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ON THE GENERALIZATION OF THE APPROXIMATE CONTINUITY

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Abstract. This paper contains the concept of the generalization of the approximate continuity. The main result concerns that this continuity is equivalent to continuity with respect to some type density topology.

Let l be the standard Lebesgue measure in the real line R, and $\mathcal{L}$ be the σ–algebra of Lebesgue measurable sets of R. By N we shall denote the set of all positive integers and by S the family of all unbounded and nondecreasing sequences of positive numbers. We shall denote a sequence $\{s_n\}_{n\in N} \in S$ by $\langle s\rangle$. We shall also write $A \sim B$ if and only if $l(A \triangle B) = 0$ for measurable sets $A, B \subset R$ (where $A \triangle B = (A \setminus B) \cup (B \setminus A)$).

Definition 1. (See [3]). *We shall say that $x \in R$ is a density point of a set $A \in \mathcal{L}$ with respect to a sequence $\{s_n\}_{n\in N} \in S$ (in abr. $\langle s\rangle$-density point) if*

$$\lim_{n\to\infty} \frac{l(A \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

Let $\langle s\rangle \in S$ and $A \in \mathcal{L}$. Putting

$$\Phi_{\langle s\rangle}(A) = \{x \in R : x \ \text{ is } \langle \text{s}\rangle - \text{density point of A}\},$$

we have the following result.

Theorem 1. (See [3]). *For each $A, B \in \mathcal{L}$ and $\langle s\rangle \in S$*

(1) $\Phi_{\langle s\rangle}(\emptyset) = \emptyset$, $\Phi_{\langle s\rangle}(R) = R$,
(2) $\Phi_{\langle s\rangle}(A \cap B) = \Phi_{\langle s\rangle}(A) \cap \Phi_{\langle s\rangle}(B)$,
(3) *if* $A \subset B$ *then* $\Phi_{\langle s\rangle}(A) \subset \Phi_{\langle s\rangle}(B)$,
(4) *if* $A \sim B$ *then* $\Phi_{\langle s\rangle}(A) = \Phi_{\langle s\rangle}(B)$,
(5) $\Phi_{\langle s\rangle}(A) \sim A$.

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We get that for every $\langle s\rangle \in S$ the family $\mathcal{T}_{\langle s\rangle} = \{A \in \mathcal{L} : A \subset \phi_{\langle s\rangle}(A)\}$ forms a topology on the real line R (see [3]). Of course, if $\langle s\rangle = \{n\}_{n\in N}$ then $\mathcal{T}_{\langle s\rangle}$ is simply the clasical ordinary density topology which is denoted by $\mathcal{T}_d$ (see [2]) and then for every $A \in \mathcal{L}$

$$\Phi(A) = \{x \in R : x \text{ is a ordinary density point of the set } A\}.$$

We also recall the following result presented in [3].

Theorem 2. *Let $\langle s\rangle \in S$. Then*

$$\mathcal{T}_{\langle s\rangle} = \mathcal{T}_d \quad \textit{iff} \quad \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0.$$

Let $S_0 = \{\langle s\rangle \in S : \liminf\limits_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}$.

Definition 2. *Let $f : R \longrightarrow R$ and $\langle s\rangle \in S$. We shall say that f is $\langle s\rangle$–approximately continuous at point $x_0 \in R$ if there exsists a set $A_{x_0} \in \mathcal{L}$ such that*

$$x_0 \in \Phi_{\langle s\rangle}(A_{x_0}), \quad \text{and} \quad f(x_0) = \lim_{\substack{x\to x_0,\\ x\in A_{x_0}}} f(x). \tag{1}$$

Let us recall the concept of appproximate continuity.

Definition 3. *Let $f : R \longrightarrow R$ and $\langle s\rangle \in S$. We shall say that f is approximately continuous at point $x_0 \in R$ if there exsists a set $A_{x_0} \in \mathcal{L}$ such that*

$$x_0 \in \Phi(A_{x_0}), \quad \text{and} \quad f(x_0) = \lim_{\substack{x\to x_0,\\ x\in A_{x_0}}} f(x). \tag{2}$$

Our further results are well known for approximate continuity (see [6]). We do generalization for $\langle s\rangle$-approximate continuity. In fact, by Theorem 2 the results are evidently extension of the classical results if $\langle s\rangle \in S \setminus S_0$.

Theorem 3. *Let $f : R \longrightarrow R$ be a Lebesgue measurable function and let $\langle s\rangle \in S$. The function f is $\langle s\rangle$-approximately continuous at a point $x_0 \in R$ if and only if the following condition is satisfied*

$$\forall_{\varepsilon>0}\ x_0 \in \Phi_{\langle s\rangle}(\{x \in R : \mid f(x) - f(x_0) \mid < \varepsilon\}). \tag{3}$$

The following lemma will be useful to prove this theorem. The condition presented in this lemma is called the condition (J_2) of J. M. Jędrzejewski (see [5]).

Lemma 1. *Let $\langle s\rangle \in S$. For every decreasing sequence $\{E_n\}_{n\in N}$ of Lebesgue measurable sets and for every point $x_0 \in R$ such that for every $n \in N$ we have $x_0 \in \Phi_{\langle s\rangle}(E_n)$, there exists a strictly decreasing sequence $\{k_n\}_{n\in N}$ of* *positive numbers less then 1 and convergent to 0 such that* x_0 *is* $\langle s\rangle$*–density point of set* $A_{x_0} = \bigcup_{n=1}^{\infty}(E_n \setminus (x_0 - k_n, x_0 + k_n))$.

Proof. Let $\langle s\rangle \in S$ and let $\{E_n\}_{n\in N}$ be a decreasing sequence of Lebesgue measurable sets. Let us fix $x_0 \in R$ such that $x_0 \in \Phi_{\langle s\rangle}(E_n)$ for each $n \in N$. Let $\{\varepsilon_n\}_{n\in N}$ be a strictly decreasing sequence of positive numbers convergent to 0. Since for every $n \in N$ we have $x_0 \in \Phi_{\langle s\rangle}(E_n)$ then

$$\forall_{n\in N}\exists_{k(n)\in N}\forall_{k\geq k(n)}\frac{l(E_n \cap [x_0 - \frac{1}{s_k}, x_0 + \frac{1}{s_k}])}{\frac{2}{s_k}} > 1 - \varepsilon_n. \tag{4}$$

We can assume that sequence $\{k(n)\}_{n\in N}$ is increasing. Let $\{h_n\}_{n\in N}$ be a subsequence of the sequence $\{s_n\}_{n\in N}$ expressed in the form $h_n = s_{k(n)}$ for each $n \in N$. Putting $k_n = \varepsilon_n \cdot \frac{1}{h_{n+1}}$ for each $n \in N$, we obtain that $\{k_n\}_{n\in N}$ is strictly decreasing sequence and $\lim_{n\to\infty} k_n = 0$. Let $A_{x_0} = \bigcup_{n=1}^{\infty}(E_n \setminus (x_0 - k_n, x_0 + k_n))$ and fix $\varepsilon > 0$. There exists $n_0 \in N$ such that $1 - 2\varepsilon_n > 1 - \varepsilon$ for any $n > n_0$. Obviously, there exists $k_0 \in N$ such that $\frac{1}{s_k} < \frac{1}{h_{n_0+1}}$ for any $k > k_0$. Let $k > k_0$. There exsists $n_1 > n_0$ such that $\frac{1}{s_k} \in [\frac{1}{h_{n_1+1}}, \frac{1}{h_{n_1}}]$. This fact with condition (4) implies that

$$\begin{aligned}
&\frac{l(A_{x_0} \cap [x_0 - \frac{1}{s_k}, x_0 + \frac{1}{s_k}])}{\frac{2}{s_k}} \geq \\
\geq\ & \frac{l((E_{n_1} \cap [x_0 - \frac{1}{s_k}, x_0 + \frac{1}{s_k}]) \setminus (x_0 - k_{n_1}, x_0 + k_{n_1}))}{\frac{2}{s_k}} \geq \\
\geq\ & \frac{l(E_{n_1} \cap [x_0 - \frac{1}{s_k}, x_0 + \frac{1}{s_k}])}{\frac{2}{s_k}} - \frac{2k_{n_1}}{\frac{2}{s_k}} > \\
>\ & 1 - \varepsilon_{n_1} - \frac{\varepsilon_{n_1} \cdot \frac{1}{h_{n_1+1}}}{\frac{1}{s_k}} \geq 1 - \varepsilon_{n_1} - \varepsilon_{n_1} > 1 - \varepsilon.
\end{aligned}$$

Finally we have that $\lim_{k\to\infty} \frac{l(A_{x_0}\cap[x_0-\frac{1}{s_k},x_0+\frac{1}{s_k}])}{\frac{2}{s_k}} = 1$, which means that x_0 is $\langle s\rangle$-density point of set A_{x_0}. □

Proof of Theorem 3. Let $\langle s\rangle \in S$.

Necessity. Let $x_0 \in R$ and suppose that f is $\langle s\rangle$-approximately continuous at a point $x_0 \in R$. Thus there exists a set $A_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi_{<s>}(A_{x_0})$

and

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in A_{x_0}}\ x\in(x_0-\delta,x_0+\delta)\Rightarrow\mid f(x_0)-f(x)\mid<\varepsilon.$$

Hence

$$\forall_{\varepsilon>0}\exists_{\delta>0}\{x\in R:\mid f(x_0)-f(x)\mid<\varepsilon\}\supset A_{x_0}\cap(x_0-\delta,x_0+\delta).$$

Since

$$x_0\in\Phi_{\langle s\rangle}(A_{x_0}\cap(x_0-\delta,x_0+\delta))\quad\text{and}\quad\{x\in R:\mid f(x_0)-f(x)\mid<\varepsilon\}\in\mathcal{L}$$

for each $\varepsilon>0$ then

$$\forall_{\varepsilon>0}\ x_0\in\Phi_{\langle s\rangle}(\{x\in R:\mid f(x_0)-f(x)\mid<\varepsilon\}).$$

Sufficiency. We assume that condition (3) holds and put for every $n\in N$

$$E_n=\{x\in R:\mid f(x_0)-f(x)\mid<\frac{1}{n}\}.$$

Of course, $\{E_n\}_{n\in N}$ is a decreasing sequence of Lebesgue measurable sets and x_0 is $\langle s\rangle$-density point of set E_n for every $n\in N$. Lemma 1 implies that there exists a strictly decreasing sequence $\{k_n\}_{n\in N}$ of positive numbers convergent to 0 such that x_0 is $\langle s\rangle$-density point of a set $A_{x_0}=\bigcup_{n=1}^{\infty}(E_n\setminus(x_0-k_n,x_0+k_n))$. Obviously, $A_{x_0}\in\mathcal{L}$ and

$$f(x_0)=\lim_{\substack{x\to x_0,\\ x\in A_{x_0}}}f(x).$$

Indeed, fix $\varepsilon>0$. There exists $n_0\in N$ such that $\frac{1}{n}<\varepsilon$ for each $n>n_0$. Let $\delta=\frac{1}{k_{n_0}}$. Since $\{k_n\}_{n\in N}$ is decreasing so, if $x\in A_{x_0}\cap(x_0-k_{n_0},x_0+k_{n_0})$ then $x\in\bigcup_{n=n_0+1}^{\infty}(E_n\setminus(x_0-k_n,x_0+k_n))$. Consequently, if $x\in A_{x_0}\cap(x_0-\delta,x_0+\delta)$ then there exists $n_1>n_0$ such that $x\in E_{n_1}$, which means that $\mid f(x)-f(x_0)\mid<\frac{1}{n_1}<\varepsilon$. And we have that $f(x_0)=\lim_{\substack{x\to x_0,\\ x\in A_{x_0}}}f(x)$. $\square$

Definition 4. *Let $f:R\longrightarrow R$ and $\langle s\rangle\in S$. We shall say that f is $\langle s\rangle$-approximately continuous if f is $\langle s\rangle$-approximately continuous at every point $x\in R$.*

Theorem 4. *A function $f:R\longrightarrow R$ is Lebesgue measurable if and only if there exists a sequence $\langle s\rangle\in S$ such that f is $\langle s\rangle$-approximately continuous a.e.*

Proof. Necessity. Let f be a Lebesgue measurable function. Then f is approximately continuous a.e. (see [1]). Hence it follows that f is $\langle s\rangle$-approximately continuous a.e. for every $\langle s\rangle\in S$.

Sufficiency. We suppose that there exists $\langle s\rangle \in S$ such that f is $\langle s\rangle$-approximately continuous a.e. Let $\langle s\rangle \in S$ and $\alpha \in R$. Put $E = \{x \in R: \ f(x) < \alpha\}$. Let F be a set of all points $x \in R$ such that f is $\langle s\rangle$-approximately continuous at point x. Then we have $E = (E \cap F) \cup (E \setminus F)$ and $l(E \setminus F) = 0$. We shall show that $F \cap E$ belongs to $\mathcal{L}$. Firstly we shall show that the following condition is fulfilled:

$$\forall_{x \in E \cap F} \exists_{E_x \in \mathcal{L}} (E_x \subset E \cap F \wedge x \in \Phi_{\langle s\rangle}(E_x)). \tag{5}$$

Indeed, let $x_0 \in E \cap F$. Then we have $f(x_0) < \alpha$ and there exists a set A_{x_0} such that

$$x_0 \in \Phi_{\langle s\rangle}(A_{x_0}) \quad \text{and} \quad f(x_0) = \lim_{\substack{x \to x_0,\\ x \in A_{x_0}}} f(x).$$

We can also assume that $x_0 \in A_{x_0}$. Consequently, there exists $\delta > 0$ such that $f(x) < \alpha$ for each $x \in A_{x_0} \cap (x_0 - \delta, x_0 + \delta)$. Let $E_{x_0} = A_{x_0} \cap (x_0 - \delta, x_0 + \delta) \cap F$. Obviously $E_{x_0} \in \mathcal{L}$ and $E_{x_0} \subset E \cap F$. Moreover $x_0 \in \Phi_{\langle s\rangle}(E_{x_0})$. Indeed, $x_0 \in \Phi_{\langle s\rangle}(A_{x_0} \cap (x_0 - \delta, x_0 + \delta))$ and

$$\lim_{n \to \infty} \frac{l\left(F \cap \left[x - \frac{1}{s_n}, x + \frac{1}{s_n}\right]\right)}{\frac{2}{s_n}} = \lim_{n \to \infty} \frac{l\left(R \cap \left[x - \frac{1}{s_n}, x + \frac{1}{s_n}\right]\right)}{\frac{2}{s_n}} = 1,$$

so $x_0 \in \Phi_{\langle s\rangle}(F)$. From the above and Theorem 1 we have that $x_0 \in \Phi_{\langle s\rangle}(E_{x_0})$. And condition (5) is satisfied.

Let B be a Lebesgue measurable kernel of the set $E \cap F$. For every $x \in E \cap F$, by condition (5) we have that $l(E_x \setminus B) = 0$ and $E_x \subset (E_x \setminus B) \cup B \sim B$. From the above and Theorem 1 we have that $\Phi_{\langle s\rangle}(E_x) \subset \Phi_{\langle s\rangle}(B)$ for every $x \in E \cap F$. Moreover, condition (5) implies that $E \cap F \subset \bigcup_{x \in E \cap F} \Phi_{\langle s\rangle}(E_x)$. Consequently,

$$B \subset E \cap F \subset \bigcup_{x \in E \cap F} \Phi_{\langle s\rangle}(E_x) \subset \Phi_{\langle s\rangle}(B).$$

Simultaneously $l(\Phi_{\langle s\rangle}(B) \setminus B) = 0$. Thus $E \cap F \in \mathcal{L}$ and consequently E is a Lebesgue measurable set. □

Corollary 1. *A function $f : R \longrightarrow R$ is approximately continuous a.e. if and only if there exists a sequence $\langle s\rangle \in S$ such that f is $\langle s\rangle$-approximately continuous a.e.*

Theorem 5. *Let $f : R \longrightarrow R$ and $\langle s\rangle \in S$. A function f is $\langle s\rangle$-approximately continuous if and only if the sets $E_\alpha = \{x \in R: \ f(x) < \alpha\}$ and $E^\alpha = \{x \in R: \ f(x) > \alpha\}$ belong to topology $\mathcal{T}_{\langle s\rangle}$ for every $\alpha \in R$.*

Proof. Let $f : R \longrightarrow R$ and $\langle s\rangle \in S$.

Necessity. We have, by Theorem 4, that $E^\alpha,\ E_\alpha \in \mathcal{L}$ for each $\alpha \in R$. Let us fix $\alpha \in R$ and let $x_0 \in E_\alpha$. Then, by Definition 4, there exists $A_{x_0} \in \mathcal{L}$ such that

$$x_0 \in \Phi_{\langle s\rangle}(A_{x_0}), \quad \text{and} \quad f(x_0) = \lim_{\substack{x\to x_0,\\ x\in A_{x_0}}} f(x).$$

Since $f(x_0) < \alpha$ then there exists $\delta > 0$ such that $A_{x_0} \cap (x_0-\delta, x_0+\delta) \subset E_\alpha$. Consequently,

$$x_0 \in \Phi_{\langle s\rangle}(A_{x_0} \cap (x_0-\delta, x_0+\delta)) \subset \Phi_{\langle s\rangle}(E_\alpha).$$

And we obtain that $E_\alpha \subset \Phi_{\langle s\rangle}(E_\alpha)$, which implies that $E_\alpha \in \mathcal{T}_{\langle s\rangle}$ for every $\alpha \in R$.
Similary we can show that $E^\alpha \in \mathcal{T}_{\langle s\rangle}$

Sufficiency. Let $x_0 \in R$, fix $\varepsilon > 0$ and suppose that sets E_α and E^α belong to topology $\mathcal{T}_{\langle s\rangle}$ for every $\alpha \in R$. Consequently, f is Lebesgue measurable function and

$$\{x \in R : \ f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon\} \in \mathcal{T}_{\langle s\rangle}.$$

Hence, it follows that x_0 is $\langle s\rangle$-density point of the set $\{x \in R : \mid f(x) - f(x_0) \mid < \varepsilon\}$. By Theorem 3 we get, that the function f is $\langle s\rangle$-approximately continuous at point x_0. It follows that f is $\langle s\rangle$-approximately continuous. $\square$

We have the below theorem

Theorem 6. *Let $f : R \longrightarrow R$ and $\langle s\rangle \in S$. A function f is $\langle s\rangle$-approximately continuous if and only if f is continuous with respect to topology $\mathcal{T}_{\langle s\rangle}$ (in abr. $\langle s\rangle$-continuous function).*

Let $\langle s\rangle \in S$. By $C(\mathcal{T}_{\langle s\rangle})$ we shall denote the family of $\langle s\rangle$-continuous function. In the paper [4] the following result is presented

Theorem 7. *Let $\langle s\rangle, \langle t\rangle \in S$. Then $C(\mathcal{T}_{\langle s\rangle}) = C(\mathcal{T}_{\langle t\rangle})$ if and only if $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle}$.*

From the above and Theorems 2 and 6 we have that for every sequence $\langle s\rangle \in S_0$ there exists a function f and a point $x_0 \in R$ such that f is $\langle s\rangle$-approximately continuous at point x_0 and is not approximately continuous at point x_0.

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