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TOPOLOGIES OF HASHIMOTO TYPE WITH RESPECT TO SIGMA-IDEAL OF COUNTABLE SETS

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Abstract. The paper deals with properties of and interrelations between three topologies on the real line finer than the natural topology. Also classes of real functions of a real variable, which are continuous, when the domain and the range are equipped with one of these topologies, are examined. Topologies are similar to that studied in [3] and some properties of classes of continuous functions resemble that in [1].

In the sequel $\mathcal{T}$ will stand for a natural topology on the real line $\mathbf{R}$, J_0-for the σ-ideal of countable subsets of $\mathbf{R}$, and $\mathbf{Q}$-for the set of rational numbers.

Proposition 1. (See [3]). *The family of sets*

$$\mathcal{T}_1 = \left\{G \subset \mathbf{R} : \exists_{G_o \in \mathcal{T}} \exists_{I \in J_0} G = G_0 \setminus I\right\}$$

is a topology finer than the natural topology.

Let $\{A_s\}_{s\in S} = \mathbf{R}/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y$ iff $x - y \in \mathbf{Q}$. Recall that $\mathrm{card}(S) = $ continuum.

Proposition 2. *The family of sets*

$$\mathcal{B} = \{B \subset \mathbf{R} : B = (a,b) \setminus \bigcup_{n=1}^{\infty} A_{s_n} \text{ for some } a,b \in \mathbf{R} \text{ and some sequence } \{s_n\}_{n\in N} \text{ of elements of } S\}$$

is a basis for a topology on the real line.

Proposition 3. *The family of sets*

$$\mathcal{C} = \{C \subset \mathbf{R} : C = (a,b) \setminus \bigcup_{i=1}^{n} A_{s_i} \text{ for some } a,b \in \mathbf{R} \text{ and some finite sequence } \{s_i\}_1^n \text{ of elements of } S\}$$

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is a basis for a topology on the real line.

The proofs of Propositions 2 and 3 are straightforward. Observe that for each $B_1, B_2 \in \mathcal{B}$ ($C_1, C_2 \in \mathcal{C}$, respectively) if $B_1 \cap B_2 \neq \emptyset$ ($C_1 \cap C_2 \neq \emptyset$, respectively), then $B_1 \cap B_2 \in \mathcal{B}$ ($C_1 \cap C_2 \in \mathcal{C}$, resp.). For properties of a basis see, for example [2].

In the sequel we shall denote by $\mathcal{T}_2$ ($\mathcal{T}_3$, resp.) a topology generated by $\mathcal{B}$ ($\mathcal{C}$, resp.). Obviously $\mathcal{T}_2$ and $\mathcal{T}_3$ are finer than a natural topology.

Proposition 4.

$$\mathcal{T} \subset_{\neq} \mathcal{T}_3 \subset_{\neq} \mathcal{T}_2 \subset_{\neq} \mathcal{T}_1.$$

Proof. The inclusions are obvious. To prove the inequalities observe, that if

$$G = (-1, 1) \setminus [\mathbf{Q} \setminus \{0\}],$$

then $G \in \mathcal{T}_1 \setminus \mathcal{T}_2$. Indeed, $G \in \mathcal{T}_1$ by definition. Suppose now, that $G \in \mathcal{T}_2$. Then G is an union of elements of $\mathcal{B}$, $G = \bigcup_{t \in T} B_t$ for some T, where $B_t \in \mathcal{B}$ for $t \in T$. Consequently, there exists $t_0 \in T$ such that $0 \in B_{t_0}$. Since $B_{t_0} = (a_0, b_0) \setminus \bigcup_{i=1}^{\infty} A_{s_i}$ for some $a_0, b_0 \in \mathbf{R}$ and $\{s_i\}_{i \in \mathbf{N}}$ from S, then B_{t_0} contains all rational numbers from the interval (a_0, b_0). But this contradicts the inclusion $B_{t_0} \subset G$ and G does not contain any rational number different from 0 .

Suppose now that $B = (0, 1) \setminus \bigcup_{i=1}^{\infty} A_{s_i}$, where $s_i \neq s_j$ for $i \neq j$. It is not difficult to see that $B \in \mathcal{T}_2 \setminus \mathcal{T}_3$. □

Proposition 5. T_i *is Hausdorff and not regular for* $i \in \{1, 2, 3\}$.

Proof. All considered topologies are Hausdorff, because they are finer than the natural topology. To prove the second assertion take $F = [1, 2] \cap \mathbf{Q}$ and $x_0 = \sqrt{2}$. Obviously F is closed in $\mathcal{T}_i$ (for $i \in \{1, 2, 3\}$).

Suppose that there exist $U_1, U_2 \in \mathcal{T}_1$ such that $F \subset U_1$, $\sqrt{2} \in U_2$, $U_1 \cap U_2 = \emptyset$. Hence $U_2 = G_2 - I_2$, where G_2 is open in the natural topology and I_2 is a countable set. Let $q \in [1, 2] \cap \mathbf{Q} \cap G_2$. Then there exists G_1-open in the natural topology and a countable set I_1 such that $q \in G_1 - I_1 \subset U_1$. Therefore $q \in (G_2 \cap G_1)$, so $(G_2 \cap G_1)$-nonempty open set. Then

$$(G_2 - I_2) \cap (G_1 - I_1) = (G_2 \cap G_1) \setminus (I_2 \cup I_1) \neq \emptyset,$$

so $U_1 \cap U_2 \neq \emptyset$ is a contradiction.

Also F and x_0 cannot be separated in coarser topologies $\mathcal{T}_2$ and $\mathcal{T}_3$. □

Let $C(\mathcal{T}, \mathcal{T}_i)$ be a class of all continuous functions $f : (\mathbf{R}, \mathcal{T}) \to (\mathbf{R}, \mathcal{T}_i)$ for $i \in \{1, 2, 3\}$.

Proposition 6. $C(\mathcal{T}, \mathcal{T}_3)$ *is a class of all constant functions.*

Proof. Suppose that there exists a function $f \in C(\mathcal{T}, \mathcal{T}_3)$, which is not constant. So $f(x_1) < f(x_2)$ for some $x_1 \neq x_2$. For instance suppose that $x_1 < x_2$. Let G be the set of all irrational numbers. Then $G \in \mathcal{T}_3$ and $f^{-1}(G) \in \mathcal{T}$. Hence $f^{-1}(G) = \bigcup_n (a_n, b_n)$, where the union is at most countable. Since $\mathcal{T} \subset \mathcal{T}_3$, $f \in C(\mathcal{T}, \mathcal{T})$. If f were constant on each (a_n, b_n), then $f(f^{-1}(G))$ would be at most countable, but it is impossible, because $f(f^{-1}(G)) \supset [f(x_1), f(x_2)] \setminus \mathbf{Q}$. Then there exists n_0 such that $f((a_{n_0}, b_{n_0}))$ is a non-degenerate interval, which is impossible since $f((a_{n_0}, b_{n_0})) \subset f(f^{-1}(G)) \subset G$. This contradiction ends the proof. □

Corollary 1. *$C(\mathcal{T}, \mathcal{T}_1)$ and $C(\mathcal{T}, \mathcal{T}_2)$ consist only of constant functions.*

Proof. The proof is obvious by virtue of Proposition 4. □

From Theorem 34.1 in [4], p. 78 it follows immediately, that

Proposition 7. *$C(\mathcal{T}_1, \mathcal{T}) = C(\mathcal{T}, \mathcal{T})$.*

Corollary 2. *$C(\mathcal{T}_2, \mathcal{T}) = C(\mathcal{T}_3, \mathcal{T}) = C(\mathcal{T}, \mathcal{T})$.*

Proposition 8. *The topological spaces $(\mathbf{R}, \mathcal{T}_1)$ and $(\mathbf{R}, \mathcal{T}_2)$ are not homeomorphic.*

Proof. Suppose that $f : (\mathbf{R}, \mathcal{T}_2) \to (\mathbf{R}, \mathcal{T}_1)$ is a homeomorphism. Then f is bijective and ordinary continuous function, by virtue of Corollary 2, since $f \in C(\mathcal{T}_2, \mathcal{T})$. Take $G = (-1, 1) \setminus [\mathbf{Q} \setminus \{0\}]$. Then $G \notin \mathcal{T}_2$, by virtue of proof of Proposition 4. But $f(G) = f((-1, 1)) \setminus f([\mathbf{Q} \setminus \{0\}])$ is a difference of a non-degenerate open interval and a countable set, so $f(G) \in \mathcal{T}_1$-a contradiction. □

Proposition 9. *The spaces $(\mathbf{R}, \mathcal{T}_2)$ and $(\mathbf{R}, \mathcal{T}_3)$ are not homeomorphic. Moreover, the class $C(\mathcal{T}_3, \mathcal{T}_2)$ is equal to the class of all constant functions.*

Proof. Let $f \in C(\mathcal{T}_3, \mathcal{T}_2)$. Let $\{s_1, s_2, \dots\}$ be a sequence of different elements of a set S, described in the connection with $\mathbf{R}/\sim$. Observe, that:

(1) $\bigcup_{i=1}^{\infty} A_{s_i}$ is not closed in $\mathcal{T}_3$, moreover, the closure of this set (in $\mathcal{T}_3$) is equal to $\mathbf{R}$.

(2) The set $F = \bigcup_{i=1}^{\infty} f(A_{s_i})$ is at most countable. Then $F^* = \bigcup_{x \in F} [x]_\sim$ is closed in $\mathcal{T}_2$. From the facts that $\bigcup_{i=1}^{\infty} A_{s_i} \subset f^{-1}(F^*)$, and $f^{-1}(F^*)$ is closed in $\mathcal{T}_3$, and (1) we have $f^{-1}(F^*) = \mathbf{R}$.

To sum up: $f \in C(\mathcal{T}, \mathcal{T})$, and $f(\mathbf{R})$ is countable. Then f is constant. □

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