f(x)+b_2(x,u,\lambda)f(u), \qquad \text{if } p\neq 0, r=0,$$
$$f\left(u+\lambda\eta(x,u)\right)\leq$$
$$\leq \log\left(b_1(x,u,\lambda)e^{rf(x)}+b_2(x,u,\lambda)e^{rf(u)}\right)^{\frac{1}{r}}, \qquad \text{if } p=0, r\neq 0,$$
$$f\left(u+\lambda\eta(x,u)\right)\leq b_1(x,u,\lambda)f(x)+b_2(x,u,\lambda)f(u), \qquad \text{if } p=0, r=0,$$

**Remark 2.** *All classes of functions which were defined by (6) according to Definition 3 are called $B$-$(p,r)$-pre-invex functions with respect to $\eta$. But one may use the following terminology:*

- *in the case $p\neq 0, r=0$, functions defined by (6) are called $B$-$(p,0)$-pre-invex with respect to $\eta$, $b_1$, $b_2$;*
- *in the case $p=0, r\neq 0$, functions defined by (6) are called $B$-$(0,r)$-pre-invex with respect to $\eta$, $b_1$, $b_2$;*
- *in the case $p=0, r=0$, functions defined by (6) are called $B$-$(0,0)$-pre-invex with respect to $\eta$, $b_1$, $b_2$ (or shortly $B$-pre-invex functions with respect to $\eta$, which were introduced by Suneja, Singh and Bector* [10]*).*

*In the case $b_1(x,u,\lambda)\equiv\lambda$ and $b_2(x,u,\lambda)\equiv 1-\lambda$ functions defined by (6) are called $(p,r)$-pre-invex with respect to $\eta$ (this class of functions was introduced by Antczak* [1]*).*

*In the case $\eta(x,u)=x-u$ functions defined by (6) we will called $B$-$(p,r)$-convex. If, extra $b_1(x,u,\lambda)\equiv\lambda$ and $b_2(x,u,\lambda)\equiv 1-\lambda$, we obtain a definition of a class of $(p,r)$-convex functions (see* [3]*).*

An analogous terminology holds in the case of $B$-$(p,r)$-pre-incave functions with respect to $\eta$, $b_1$, $b_2$, for which the direction of inequalities (6) should be changed to the opposite ones.

**Definition 4.** *Let $X\subset R^n$ be a $p$-invex set with respect $\eta$. We say that a function $f: X\to R$ is strictly $B$-$(p,r)$-pre-invex (strictly $B$-$(p,r)$-pre-incave) with respect to $\eta$, $b_1$, $b_2$ at $u\in X$ on $X$ if inequalities (6) are sharp and they hold for all $x\neq u\in X$ and any $\lambda\in(0,1)$.*
*If inequalities (6) are satisfied at any point $u\in X$, then $f$ is said to be*

*strictly $B$-$(p,r)$-pre-invex (strictly $B$-$(p,r)$-pre-incave) with respect to $\eta$, $b_1$, $b_2$ on $X$.*

**Definition 5.** *Let $X \subset R^n$ be a p-invex set with respect $\eta$. We say that a function $f : X \to R$ is weakly $B$-$(p,r)$-pre-invex (weakly $B$-$(p,r)$-pre-incave) with respect to $\eta$, $b_1$, $b_2$ at $u \in X$ on $X$ if the respective inequality (6) holds for some $\bar{\lambda} \in (0,1)$.*

**Remark 3.** *Every convex function is $B$-$(0,0)$-pre-invex with respect to $\eta$, $b_1$, $b_2$, where*

$$\eta(x,u) = x - u, \quad b_1 = \lambda, \quad b_2 = 1 - \lambda.$$

*But the converse is not true.*

**Remark 4.** *Every pre-invex function with respect to $\eta$ [5] defined by*

$$f\left(u + \lambda\eta(x,u)\right) \leq \lambda f(x) + (1-\lambda) f(u)$$

*is also $B$-$(0,0)$-pre-invex function with respect to the same function $\eta$ and with respect to $b_1$, $b_2$ defined by*

$$b_1 = \lambda, \quad b_2 = 1 - \lambda. \tag{6}$$

*It is not difficult to see that there exist $B$-$(p,r)$-pre-invex functions with respect to $\eta$ and with respect to $b_1$, $b_2$ defined by (6) which are not pre-invex with respect to $\eta$.*

**Remark 5.** *Every $(p,r)$-pre-invex function with respect to $\eta$ introduced by Antczak [1] is also $B$-$(p,r)$-pre-invex function with respect to the same function $\eta$ and with respect to $b_1$, $b_2$ defined by (6). However, the converse is not true. There exist $B$-$(p,r)$-pre-invex functions with respect to $\eta$, $b_1$, $b_2$, which are not $(p,r)$-pre-invex with respect to $\eta$.*

**Remark 6.** *Every $B$-pre-invex function with respect to $\eta$, $b_1$, $b_2$ introduced by Suneja, Singh and Bector [10] is also $B$-$(0,0)$-pre-invex function with respect to the same function $\eta$ and with respect to $b_1$, $b_2$ defined by (6). However, the converse is not true. There exist $B$-$(p,r)$-pre-invex functions with respect to $\eta$, $b_1$, $b_2$, which are not $B$-pre-invex with respect to $\eta$, $b_1$, $b_2$.*

In the following example we give an example of $B$-$(p,r)$-pre-invex function with respect to $\eta$, $b_1$, $b_2$ which is neither $(p,r)$-pre-invex with respect to $\eta$ nor $B$-pre-invex with respect to $\eta$, $b_1$, $b_2$.

**Example 2.** *Let $X = \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$. Note $X$ is a pre-invex set with respect to $\eta$ defined by*

$$\eta(x,u) = \begin{cases} 0, & \text{if } 0 < x \leq u < \frac{\pi}{2} \vee \frac{\pi}{2} < x \leq u < \pi, \\ x - u, & \text{in other cases.} \end{cases} \tag{7}$$

*We consider $f : X \to R$ defined by $f(x) = \log(1 - \sin x)$. It is not difficult to show that $f$ is $B$-$(1,1)$-pre-invex function with respect to $\eta$, $b_1$, $b_2$. Indeed, if we put*

$$(8) \quad b_1 = \begin{cases} 1, & \text{if } 0 < x \le u < \frac{\pi}{2} \vee \frac{\pi}{2} < x \le u < \pi, \\ \frac{\sin(\lambda x + (1-\lambda)u) - \sin u}{\sin x - \sin u}, & \text{in other cases,} \end{cases}$$

$$(9) \quad b_2 = \begin{cases} 0, & \text{if } 0 < x \le u < \frac{\pi}{2} \vee \frac{\pi}{2} < x \le u < \pi, \\ \frac{\sin x - \sin(\lambda x + (1-\lambda)u)}{\sin x - \sin u}, & \text{in other cases,} \end{cases}$$

*then $f$ is $B$-$(0,1)$-pre-invex with respect to $\eta$, $b_1$, $b_2$. But $f$ is neither $(p,r)$-pre-invex with respect to $\eta$ nor $B$-pre-invex with respect to $\eta$, $b_1$, $b_2$ given above in (7)-(9).*

Now we give some example of a nondifferentiable $B$-$(p,r)$-pre-invex function defined on $R$.

**Example 3.** *Let $f : R \to R$ be defined by*

$$f(x) = \begin{cases} \log(1 - e^x), & \text{if } x < 0, \\ 0, & \text{if } x \ge 0. \end{cases}$$

*We set*

$$\eta(x,u) = \begin{cases} x - u, & \text{if } xu > 0, \\ 0, & \text{if } xu \le 0. \end{cases}$$

$$(10) \quad b_1 = \begin{cases} \frac{e^{\lambda x + (1-\lambda)u} - e^u}{e^x - e^u}, & \text{if } xu > 0, \\ 1, & \text{if } xu \le 0, \end{cases}$$

$$(11) \quad b_2 = \begin{cases} \frac{e^u - e^{\lambda x + (1-\lambda)u}}{e^x - e^u}, & \text{if } xu > 0, \\ 0, & \text{if } xu \le 0. \end{cases}$$

*Then by Definition 3 $f$ is a $B$-$(1,1)$-pre-function on $R$ with respect to $\eta$, $b_1$, $b_2$ defined above.*

A convex function defined on a nonempty open set is a continuous function [6]. But it can be seen from the following example that it is not necessarily true for a $B$-$(p,r)$-invex function.

**Example 4.** *Let $X = (-2,0) \cup (0,2)$. Define a function $f : X \to R$ by*

$$f(x) = \begin{cases} \log\left(x^2 + 1\right), & \text{if } -1 < x < 0 \vee 0 < x < 1, \\ 0, & \text{if } -2 < x \le -1 \vee 1 \le x < 2. \end{cases}$$

*Let $b_1 : X \times X \times [0,1] \to R_+$ and $b_2 : X \times X \times [0,1] \to R_+$ be defined by*

$$b_1(x,u,\lambda) = \begin{cases} \frac{(\lambda x + (1-\lambda)u)^2 - u^2}{x^2 - u^2}, & \text{if } (x,u \in (-2;-1) \vee x,u \in (1,2)) \wedge x \ne u \\ 1 - (1-\lambda)^2, & \text{in other cases,} \end{cases}$$

$$b_2(x,u,\lambda) = \begin{cases} \frac{x^2-(\lambda x+(1-\lambda)u)^2}{x^2-u^2}, & \text{if } (x,u \in (-2;-1) \vee x,u \in (1,2)) \wedge x \neq u \\ (1-\lambda)^2, & \text{in other cases,} \end{cases}$$

*We also put*

$$\eta(x,u) = \begin{cases} x-u, & \text{if } x,u \in (-2;-1) \vee x,u \in (1,2), \\ 0, & \text{in other cases.} \end{cases}$$

*As follows from Example 2, $X$ is an 1-invex set with respect to $\eta$ defined above. Further, it can be seen that $f$ is $B$-$(1,1)$-pre-invex with respect to $\eta$ on $X$, but $f$ is not continuous at $x = \pm 1$.*

The next theorems concern the relationships between the classes of $B$-$(p,r)$-pre-invex ($B$-$(p,r)$-pre-incave) functions with respect to the same functions $\eta$ with different values of the exponents $p$ and $r$. To prove them, we recall some useful lemma concerning the well-known relation between weighted means of an arbitrary sequence of nonnegative numbers of different orders [7].

**Lemma 1.** *If $a_1 = ... = a_m = a_0$, then $M_r(a;q) = M_r(a_1,...,a_m) = a_0$. Otherwise, $M_r(a;q)$ is a strictly increasing function of the variable $r$, that is, for $-\infty \leq r < s \leq \infty$, the inequality*

$$M_r(a;q) < M_s(a;q)$$

*holds for all weight values $q = (q_1,...,q_m)$.*

**Theorem 1.** *Let $f : X \to R$ be a $B$-$(p,r)$-pre-invex ($B$-$(p,r)$-pre-incave) function with respect $\eta$, $b_1$, $b_2$ on $X \subset R^n$. Then, it is a $B$-$(p,s)$-pre-invex ($B$-$(p,s)$-pre-incave) function with respect to the same function $\eta$, $b_1$, $b_2$ for all $s > r$ ($s < r$) on $X$.*

*Proof.* Follows from Lemma 1. □

**Theorem 2.** *Let $f : X \to R$ be a $B$-$(p,r)$-pre-invex ($B$-$(p,r)$-pre-incave) function with respect $\eta$, $b_1$, $b_2$ on $X \subset R^n$. Moreover, we assume that $f$ is a nondecreasing (nonincreasing) function on $X$. Then, it is a $B$-$(t,s)$-pre-invex ($B$-$(t,s)$-pre-incave) function with respect to the same function $\eta$, $b_1$, $b_2$ for all $t < p$ and $s > r$ ($t > p$ and $s < r$) on $X$.*

*Proof.* Follows from Lemma 1. □

## 3. Geometric properties of $B$-$(p,r)$-pre-invex functions

On the basis of the definitions of $p$-invex sets given in the proceeding section, we shall give the necessary and sufficient (geometric) conditions for $B$-$(p,r)$-pre-invexity with respect to $\eta$, $b_1$, $b_2$.

**Theorem 3.** *If $f : X \to R$ is a $B$-$(p,r)$-pre-invex function with respect to $\eta$, $b_1$, $b_2$ on $X \subset R^n$, then the level set $L_\alpha = \{x \in S : f(x) \leq \alpha\}$ is $p$-invex with respect to the same function $\eta$ for every $\alpha$.*

*Proof.* Let $f$ be a $B$-$(p,r)$-pre-invex function with respect to $\eta$, $b_1$, $b_2$ on $X$ and we assume that $x, u \in L_\alpha$. By the definition of a level set, the inequalities $f(x) \leq \alpha$ and $f(u) \leq \alpha$ are true. Hence, we have

a) for $p \neq 0$, $r \neq 0$:

$$\begin{aligned} & f\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{1/p}\right) \leq \\ & \leq \log\left(b_1(x,u,\lambda)\,\mathrm{e}^{rf(x)} + b_2(x,u,\lambda)\,\mathrm{e}^{rf(u)}\right)^{1/r} \leq \\ & \leq \log\left(b_1(x,u,\lambda)\,\mathrm{e}^{r\alpha} + b_2(x,u,\lambda)\,\mathrm{e}^{r\alpha}\right)^{1/r} \leq \\ & \leq \log\left(\mathrm{e}^{r\alpha}\right)^{1/r} = \frac{1}{r} r\alpha = \alpha. \end{aligned}$$

Thus we have proved that the relation $\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{1/p} \in L_\alpha$ holds for any $0 \leq \lambda \leq 1$ and any $\alpha$;

b) for $p = 0$, $r \neq 0$:

$$\begin{aligned} & f(u + \lambda\eta(x,u)) \leq \log\left(b_1(x,u,\lambda)\,\mathrm{e}^{rf(x)} + b_2(x,u,\lambda)\,\mathrm{e}^{rf(u)}\right)^{1/r} \leq \\ & \leq \log\left(b_1(x,u,\lambda)\,\mathrm{e}^{r\alpha} + b_2(x,u,\lambda)\,\mathrm{e}^{r\alpha}\right)^{1/r} = \log\left(\mathrm{e}^{r\alpha}\right)^{1/r} = \alpha; \end{aligned}$$

Thus we have proved that the relation $u + \lambda\eta(x,u) \in L_\alpha$ holds for any $0 \leq \lambda \leq 1$ and any $\alpha$;

c) for $p \neq 0$, $r = 0$:

$$\begin{aligned} & f\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{1/p}\right) \leq \\ & \leq b_1(x,u,\lambda)\,f(x) + b_2(x,u,\lambda)\,f(u) \leq \\ & \leq b_1(x,u,\lambda)\,\alpha + b_2(x,u,\lambda)\,\alpha = \alpha. \end{aligned}$$

Thus we have proved that the relation $\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{1/p} \in L_\alpha$ holds for any $0 \leq \lambda \leq 1$ and any $\alpha$;

d) for $p = 0$ , $r = 0$:

$$\begin{aligned} f(u + \lambda\eta(x,u)) & \leq b_1(x,u,\lambda)\,f(x) + b_2(x,u,\lambda)\,f(u) \leq \\ & \leq b_1(x,u,\lambda)\,\alpha + b_2(x,u,\lambda)\,\alpha = \alpha. \end{aligned}$$

On the basis of the definition of a $p$-invex set with respect to $\eta$ this means that the set $L_\alpha$ is $p$-invex with respect to $\eta$ (in the case when $p = 0$, it is invex with respect to $\eta$) for any $\alpha$. □

We introduce the definition of a $B$-$(p,r)$-invex set with respect to $\eta$, $b_1$, $b_2$, which will enable us to give another geometric properties of $B$-$(p,r)$-pre-invex functions with respect to $\eta$, $b_1$, $b_2$.

**Definition 6.** *Let $X \subset R^n$, $Y \subset R^m$ and $\eta : R^n \times R^n \to R^n$ be a vector function. Then $T := X \times Y = \{(x,y) : x \in X,\ y \in Y\}$ is said to be a $B$-$(p,r)$-invex set with respect to $\eta$, $b_1$, $b_2$ if the relation*

$$\left(\log\left[M_p\left(\mathrm{e}^{\eta(x^1,x^2)+x^2}, \mathrm{e}^{x^2};\, q\right)\right], \log\left[M_r\left(\mathrm{e}^{y^1}, \mathrm{e}^{y^2}; b\right)\right]\right) \in T$$

*is true for any $\left(x^1, y^1\right) \in T$, $\left(x^2, y^2\right) \in T$, any $q_1 \geq 0$, $q_2 \geq 0$, $q_1 + q_2 = 1$, and any $b_1$, $b_2$ satisfying (3).*

**Definition 7.** *Taking into account in the above definition of a $B$-$(p,r)$-invex set with respect to $\eta$, $b_1$, $b_2$ the form of a weighted $r$-mean, we get according to the values of $p$ and of $r$ the following definitions of*

*a) $B$-$(p,r)$-invex set with respect to $\eta$, $b_1$, $b_2$ in the case when $p \neq 0$, $r \neq 0$:*

$$\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{\frac{1}{p}}, \log\left(b_1\left(x,u,\lambda\right)\mathrm{e}^{r\alpha} + b_2\left(x,u,\lambda\right)\mathrm{e}^{r\beta}\right)^{\frac{1}{r}}\right) \in T,$$

*b) $B$-$(0,r)$-invex set with respect to $\eta$, $b_1$, $b_2$ in the case when $p = 0$, $r \neq 0$:*

$$\left(u + \lambda\eta(x,u), \log\left(b_1\left(x,u,\lambda\right)\mathrm{e}^{r\alpha} + b_2\left(x,u,\lambda\right)\mathrm{e}^{r\beta}\right)^{\frac{1}{r}}\right) \in T,$$

*c) $B$-$(p,0)$-invex set with respect to $\eta$ in the case when $p \neq 0$, $r = 0$:*

$$\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{\frac{1}{p}}, b_1\left(x,u,\lambda\right)\alpha + b_2\left(x,u,\lambda\right)\beta\right) \in T,$$

*d) $B$-$(0,0)$-invex set with respect to $\eta$, $b_1$, $b_2$ in the case when $p = 0$, $r = 0$:*

$$\left(u + \lambda\eta(x,u), b_1\left(x,u,\lambda\right)\alpha + b_2\left(x,u,\lambda\right)\beta\right) \in T.$$

**Definition 8.** *The epigraph of $f$ is the set*

$$epi\, f = \left\{(x,\alpha) \in X \times R : f(x) \leq \alpha\right\}.$$

**Theorem 4.** *A function $f : X \to R$ is $B$-$(p,r)$-pre-invex with respect to $\eta$, $b_1$, $b_2$ on $X \subset R^n$ if and only if its epigraph is a $B$-$(p,r)$-invex set with respect to $\eta$, $b_1$, $b_2$.*

*Proof.* There exists a vector function $\eta : S \times S \to R^n$, such that one of relations (6) holds. It is equivalent, by the definition of the epigraph, to one of the following relations

$$\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)}+(1-\lambda)\mathrm{e}^{pu}\right)^{\frac{1}{p}},\right.$$
$$\left.\log\left(b_1(x,u,\lambda)\,\mathrm{e}^{rf(x)}+b_2(x,u,\lambda)\,\mathrm{e}^{rf(u)}\right)^{\frac{1}{r}}\right)\in E(f),\text{if } p\neq 0, r\neq 0,$$

$$\left(u+\lambda\eta(x,u),\right.$$
$$\left.\log\left(b_1(x,u,\lambda)\,\mathrm{e}^{rf(x)}+b_2(x,u,\lambda)\,\mathrm{e}^{rf(u)}\right)^{\frac{1}{r}}\right)\in E(f),\text{if } p=0, r\neq 0,$$

$$\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)}+(1-\lambda)\mathrm{e}^{pu}\right)^{\frac{1}{p}},\right.$$
$$\left.b_1(x,u,\lambda)\,f(x)+b_2(x,u,\lambda)\,f(u)\right)\in E(f),\qquad \text{if } p\neq 0, r=0,$$

$$(u+\lambda\eta(x,u),b_1(x,u,\lambda)\,f(x)+b_2(x,u,\lambda)\,f(u))\in E(f),\quad \text{if } p=0, r=0,$$

by which we conclude that the epigraph of $f$ is a $B$-$(p,r)$-invex set with respect to $\eta$, $b_1$, $b_2$. □

## 4. Fundamental property of $B$-$(p,r)$-pre-invex functions

As it is known [11], a characteristic property of the class of pre-invex functions with respect to $\eta$ is the fact that each local minimum of any function belonging to this class is its global minimum. It turns out that this is also the case for the class of $B$-$(p,r)$-pre-invex functions with respect to $\eta$, $b_1$, $b_2$. Moreover, the set of points of a global minimum of a function of this type is $p$-invex with respect to $\eta$.

**Definition 9.** *We say that $u\in S\subset R^n$ is a local (strict local) minimum point of the function $f:S\to R$, if there exists a number $\varepsilon>0$ such that, the inequality $f(x)\geq f(u)$ $(f(x)>f(u))$ is satisfied for all points $x\in S\cap B(u;\varepsilon)$ $(x\in S\cap B(u;\varepsilon)\setminus\{u\})$.*

*We say that $u\in S\subset R^n$ is a global (strict global) minimum point of the function $f:S\to R$, if the inequality $f(x)\geq f(u)$ $(f(x)>f(u))$ holds for all points $x\in S$ $(x\in S,\ x\neq u)$.*

**Theorem 5.** *Let $f:X\to R$ be a $B$-$(p,r)$-pre-invex function with respect to $\eta$, $b_1$, $b_2$ on $X$, and, moreover, we assume that $\eta$ satisfies the following condition: $\eta(x,u)\neq 0$, if $x\neq u$. Then each point of a local minimum of the function $f$ is its point of global minimum. The set of points which are global minima of $f$ is a $p$-invex set with respect to $\eta$.*

*Proof.* The theorem will be proved only in the case when $p \neq 0$, $r \neq 0$ (other cases can be dealt with likewise the only changes arise from the form of inequalities defining the class of $B$-$(p,r)$-pre-invex functions with respect to $\eta$, $b_1$, $b_2$ for given $p$ and $r$).

Assume that $u \in X$ is a point of local minimum of $f$ which is not a point of global minimum. Hence, there exists a point $\bar{x} \in X$ such that $f(\bar{x}) < f(u)$. By assumption, $f$ is $B$-$(p,r)$-pre-invex with respect to $\eta$, $b_1$, $b_2$ on $X$. Thus by definition, for all $x, u \in X$ and any $\lambda \in [0,1]$ , the inequality

$$f\left(\log\left(\lambda \mathrm{e}^{p(\eta(x,u)+u)} + (1-\lambda)\mathrm{e}^{pu}\right)^{1/p}\right) \leq$$
$$\leq \log\left(b_1\left(x,u,\lambda\right)\mathrm{e}^{rf(x)} + b_2\left(x,u,\lambda\right)\mathrm{e}^{rf(u)}\right)^{1/r}$$

is true. In particular, the above inequality holds also in the case when $x = \bar{x}$ (with the left-hand side being transformed with the aid of the well-known theorems concerning logarithms):

$$f\left(u + \log\left(\lambda \mathrm{e}^{p\eta(\bar{x},u)} + (1-\lambda)\right)^{1/p}\right) \leq$$
$$\leq \log\left(b_1\left(x,u,\lambda\right)\mathrm{e}^{rf(\bar{x})} + b_2\left(x,u,\lambda\right)\mathrm{e}^{rf(u)}\right)^{1/r}.$$

Taking into account the fact that $f(\bar{x}) < f(u)$, we get

$$f\left(u + \log\left(\lambda \mathrm{e}^{p\eta(\bar{x},u)} + (1-\lambda)\right)^{1/p}\right) \leq$$
$$\leq \log\left(b_1\left(x,u,\lambda\right)\mathrm{e}^{rf(\bar{x})} + b_2\left(x,u,\lambda\right)\mathrm{e}^{rf(u)}\right)^{1/r} <$$
$$< \log\left(b_1\left(x,u,\lambda\right)\mathrm{e}^{rf(u)} + b_2\left(x,u,\lambda\right)\mathrm{e}^{rf(u)}\right)^{1/r} =$$
$$= \log\left(\mathrm{e}^{rf(u)}\right)^{1/r} = \frac{1}{r}rf(u) = f(u).$$

Thus we have we shown that, for any $0 \leq \lambda \leq 1$, the inequality

$$f\left(u + \log\left(\lambda \mathrm{e}^{p\eta(\bar{x},u)} + (1-\lambda)\right)^{1/p}\right) < f(u)$$

is satisfied, which is a contradiction to the assumption that $u$ is a local minimum point.

Denote by $A$ the set of points of global minimum of $f$ and let $x$ and $u$ be arbitrary points belonging to $A$. In order to prove that $A$ is a $p$-invex set with respect to $\eta$, we have to show that, for any $0 \leq \lambda \leq 1$, the relation

$\log\left(\lambda e^{p(\eta(x,u)+u)} + (1-\lambda)e^{pu}\right)^{1/p} \in A$ is true. Since $f$ is $B$-$(p,r)$-invex with respect to $\eta$, $b_1$, $b_2$ and $f(x) = f(u)$ (because $x, u \in A$ ), we have

$$f\left(\log\left(\lambda e^{p(\eta(x,u)+u)} + (1-\lambda)e^{pu}\right)^{1/p}\right) \leq$$
$$\leq \log\left(b_1(x,u,\lambda)\, e^{rf(x)} + b_2(x,u,\lambda)\, e^{rf(u)}\right)^{1/r} =$$
$$= \log\left(b_1(x,u,\lambda)\, e^{rf(x)} + b_2(x,u,\lambda)\, e^{rf(x)}\right)^{1/r} =$$
$$= \log\left(e^{rf(x)}\right)^{1/r} = \frac{1}{r} r f(x) = f(x).$$

We have shown that

$$f\left(\log\left(\lambda e^{p(\eta(x,u)+u)} + (1-\lambda)e^{pu}\right)^{1/p}\right) \leq f(x) = f(u), \qquad \forall \lambda \in [0,1].$$

Since $x$ and $u$ are points of global minimum of $f$, it follows that

$$\log\left(\lambda e^{p(\eta(x,u)+u)} + (1-\lambda)e^{pu}\right)^{1/p} \in A, \qquad \forall \lambda \in [0,1].$$

□

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