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NET-DEFINABILITY OF COMPUTABLE FUNCTIONS

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Abstract. This paper provides two definitions: definition of neural network, and definition of net–definable function. Comparing to usual meaning of neural network, this definition concerns to a topological structure of the net. A construction of net–definable funcitons and a proof (Theorem 10) that this construction leads to the Turing–machine is a main result of the paper. Most of concepts of this paper have been taken from A. Church (see [1]).

For any set A we shall use the notation A^* for a set of finite sequences of elements of A and λ for an empty sequence ($= \emptyset$). We assume that there is given a set $\mathrm{AL} = \{ a_n; n \in \mathbb{N} \}$ called *alphabet* satisfing following conditions: a) $a_i \neq a_j$ for $i \neq j$, b) if a belongs to AL then $(a, \emptyset)$ does not belong to AL, c) if a and b belong to AL then (a, b) does not belong to AL. Function $\mathbb{N} \ni n \mapsto a_n$ will be called to be an *ordering of alphabet*. For two finite sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^k$ we denote finite sequence $(a_i)_{i=1}^n \bullet (b_i)_{i=1}^k = (c_i)_{i=1}^{n+k}$ by

$$c_i = \begin{cases} a_i, & \text{if } i \in \{ 1, \dots, n \}, \\ b_{i-n}, & \text{if } i \in \{ n+1, \dots, n+k \}. \end{cases}$$

By *number* we always mean a natural number (element of the set $\{ 1, 2, \dots \}$).

Definition 1. *Let* $\mathcal{NN}_1 = \{ (a, \emptyset); a \in \mathrm{AL} \}$ *and for* $k \in \{ 2, 3, \dots \}$ *let*

$\mathcal{NN}_k = \mathcal{NN}_{k-1} \cup \{ (S, a); S \in \mathcal{NN}_{k-1}, a \in \mathrm{AL} \} \cup \{ (S, T); S, T \in \mathcal{NN}_{k-1} \}$.

We define

$$\mathcal{NN} = \bigcup_{k=1}^{\infty} \mathcal{NN}_k.$$

Any S is said to be a neural network, or simply a net, if $S \in \mathcal{NN}$.

Proposition 1. *Let $Z \subset SN$ be a set satisfing following statements:*

1° $\forall_{a \in \mathrm{AL}} \big((a, \emptyset) \in Z \big)$,

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2° $\forall_{a \in \mathrm{AL}} \forall_{S,T \in Z}\, ((S,a) \in Z, (S,T) \in Z)$.

Then $Z = \mathcal{N}N$. □

Definition 2. *Let us define five mappings* V, B', B, F, C *of* $\mathcal{N}N$ *into* 2^{AL} *as follows:*

$$\begin{aligned}
V((a,\emptyset)) &= \{a\}, \quad \textit{for} \quad a \in \mathrm{AL},\\
V((S,a)) &= V(S), \quad \textit{for} \quad S \in \mathcal{N}N, a \in \mathrm{AL},\\
V((S_1,S_2)) &= V(S_1) \cup V(S_2), \quad \textit{for} \quad S_1, S_2 \in \mathcal{N}N.
\end{aligned}$$

$$\begin{aligned}
B'((a,\emptyset)) &= \emptyset, \quad \textit{for} \quad a \in \mathrm{AL},\\
B'((S,a)) &= B'(S) \cup \{a\}, \quad \textit{for} \quad S \in \mathcal{N}N, a \in \mathrm{AL},\\
B'((S_1,S_2)) &= B'(S_1) \cup B'(S_2), \quad \textit{for} \quad S_1, S_2 \in \mathcal{N}N.
\end{aligned}$$

$$B(S) = B'(S) \cap V(S).$$

$$\begin{aligned}
F((a,\emptyset)) &= \{a\}, \quad \textit{for} \quad a \in \mathrm{AL},\\
F((S,a)) &= F(S) \setminus \{a\}, \quad \textit{for} \quad S \in \mathcal{N}N, a \in \mathrm{AL},\\
F((S_1,S_2)) &= F(S_1) \cup F(S_2), \quad \textit{for} \quad S_1, S_2 \in \mathcal{N}N.
\end{aligned}$$

$$\begin{aligned}
C((a,\emptyset)) &= \{a\} \quad \textit{for} \quad a \in \mathrm{AL},\\
C((S,a)) &= \{a\} \cup C(S) \quad \textit{for} \quad S \in \mathcal{N}N, a \in \mathrm{AL},\\
C((S,S')) &= C(S) \cup C(S') \quad \textit{for} \quad S, S' \in \mathcal{N}N.
\end{aligned}$$

Definition 3. *Let* $S \in \mathcal{N}N$, $x, y \in \mathrm{AL}$. *Let us define* S^x_y *by*

$$\begin{aligned}
(a,\emptyset)^x_y &= \begin{cases} (x,\emptyset), & \textit{if } a = y,\\ (a,\emptyset), & \textit{else.}\end{cases}\\
(T,a)^x_y &= \begin{cases} (S^x_y, x), & \textit{if } a = y,\\ (S^x_y, a), & \textit{else.}\end{cases}\\
(T_1,T_2)^x_y &= ({T_1}^x_y, {T_2}^x_y).
\end{aligned}$$

Clearly, for any $S \in \mathcal{N}N$ *and any* $x, y \in \mathrm{AL}$ *we have* $S^x_y \in \mathcal{N}N$.

Proposition 2. *For any* $x, y \in \mathrm{AL}$, $S \in \mathcal{N}N$ *we have that* y *does not belong to* $C(S^x_y)$. □

Proposition 3. *For any* $S \in \mathcal{N}N$, $x \in \mathrm{AL}$ *and* $y \in \mathrm{AL} \setminus C(S)$ *we have* $(S^y_x)^x_y = S$.

Proof. i) let $x, y, a \in \mathrm{AL}$ and $y \notin C((a, \emptyset))$. Then

$$\begin{aligned} ((a,\emptyset)^y_x)^x_y &= \begin{cases} (y,\emptyset)^x_y, & \text{if } a = x, \\ (a,\emptyset)^x_y, & \text{else.} \end{cases} \\ &= \begin{cases} (x,\emptyset), & \text{if a = x,} \\ (a,\emptyset), & \text{else.} \end{cases} \\ &= (a,\emptyset). \end{aligned}$$

ii) suppose that some $S \in \mathcal{N}N$ satisfies the thesis. Let $x, y, a \in \mathrm{AL}$ and $y \notin C((S, a))$. We have $y \notin C(S), y \neq a$, so

$$\begin{aligned} ((S,a)^y_x)^x_y &= \begin{cases} (S^y_x, y)^x_y, & \text{if } a = x, \\ (S^y_x, a)^x_y, & \text{else.} \end{cases} \\ &= \begin{cases} (S,x), & \text{if } a = x, \\ (S,a), & \text{else.} \end{cases} \\ &= (S,a). \end{aligned}$$

iii) suppose that thesis is satisfied for some $S_1, S_2 \in \mathcal{N}N$. Let $y \in \mathrm{AL}$ such that $y \notin C((S_1, S_2))$. Then $y \notin C(S_1), y \notin C(S_2)$ and

$$((S_1,S_2)^y_x)^x_y = ({S_1}^y_x, {S_2}^y_x)^x_y = (({S_1}^y_x)^x_y, ({S_2}^y_x)^x_y) = (S_1, S_2).$$

By virtue of Proposition 1 the proof is complete. □

Definition 4. *Let Sub be a mapping from* : $\mathcal{N}N \times \mathrm{AL} \times \mathrm{AL}$ *into* $\mathcal{N}N$ *defined by*

$$Sub(S,x,y) = \begin{cases} S^x_y, & \textit{if } x \notin C(S), y \notin F(S); \\ S, & \textit{else.} \end{cases}$$

Definition 5. *Let* $\longrightarrow^{\mathrm{I}}$ *be a relation on* $\mathcal{N}N \times \mathcal{N}N$ *by*

$$\begin{aligned} S \stackrel{\mathrm{I}}{\longrightarrow} T \iff \Big\{ & \Big[\exists_{a,b\in\mathrm{AL}}\big(Sub(S,a,b) = T\big)\Big] \\ \vee\ & \Big[\exists_{S',T'\in\mathcal{N}N}\exists_{a\in\mathrm{AL}}\big(S' \stackrel{\mathrm{I}}{\longrightarrow} T', S = (S',a), T = (T',a)\big)\Big] \\ \vee\ & \Big[\exists_{S',T',S'',T''\in\mathcal{N}N}\big(S' \stackrel{\mathrm{I}}{\longrightarrow} T', S = (S',S''), T = (T',T'')\big)\Big] \\ \vee\ & \Big[\exists_{S',T',S'',T''\in\mathcal{N}N}\big(S'' \stackrel{\mathrm{I}}{\longrightarrow} T'', S = (S',S''), T = (T',T'')\big)\Big] \Big\}. \end{aligned}$$

Proposition 4. *For all* $x, y \in \mathrm{AL}$ *and* $S \in \mathcal{N}N$, *if* $y \notin F(S), x \notin C(S)$ *then* $x \notin F(S^x_y)$. □

Proposition 5. *The relation* $\longrightarrow^{\mathrm{I}}$ *is reflexive and symmetric.*

Proof. Let $S \in \mathcal{N}N$ and $a \in \mathrm{AL}$. Since for any $a \in \mathrm{AL}$ is $S^a_a = S$, we have $\mathrm{Sub}(S, a, a) = S$. To show symmetry suppose that $S, S' \in \mathcal{N}N$ and $\mathrm{Sub}(S, x, y) = S'$ for $x, y \in \mathrm{AL}$. If $S = S'$ then the thesis is satisfied; also suppose that $S \neq S'$. Therefore $x \notin C(S)$ and $y \notin F(S)$. From Propositions 2 and 4, we have that

$$x \notin F(S^x_y), \quad \text{and} \quad y \notin C(S^x_y).$$

By Proposition 3 we have $(S^x_y)^y_x = S$, and therefore

$$\mathrm{Sub}(S', y, x) = \mathrm{Sub}(S^x_y, y, x) = S.$$

□

Definition 6. *Let* $\longrightarrow^{\mathrm{convI}}$ *be a relation on* $\mathcal{N}N \times \mathcal{N}N$ *by*

$$S_1 \overset{\mathrm{convI}}{\longrightarrow} S_2 \iff \Big(\big(S_1 \overset{\mathrm{I}}{\longrightarrow} S_2\big) \vee \exists_{k \in \mathbb{N}} \exists_{T_1, \ldots, T_k \in \mathcal{N}N} \big(S_1 \overset{\mathrm{I}}{\longrightarrow} T_1, T_k \overset{\mathrm{I}}{\longrightarrow} S_2, \\ , \forall_{n \in \{1, \ldots, k-1\}} (T_n \overset{\mathrm{I}}{\longrightarrow} T_{n+1})\big)\Big).$$

Proposition 6. *Relation* $\longrightarrow^{\mathrm{convI}}$ *is an equivalence relation.* □

Definition 7. *Let* $S, T \in \mathcal{N}N$, $x \in \mathrm{AL}$*. We shall define* S^T_x *as follows:*

$$\begin{aligned}
(a, \emptyset)^T_x &= \begin{cases} T, & \text{if } a = x; \\ (a, \emptyset), & \text{else.} \end{cases} \\
(S, a)^T_x &= \begin{cases} (S^T_x, a), & \text{if } a \neq x; \\ S^T_x, & \text{else.} \end{cases} \\
(S_1, S_2)^T_x &= (S_1{}^T_x, S_2{}^T_x).
\end{aligned}$$

Clearly, for any $S, T \in \mathcal{N}N$ *and any* $x \in \mathrm{AL}$ *we have* $S^T_x \in \mathcal{N}N$.

Definition 8. *Let us define a relation* $\longrightarrow^{\mathrm{II}}$ *on* $\mathcal{N}N \times \mathcal{N}N$ *by*

$$\begin{aligned}
S \overset{\mathrm{II}}{\longrightarrow} T \quad \iff \Big\{ & \Big[\exists_{S', T' \in \mathcal{N}N} \exists_{a \in \mathrm{AL}} \big(S = ((S', a), T'), T = S'^{T'}_a \\
& , a \notin B(S'), B(S') \cap F(T') = \emptyset\big)\Big] \\
& \vee \Big[\exists_{S', T' \in \mathcal{N}N} \exists_{a \in \mathrm{AL}} \big(S' \overset{\mathrm{II}}{\longrightarrow} T', S = (S', a), T = (T', a)\big)\Big] \\
& \vee \Big[\exists_{S', S'', T', T'' \in \mathcal{N}N} \big(S' \overset{\mathrm{II}}{\longrightarrow} T', S = (S', S''), T = (T', T'')\big)\Big] \\
& \vee \Big[\exists_{S', S'', T', T'' \in \mathcal{N}N} \big(S'' \overset{\mathrm{II}}{\longrightarrow} T'', S = (S', S''), T = (T', T'')\big)\Big] \Big\}.
\end{aligned}$$

Definition 9. *Let* $\longrightarrow^{\text{convII}}$ *be a relation on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \overset{\text{convII}}{\longrightarrow} S_2 \iff \Big[\big(S_1 \overset{\text{II}}{\longrightarrow} S_2\big) \vee \exists_{k\in\mathbb{N}}\exists_{T_1,\dots,T_k\in\mathcal{N}\mathcal{N}}\big(S_1 \overset{\text{II}}{\longrightarrow} T_1 , T_k \overset{\text{II}}{\longrightarrow} S_2, \forall_{n\in\{1,\dots,k-1\}}(T_n \overset{\text{II}}{\longrightarrow} T_{n+1})\big)\Big].$$

Definition 10. *Let us define a relation* $\longrightarrow^{\text{III}}$ *on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \overset{\text{III}}{\longrightarrow} S_2 \iff S_2 \overset{\text{II}}{\longrightarrow} S_1.$$

Definition 11. *Let us define a relation* $\longrightarrow^{\text{convIII}}$ *on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \overset{\text{convIII}}{\longrightarrow} S_2 \iff S_2 \overset{\text{convII}}{\longrightarrow} S_1.$$

Definition 12. *Let* $\longrightarrow^{\text{I,II,III}}$ *be a relation on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *defined as follows:*

$$S_1 \overset{\text{I,II,III}}{\longrightarrow} S_2 \iff \big(S_1 \overset{\text{I}}{\longrightarrow} S_2 \vee S_1 \overset{\text{II}}{\longrightarrow} S_2 \vee S_1 \overset{\text{III}}{\longrightarrow} S_2\big).$$

Definition 13. *Let us define a relation* $\longrightarrow$ *on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \longrightarrow S_2 \iff \Big[\big(S_1 \overset{\text{I,II,III}}{\longrightarrow} S_2\big) \vee \exists_{k\in\mathbb{N}}\exists_{T_1,\dots,T_k\in\mathcal{N}\mathcal{N}}\big(S_1 \overset{\text{I,II,III}}{\longrightarrow} T_1 , T_k \overset{\text{I,II,III}}{\longrightarrow} S_2, \forall_{n\in\{1,\dots,k-1\}}(T_n \overset{\text{I,II,III}}{\longrightarrow} T_{n+1})\big)\Big].$$

Proposition 7. *The relation* $\longrightarrow$ *is an equivalence relation.* □

Definition 14. *Let us define a relation* $\longrightarrow^{\text{convI,II}}$ *on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \overset{\text{convI,II}}{\longrightarrow} S_2 \iff \Big[\big(S_1 \overset{\text{convI}}{\longrightarrow} S_2 \vee S_1 \overset{\text{convII}}{\longrightarrow} S_2\big) \vee\exists_{k\in\mathbb{N}}\exists_{T_1,\dots,T_k\in\mathcal{N}\mathcal{N}}\big((S_1 \overset{\text{convI}}{\longrightarrow} T_1 \vee S_1 \overset{\text{convII}}{\longrightarrow} T_1) , (T_k \overset{\text{convI}}{\longrightarrow} S_2 \vee T_k \overset{\text{convII}}{\longrightarrow} S_2) , \forall_{n\in\{1,\dots,k-1\}}(T_n \overset{\text{convI}}{\longrightarrow} T_{n+1} \vee T_n \overset{\text{convII}}{\longrightarrow} T_{n+1})\big)\Big].$$

Definition 15. *Let* $\longrightarrow^{\text{convI,III}}$ *be a relation on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \overset{\text{convI,III}}{\longrightarrow} S_2 \iff \Big[\big(S_1 \overset{\text{convI}}{\longrightarrow} S_2 \vee S_1 \overset{\text{convIII}}{\longrightarrow} S_2\big) \vee\exists_{k\in\mathbb{N}}\exists_{T_1,\dots,T_k\in\mathcal{N}\mathcal{N}}\big((S_1 \overset{\text{convI}}{\longrightarrow} T_1 \vee S_1 \overset{\text{convIII}}{\longrightarrow} T_1) , (T_k \overset{\text{convI}}{\longrightarrow} S_2 \vee T_k \overset{\text{convIII}}{\longrightarrow} S_2) , \forall_{n\in\{1,\dots,k-1\}}(T_n \overset{\text{convI}}{\longrightarrow} T_{n+1} \vee T_n \overset{\text{convIII}}{\longrightarrow} T_{n+1})\big)\Big].$$

Definition 16. *Let* $\longrightarrow^{\text{exp}}$ *be a relation on* $\mathcal{N}\mathcal{N} \times \mathcal{N}\mathcal{N}$ *by*

$$S_1 \overset{\text{exp}}{\longrightarrow} S_2 \iff \exists_{T_1,T_2\in\mathcal{N}\mathcal{N}}(S_1 \overset{\text{convI}}{\longrightarrow} T_1, T_1 \overset{\text{III}}{\longrightarrow} T_2, T_1 \overset{\text{convI}}{\longrightarrow} S_2).$$

We shall say for if $S_1 \longrightarrow^{\text{exp}} S_2$ *that* S_2 *is an expansion of* S_1*.*

Definition 17. *Let* $\longrightarrow^{\text{red}}$ *be a relation on* $\mathcal{NN} \times \mathcal{NN}$ *by*

$$S_1 \stackrel{\text{red}}{\longrightarrow} S_2 \Longleftrightarrow \exists_{T_1,T_2 \in \mathcal{NN}} (S_1 \stackrel{\text{convI}}{\longrightarrow} T_1, T_1 \stackrel{\text{II}}{\longrightarrow} T_2, T_1 \stackrel{\text{convI}}{\longrightarrow} S_2).$$

We shall way for if $S_1 \longrightarrow^{\text{red}} S_2$ *that* S_2 *is a reduction of* S_1.

Definition 18. *Let* $\longrightarrow^{\text{nred}}$ *be a relation on* $\mathcal{NN} \times \mathcal{NN}$ *by*

$$S_1 \stackrel{\text{nred}}{\longrightarrow} S_2 \quad \Longleftrightarrow \quad \Big[S_1 \stackrel{\text{red}}{\longrightarrow} S_2 \vee \exists_{k \in \mathbb{N}} \exists_{T_1,\ldots,T_k \in \mathcal{NN}} \big(S_1 \stackrel{\text{red}}{\longrightarrow} T_1, T_k \stackrel{\text{red}}{\longrightarrow} S_2$$
$$, \forall_{n \in \{1,\ldots,k-1\}} (T_n \stackrel{\text{red}}{\longrightarrow} T_{n+1}) \big) \Big].$$

Definition 19. *Let us define a subset* $F\mathcal{NN}$ *of* $\mathcal{NN}$ *by*

$$S \in F\mathcal{NN} \Longleftrightarrow \Big(S \in \mathcal{NN}, \sim \Big(\exists_{T_1, S_1 \in \mathcal{NN}} \exists_{a \in \text{AL}} \big((T_1, a), S_1 \big) \in \overline{S} \big) \Big) \Big),$$

where $\overline{} : \mathcal{NN} \to 2^{\mathcal{NN}}$ *as follows:*

$$\begin{aligned} \overline{(a, \emptyset)} &= \{ (a, \emptyset) \}, \quad \textit{for} \quad a \in \text{AL} \\ \overline{(S, a)} &= \{ (S, a) \} \cup \overline{S}, \quad \textit{for} \quad S_1, S_2 \in \mathcal{NN} \\ \overline{(S_1, S_2)} &= \{ (S_1, S_2) \} \cup \overline{S_1} \cup \overline{S_2}, \quad \textit{for} \quad S_1, S_2 \in \mathcal{NN}. \end{aligned}$$

Definition 20. *Let* $A, B \in \mathcal{NN}$. B *is said to be a normal form of* A *if* $A \to B$ *and* $B \in F\mathcal{NN}$.

Definition 21. *Let us denote a subset* $PF\mathcal{NN}$ *of* $\mathcal{NN}$ *by*

$$S \in PF\mathcal{NN} \quad \Longleftrightarrow \quad \Big(S \in F\mathcal{NN}, B(S) \cap F(S) = \emptyset$$
$$, \forall_{n \in D_{SV(S)}} \big((SV(S))(n) = a_n \big) \Big),$$

where SV *is defined as follows:*

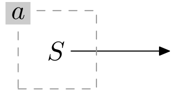
$$\begin{aligned} SV((a, \emptyset)) &= \lambda, \quad \textit{for} \quad a \in \text{AL} \\ SV((S, a)) &= (a) \bullet SV(S), \quad \textit{for} \quad a \in \text{AL}, S \in \mathcal{NN} \\ SV((S_1, S_2)) &= SV(S_1) \bullet SV(S_2), \quad \textit{for} \quad S_1, S_2 \in \mathcal{NN}. \end{aligned}$$

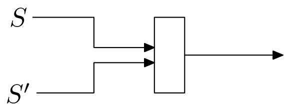
Definition 22. *Let* $A, B \in \mathcal{NN}$. B *is said to be a principial normal form of* A *if* $A \to B$ *and* $B \in PF\mathcal{NN}$.

1. Geometric interpretation

Denoting:

$(a, \emptyset)$ by

(S, a) by 

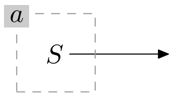
(S, S') by 

we have that every diagram build as follows:

1)

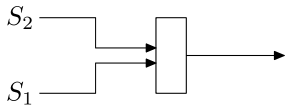
is a diagram, for $a \in \mathrm{AL}$

2) if $a \in \mathrm{AL}$, S is a diagram then



is a diagram

3) if S_1, S_2 are diagrams then



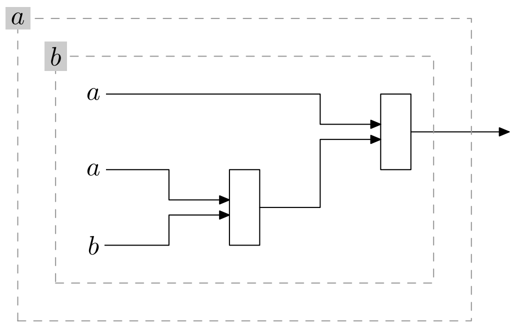
is a diagram

represents some neural network and contrary: every neural network can be represented by some diagram, builid by 1), 2) and 3) conditions.

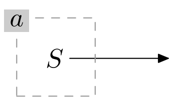
Example 1. *Neural network, which will represent number 2:*

$$((((a, \emptyset), ((a, \emptyset), (b, \emptyset))), b), c)$$

is represented by diagram

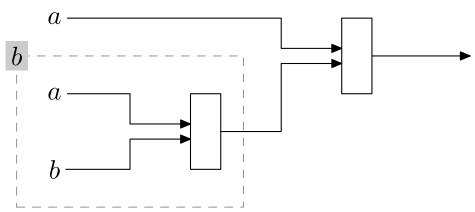


On such a diagrams parts form

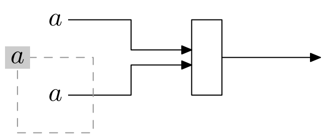


could be means as showing, where a variable a is bounded (inside of the frame) and where is free (outside).

Example 2. *In the following diagram:*



variable a is free and b is bounded. From the definition 2, follows that the same variable can be free and bounded simultaneously, as in following diagram:



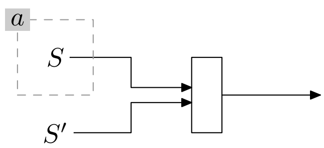
which represents a neural network

$$S = ((a, \emptyset), ((a, \emptyset), a))$$

(in fact, $a \in F(S)$ and $a \in B(S)$).

If S' is a diagram, which representes a net S then S^x_y is represented by diagram build from S' by replacing all y by x. Action $Sub(S, x, y)$ definded in 4 is such the replacing without x being free or bounded in S' and y existing in diagram S'.

If a is not bounded in a diagram S and bounded variabels of S and free of S' do not intersect then network, which is represented by diagram



is in relation $\longrightarrow^{\text{II}}$ with a network represented by a diagram

$$S_a^{S'} \longrightarrow .$$

Relaction $\longrightarrow^{\text{III}}$ is the contrary of $\longrightarrow^{\text{II}}$.

2. Normal form theorems

Definition 23. *The ordered triple $(W, \leq, f)$ is said to be a standard, indexed, net–finite binary tree if:*

1° $W \subset \{L, P\}^*$
2° $\#W < \aleph_0$

3°

$$\forall_{(w_1^1,\dots,w_1^n),(w_2^1,\dots,w_2^m)\in W} \quad \Big((w_1^1,\dots,w_1^n) \leq (w_2^1,\dots,w_2^m) \iff \exists_{c\in \mathbb{N}^\circ}\big((w_1^1,\dots,w_1^c) = (w_2^1,\dots,w_2^m)\big)\Big)$$

4° $(W, \leq)$ *is a binary tree*

5° $f\colon W \to \mathcal{N}N \cup \mathrm{AL}$

6° $(w_1^1,\dots,w_1^n) \in W \Longrightarrow (w_1^1,\dots,w_1^{n-1}) \in W.$

We shall write indexed tree instead of standard, indexed, net–finite binary tree. We use the convention that the root is a greatest element.

Proposition 8. *Let* $(W, \leq, f)$ *be an indexed tree and let* $w \in W$.

1° *If* w *is not a leaf then* $w \bullet (L) \in W$ *and* $w \bullet (P) \in W$;

2° λ *is a root.*

□

Definition 24. *For every* $S \in \mathcal{N}N$ *there exists a unique indexed tree* $(W_S, \leq_S, f_S)$ *satisfied following statements:*

1° $f_S(\lambda) = S$

2° *If* $f_S(w) = (a, \emptyset)$ *for any* $w \in W_S$ *and* $a \in \mathrm{AL}$ *then* w *is a leaf*

3° *If* $f_S(w) = (T, a)$ *for any* $w \in W_S$, $T \in \mathcal{N}N$, $a \in \mathrm{AL}$ *then* $f_S(w \bullet (L)) = T$, $f_S(w \bullet (P)) = a$

4° *If* $f_S(w) = (T, T')$ *for* $w \in W_S$, $T, T' \in \mathcal{N}N$ *then* $f_S(w \bullet (L)) = T$, $f_S(w \bullet (P)) = T'$.

The tree $(W_S, \leq_S, f_S)$ *of these properties we shall denote by* $\widehat{S}$. *For* $T, T' \in \mathcal{N}N$, *If* $\widehat{T} = \widehat{T'}$ *then* $T = T'$. □

Definition 25. *The pair* (S, T), *where* $S, T \in \mathcal{N}N$ *will be said to be I,II–action if* $S \longrightarrow^{\mathrm{I}} T$ *or* $S \longrightarrow^{\mathrm{II}} T$. *The sequence* $(S_i)_{i=1}^n$, *where* $n \in \mathbb{N}$, *and for every* $i \in \{1, \dots n\}$ *is* $S_i \in \mathcal{N}N$ *will be said to be a sequence of I,II—actions if for every* $j \in \{1, \dots, n-1\}$ (S_j, S_{j+1}) *is a I,II–action.*

Definition 26. *Let* $S \in \mathcal{N}N$ *and* $\widehat{S} = (W, \leq, f)$. *We shall denote*

$$Ind_S(w) = f(w) \quad for \quad w \in W,$$

and

$$\check{S} = \{\, (T, n); n \in W, f(n) = T \,\}.$$

Proposition 9. *The operation* $\check{}$ *has following properties*

i) $\check{}$ *is a mapping of* $\mathcal{N}N$ *into* $2^{\mathcal{N}N \times \{L,P\}^*}$

ii) *for every* $S, T \in \mathcal{N}N$, $n \in \{L, P\}^*$ *if* $(T, n) \in \check{S}$ *then* $Ind_S(n) = T$

iii) *for every* $S, T \in \mathcal{N}N$ *if* $\check{S} = \check{T}$ *then* $S = T$.

□

Proposition 10. *Let* $M, N \in \mathcal{N}N$, $x \in \mathrm{AL}$, $a \in \{L, P\}^*$, $x \notin B(M)$. *Then*

$$\mathrm{Ind}_{((M,x),N)}((L,L) \bullet n) = (x, \emptyset) \Longrightarrow \mathrm{Ind}_{M_x^N}(n) = N$$

Proof. 1° Let $M = (a, \emptyset)$, where $a \in \mathrm{AL}$.
i) if $a \neq x$ then $M_x^N = (a, \emptyset)_x^N = (a, \emptyset)$ and this implication is true
ii) if $a = x$ then $\mathrm{Ind}_{((M,x),N)}((L,L) \bullet n) = (x, \emptyset)$ implies that $n = \lambda$. Because $M_x^N = (x, \emptyset)_x^N = N$ then $\mathrm{Ind}_{M_x^N}(\lambda) = N$, and the implication is true
2° Suppose that Proposition is true for some $S \in \mathcal{N}N$. Let $a \in \mathrm{AL}$ and $M = (S, a)$.
i) if $a = x$ then if for some $n \in \{L, P\}^*$ is $\mathrm{Ind}_{(S,a)}(n) = (x, \emptyset)$ then $x \in B((S, a))$ and the implication is true
ii) if $a \neq x$ and for some $n \in \{L, P\}^*$ is $\mathrm{Ind}_{(S,a)}(n) = (x, \emptyset)$ then $n = (L) \bullet n'$, where $\mathrm{Ind}_S(n') = (x, \emptyset)$. Then from the definition $(S, a)_x^N = (S_x^N, a)$ and by hypothesis $\mathrm{Ind}_{(S,a)_x^N}((L) \bullet n') = N$.
3° Suppose that some $S, S' \in \mathcal{N}N$ satifies this Proposition. Let for some $n \in \{L, P\}^*$ be $\mathrm{Ind}_{(S,S')}(n) = (x, \emptyset)$. Suppose that $n = (L) \bullet n'$, where $n' \in \{L, P\}^*$. Then $\mathrm{Ind}_S(n') = (x, \emptyset)$ and from hypothesis $\mathrm{Ind}_{S_x^N}(n') = N$. If $x \notin B((S, S'))$, then $x \notin B(S), x \notin B(S')$ and $\mathrm{Ind}_{(S,S')_x^N}(n) = \mathrm{Ind}_{(S_x^N, S'^N_x)}(n) = N$.
Hence by virtue of Proposition 1 the proof is complete. □

Definition 27. *Let* $D = (W, \leq)$ *be a finite binary tree; let* $W_1 \subset W$ *such, that*

$$\forall_{w_1, w_1' \in W_1}\big(\sim(w_1 \leq w_1' \vee w_1' \leq w_1)\big), \forall_{w \in W}\exists_{w_1 \in W_1}\big(w \leq w_1 \vee w_1 \leq w)\big)$$

Every set W_1 *of this property will be said to generate the tree* D *and* D *to be generated by* W_1.

Proposition 11. *Let* $D = (W, \leq)$ *be a finite binary tree and let* W_1 *generate* D. *Then*

$$\Big(\big\{\, w \in W; \exists_{w_1 \in W_1}(w_1 \leq w) \,\big\}, \leq_{|\{\, w \in W; \exists_{w_1 \in W_1}(w_1 \leq w)\,\}}\Big)$$

is a unique binary subtree of D *such that the set of its leaves is equal to* W_1.

Proof. 1° We shall show that it is a binary tree. It is clear that it is a tree. To show that it is a binary tree, we have to show that every $w \in \big\{\, w \in W; \exists_{w_1 \in W_1}(w_1 \leq w) \,\big\}$ is either a leaf, or has two sons in this tree. If $w \in W_1$ then $\forall_{w_1, w_1' \in W_1} \sim (w_1 \leq w_1' \vee w_1' \leq w_1)$ implies that there is no $w' \in \{\, w \in W; \exists_{w_1 \in W_1}(w_1 \leq w)\,\}$, for which there is $w' \leq w$; so w is a

leaf. Let $w \notin W_1$. Suppose that w' and w'' are two sons of w in D. From definition, there exist w'_1 and $w''_2 \in W_1$ such that $\big((w'_1 \leq w') \vee (w' \leq w'_1)\big)$ and $\big((w''_1 \leq w'') \vee (w'' \leq w''_1)\big)$. Since $w \notin W_1$ then $w'_1 \leq w_1$ and $w''_1 \leq w''$. Hence $w', w'' \in \big\{\, w \in W; \exists_{w_1 \in W_1}(w_1 \leq w) \,\big\}$.
2° Uniqueness. Let us suppose that $(W', \leq_{|W' \times W'})$ is a subtree of D different than $(A, \leq_{|A \times A})$, where $A = \big\{\, w \in W; \exists_{w_1 \in W_1}(w_1 \leq w) \,\big\}$, for which the set of leaves is also W_1. Let $w \in W$ such that $w \in A$ and $w \notin W'$. Proof is a consequence of following lemma:

Lemma 1. *Let $D = (W, \leq)$ be a finite binary tree and let L be the a set of its leaves. The following statement is true*

$$\forall_{w \in W} \exists_{l \in L} (l \leq w)$$

Proof. By contrary, suppose that there exists $w \in W$ such that for every $l \in L$ it is false that $l \leq w$. Let also $\#W = m$. We have that $w \notin L$. But w has two sons: w_{11} and w_{12}. If $w_{11} \in L$ then $w_{11} \leq w$, what is an absurd. So w_{11} has two sons: w_{21} and w_{22}. Moreover $w \neq w_{11}$, $w_{11} \neq w_{21}$, $w_{21} \neq w$. By induction, we can find the set $\{\, w, w_{11}, w_{21}, \ldots, w_{m1} \,\} \subset W$ such that $\#\{\, w, w_{11}, w_{21}, \ldots, w_{m1} \,\} = m + 1 > \#W$, which is an absurd. □

Suppose now that for $w \in W$ we have $w \in W'$ and $w \notin A$. From hypothesis there exists $w_1 \in W_1$ such that $(w_1 \leq w) \vee (w \leq w_1)$ and $w \notin A$ that implies that $w \leq w_1$ and $w \neq w_1$. But now W_1 is not a set of leaves of W'. □

Definition 28. *Let $D = (W, \leq)$ be a tree and let $w' \in W$. We shall denote the set $\{\, w \in W; w \leq w' \,\}$ by $\Delta w'$.*

Definition 29. *Let $D_1 = (W_1, \leq_1, f_1)$, $D_2 = (W_2, \leq_2, f_2)$ be indexed trees and let $w_1 \in W_1$. We shall denote by $D_1[w_1 \leftarrow D_1] = (W, \leq, f)$ an indexed tree as follows:*

1° $W = (W_1 - \Delta w_1) \cup \{\, w_1 + w_2; w_2 \in W_2 \,\}$
2°

$$f(w) = \begin{cases} f_1(w), & \textit{if } w \in W_1 \setminus \Delta w_1; \\ f_2(v), & \textit{if } w \in \{\, w_1 + w_2; w_2 \in W_2 \,\} \textit{ and } w = w_1 + v. \end{cases}$$

□

Definition 30. *Let $D = (W, \leq)$ be a finite binary tree satisfing conditions 1°, 2°, 3°, 4°, 6° of definition 23. More, let W_1 generate D and $f\colon W_1 \to \mathcal{N}N \cup \mathrm{AL}$ such that if $f(w_1) \in \mathrm{AL}$ then $w_1 = (a_1, \ldots, a_k) \bullet (P)$, for some $(a_1, \ldots, a_k) \in \{\, L, P \,\}^*$, $k \in \mathbb{N}$ and $w_1 \in W_1$. Then there exists a unique neural network S such that*

$$\forall_{w_1 \in W_1} \big(f(w_1) = \mathrm{Ind}_S(w_1) \big)$$

The net S will be said to be generated by f and denoted by $S = \operatorname{gen}(f)$

Proof. Construction of S follows from Proposition 11 and Proposition 24. □

Definition 31. *Let $S, T \in \mathcal{NN}$, $M, N \in \mathcal{NN}$, $x \in \mathrm{AL}$, $n \in \{L, P\}^*$, and let $S \longrightarrow^{\mathrm{II}} T$. We shall call the I,II–action (S, T) to be a contraction of $(((M, x), N), n)$ if:*

1° $(((M, x), N), n) \in \breve{S}$ *and*
2° $\widehat{T} = \widehat{S}[n \leftarrow \widehat{M_x^N}]$

Proposition 12. *Let S, T be neural networks, $S \longrightarrow^{\mathrm{II}} T$, (S, T) be a contraction of $(((M, x), N), n)$ and let $A_S = \{ w \in \breve{S}; w \text{ is a leaf }, w \notin \Delta n \}$. Then $\widehat{T} = \widehat{S}[n \leftarrow \widehat{M_x^N}]$ and $A_a \cup \{n\}$ generates a tree $\widehat{T}$ (more precisely, $\mathrm{Ind}_{\breve{T}}(n) = M_x^N$).* □

Proposition 13.

$$\forall_{S,T \in \mathcal{NN}} \Big(S \xrightarrow{\mathrm{II}} T \Longrightarrow \quad \Big(\exists_{(((M,x),N),n) \in \breve{S}} \Big((S, T) \text{ is a contraction of } (((M, x), N), n) \Big) \Big) \Big).$$

Proof. Let us denote by $\Phi(S)$ the formula

$$\forall_{T \in \mathcal{NN}} \Big(S \xrightarrow{\mathrm{II}} T \Longrightarrow \quad \Big(\exists_{(((M,x),N),n) \in \breve{S}} \Big((S, T) \text{ is a contraction of } (((M, x), N), n) \Big) \Big) \Big).$$

We shall prove the formula

$$\forall_{S \in \mathcal{NN}} \big(\Phi(S) \big),$$

which is equivalent to thesis. 1° Let $S = (a, \emptyset)$, where $a \in \mathrm{AL}$. From definition, there is no $T \in \mathcal{NN}$ such that $S \longrightarrow^{\mathrm{II}} T$. So $\Phi((a, \emptyset))$.
2° Suppose that $\Phi(S')$ and let $S = (S', a)$. From definition, we have that $(S', a) \longrightarrow^{\mathrm{II}} T$ if and only if $T = (T', a)$ and $S' \longrightarrow^{\mathrm{II}} T'$. From hypothesis we have that there exists $(((M, x), N), n) \in \breve{S}'$ such that (S', T') is a contraction of $(((M, x), N), n)$. But hence $((S', a), (T', a))$ is a contraction of $(((M, x), (L) \bullet n) \in \breve{(S', a)} = \breve{S}$.

3° Suppose that $\Phi(S')$ and $\Phi(S'')$. Let $S = (S', S'')$ and $S \longrightarrow^{\mathrm{II}} T$. From definition, there are three cases
i) $S' = (A, y), T = A_y^{S''}$,

ii) $T = (T', S''), S' \longrightarrow^{\mathrm{II}} T'$,
iii) $T = (S', T''), S'' \longrightarrow^{\mathrm{II}} T''$.
Let us consider them:
i) $S' = (A, y), T = A_y^{S''}$. Then (S, T) is a contraction of $(S, \lambda) \in \breve{S}$
ii) $T = (T', S''), S' \longrightarrow^{\mathrm{II}} T'$. From hypothesis we have that (S', T') is a contraction of $(((M, x), N), n) \in \breve{S}'$. Also (S, T) is a contraction of $(((M, x), N), (L) \bullet n)$.

iii) $T = (S', T''), S'' \longrightarrow^{\mathrm{II}} T''$. From hypothesis (S'', T'') is a contraction of $(((M, x), N), n) \in \breve{S}''$. Then clearly (S, T) is a contraction of $(((M, x), N), (P) \bullet n)$.
Applying Proposition 1 concludes the proof. □

Proposition 14. *Let S, T_1, T_2 be neural networks, (S, T_1) be a contraction of $(((M_1, x_1), N_1), n_1)$ and (S, T_2) be a contraction of $(((M_2, x_2), N_2), n_2)$. Then*

$$T_1 = T_2 \Longleftrightarrow (((M_1, x_1), N_1), n_1) = (((M_2, x_2), N_2), n_2) \Longleftrightarrow n_1 = n_2$$

□

Definition 32. *Let $C = (S_1, \ldots, S_k)$, be a sequence of I, II–actions, $\{ (A_i, n_i) : i \in I \} \subset \breve{S}_1$, and let for $i_1, i_2 \in I$ such that $i_1 \neq i_2$ be $(A_{i_1}, n_{i_1}) \neq (A_{i_2}, n_{i_2})$. Let us define*

$$\mathrm{Tr}^C : \{ (A_i, n_i); i \in I \} \to 2^{\breve{S}_k}$$

by

1° *If $C = (S)$ then*

$$\mathrm{Tr}^C \big\{ (A_i, n_i) \big\} = \{ (A_i, n_i) \}$$

2° *If $C = (S_1, S_2)$, where $S_1 \longrightarrow^{\mathrm{I}} S_2$ then*

$$\mathrm{Tr}^C \big\{ (A_i, n_i) \big\} = \{ (\mathrm{Ind}_{S_2}(n_i), n_i) \} = \{ (A_{i}{}_y^x, n_i) \}$$

3° *If $C = (S_1, S_2)$, where $S_1 \longrightarrow^{\mathrm{II}} S_2$, and where (S_1, S_2) is a contraction of $(((M, x), N), n)$ and $\sim(n \leq n_i), \sim(n_i \leq n)$ then*

$$\mathrm{Tr}^C \big\{ (A_i, n_i) \big\} = \{ (A_i, n_i) \}$$

4° *If $C = (S_1, S_2)$, where $S_1 \longrightarrow^{\mathrm{II}} S_2$ and where (S_1, S_2) is a contraction of $(((M_i, x_i), N_i), n_i)$ and $A_i = ((M_i, x_i), N_i)$ then*

$$\mathrm{Tr}^C \big\{ (A_i, n_i) \big\} = \emptyset$$

5° *If* $C = (S_1, S_2)$, *where* $S_1 \longrightarrow^{\mathrm{II}} S_2$ *is a contraction of* $(((M, x), N), n)$ *and* $n \leq n_i, \sim(n = n_i)$, *then*

$$\mathrm{Tr}^C\{\,(A_i, n_i)\,\} = \{\,(\mathrm{Ind}_{S_2}(n_i), n_i)\,\} = \{\,(((M', x), N'), n_i)\,\}$$

6° *If* $C = (S_1, S_2)$, *where* $S_1 \longrightarrow^{\mathrm{II}} S_2$ *and where* (S_1, S_2) *is a contraction of* $(((M, x), N), n)$ *and* $n_i \leq n, n \neq n_i$. *Then there are possible two cases:*

i) $n_i \leq n \bullet (L)$. *Then, since* $A_i \neq (M, x)$ *and* $A_i \neq x$, *so* $n_i \leq n \bullet (L, L)$. *Also, we have* $n_i = n \bullet (L, L) \bullet r$. *Then*

$$\mathrm{Tr}^C\big\{\,(A_i, n_i)\,\big\} = \big\{\,(\mathrm{Ind}_{S_2}(n \bullet r), n \bullet r)\,\big\}$$

ii) $n_i \leq n \bullet (P)$. *Then* $\mathrm{Ind}_{S_1}(n \bullet (P)) = N$. *Let* $n_i = n \bullet (P) \bullet \eta$ *and let* $J = \big\{\,\xi; \mathrm{Ind}_{S_1}(n \bullet (L, L) \bullet \xi) = (x, \emptyset)\,\big\}$. *Then we put*

$$\mathrm{Tr}^C\big\{\,(A_i, n_i)\,\big\} = \big\{\,(\mathrm{Ind}_{S_2}(n \bullet \xi \bullet \eta), n \bullet \xi \bullet \eta); \xi \in J\,\big\}$$

From Proposition 10, we have that for any $\xi \in J$ *is* $\mathrm{Ind}_{S_2}(n \bullet \xi \bullet \eta) = A_i$.

7° *If* $C = (S_1, \ldots, S_k)$ *is a sequence of I,II–actions then let*

$$\begin{aligned} X_2 &= \mathrm{Tr}^{(S_1, S_2)}\big\{\,(A_i, n_i)\,\big\}, \\ X_3 &= \bigcup_{(B_i, y_i) \in X_2} \Big\{\,\mathrm{Tr}^{(S_2, S_3)}\,\Big\}, \\ &\ \vdots \\ X_k &= \bigcup_{(B_i, y_i) \in X_{k-1}} \Big\{\,\mathrm{Tr}^{(S_{k-1}, S_k)}\,\Big\} \quad \text{and} \end{aligned}$$

$$\mathrm{Tr}^C\{\,A_i, x_i\,\} = X_k$$

This construction is proper, because

$$\forall_{(B,y) \in \mathrm{Tr}^{(S_i, S_{i+1})}\{\,(A_i, x_i)\,\}} \exists_{P,Q \in \mathcal{NN}, r \in \mathrm{AL}}\big(B = ((P, r), Q)\big).$$

8°
$$\mathrm{Tr}^C\big\{\,(A_i, n_i); i \in I\,\big\} = \bigcup_{i \in I}\Big\{\,\mathrm{Tr}^C\big\{\,(A_i, n_i)\,\big\}\,\Big\}.$$

Remark 1. *The construction from definition 32 is proper, because if* $S \longrightarrow^{\mathrm{I}} S'$ *and* $S \longrightarrow^{\mathrm{II}} S''$ *then* $S' \neq S''$. □

Proposition 15. *Let* $C = (S_1, \ldots, S_k)$ *be a sequence of I,II–actions and let* $(A_1, n_1), (A_2, n_2) \in \breve{S}_1$ *and* $(A_1, n_1) \neq (A_2, n_2)$. *Then we have*

$$\mathrm{Tr}^C\big\{\,(A_1, n_1)\,\big\} \cap \mathrm{Tr}^C\big\{\,(A_2, n_2)\,\big\} = \emptyset$$

□

Proposition 16. *Let* $(B_1,\ldots,B_n)$ *be a sequence of I,II–actions and let* $(((M,x),N),n) \in \breve{B}_1$. *Then* $\mathrm{Tr}^{(B_1,\ldots,B_n)}\{\,(((M,x),N),n)\,\} = \{\,D_k; k \in K\,\}$, *where* $\#K < \aleph_0$ *and for* $k \in K$ *is* $D_k = (((M_k,x_k),N_k),n_k) \in \breve{B}_n$. $\Box$

Definition 33. *Let* $(A_1,\ldots,A_n)$ *be a finite sequence of neural networks such that for* $i \in \{\,1,\ldots,n-1\,\}$ *is* $A_i \longrightarrow^{\mathrm{red}} A_{i+1}$, *and* $A_i \longrightarrow^{\mathrm{I}} B_{i,1} \longrightarrow^{\mathrm{I}} \cdots \longrightarrow^{\mathrm{I}} B_{i,p_i} \longrightarrow^{\mathrm{II}} B_{i,p_i+1} \longrightarrow^{\mathrm{I}} \cdots \longrightarrow^{\mathrm{I}} B_{i,r_i} \longrightarrow^{\mathrm{I}} A_{i+1}$ *and let* $\{(((M_i,x_i),N_i),n_i) : i \in I\} \subset \breve{A}_1$. *We shall call* $(A_1,\ldots,A_n)$ *to be a sequence of contractions of* $\{\,(((M_i,x_i),N_i),n_i); i \in I\,\}$ *if for every* $i \in \{\,1,\ldots,n-1\,\}$ *is* $B_{i,p_i} \longrightarrow^{\mathrm{II}} B_{i,p_i+1}$ *and* (B_{i,p_i},B_{i,p_i+1}) *is a contraction of* S, *where* $S \in \bigcup\{\,x; \exists_{i\in I}(x = \mathrm{Tr}^{(A_1,B_{1,1},\ldots,B_{1,r_1},A_2,\ldots,B_{i,p_i})}\{\,(((M_i,x_i),N_i),n_i))\,\}\,\}$. *Moreover, if* $\bigcup\{x; \exists_{i\in I}\ (x = \mathrm{Tr}^{(A_1,B_{1,1},\ldots,B_{1,r_1},A_2,\ldots,B_{i,p_i})}\{(((M_i,x_i),N_i),n_i))\} = \emptyset$ *then we say that a sequence of contractions vanishes and* A_n *is its result.*

Remark 2. *If a sequence of contractions* $\{\,(((M_i,x_i),N_i),n_i); i \in I\,\}$ *vanishes then we assume that the sequence of following contractions is empty.*

Theorem 3. *For every* $A \in \mathcal{N}N$ *and every* $\{\,(((M_i,x_i),N_i),n_i) : i \in I\,\} \subset \breve{A}$ *there exists* $m \in \mathbb{N} \cup \{\,0\,\}$ *such that every sequence of contractions* $\{(((M_i, x_i), N_i), n_i) : i \in I\}$ *vanishes after at most* m *contractions and if* A' *and* A'' *are two results of vanished sequences of contractions then* $A' \longrightarrow^{\mathrm{convI}} A''$.

Lemma 2. *Let* $S = ((M,x),N)$ *and* $(((A,y),B),m) \in \breve{M}$. *Let also* $S \longrightarrow^{\mathrm{II}} T_1$ *and* (S,T_1) *be a contraction of* $(((M,x),N),\lambda)$, $T_1 \longrightarrow^{\mathrm{II}} T_2$ *and* (T_1,T_2) *be a contraction of* $\mathrm{Tr}^{(S,T_1)}\{\,(((A,y),B),m)\,\}$ *and* $S \longrightarrow^{\mathrm{II}} T_1'$, (S,T_1') *be a contraction of* $(((A,y),B),(L,L)\bullet m)$, $T_1' \longrightarrow^{\mathrm{II}} T_2'$ *and let* (T_1',T_2') *be a contraction of* $\mathrm{Tr}^{(S,T_1')}\{\,(((M,x),N),\lambda)\,\}$. *Then* $T_2 = T_2'$.

Proof. Clearly, there is sufficien to prove that $\mathrm{Ind}_{T_2}(m) = \mathrm{Ind}_{T_2'}(m)$. Also let $K_1 = \{\,k \in \{\,L,P\,\}^*; \mathrm{Ind}_A(k) = (x,\emptyset)\,\} = \{\,k_1^1,\ldots,k_1^{r_1}\,\}$, $K_2 = \{\,k \in \{\,L,P\,\}^*; \mathrm{Ind}_B(k) = (x,\emptyset)\,\} = \{\,k_2^2,\ldots,k_2^{r_2}\,\}$, $K_3 = \{\,k \in \{\,L,P\,\}^*; \mathrm{Ind}_B(k) = (y,\emptyset)\,\} = \{\,k_3^2,\ldots,k_3^{r_3}\,\}$. Then $\widehat{\mathrm{Ind}_{T_1}(m)} = ((\widehat{U_1,y)},V_1)$, where $\widehat{U_1} = \widehat{A}[k_1^1 \leftarrow \widehat{N}]\cdots[k_1^{r_1} \leftarrow \widehat{N}]$, $\widehat{V_1} = \widehat{B}[k_2^1 \leftarrow \widehat{N}]\cdots[k_2^{r_2} \leftarrow \widehat{N}]$. Let us denote $K_3' = \{\,k \in \{\,L,P\,\}^*; \mathrm{Ind}_{\mathrm{Ind}_{T_1}(m)}(k) = (y,\emptyset)\,\}$. Because $y \notin F(N)$ then $K_3' = K_3$. Then

$$\begin{aligned}\widehat{\mathrm{Ind}_{T_2}(m)} = \quad & \widehat{A}[k_1^1 \leftarrow \widehat{N}]\cdots[k_1^{r_1} \leftarrow \widehat{N}]\big[k_3^1 \leftarrow \widehat{B}[k_2^1 \leftarrow \widehat{N}]\cdots[k_2^{r_2} \leftarrow \widehat{N}]\big]\cdots \\ & \cdots[k_3^{r_3} \leftarrow \widehat{B}\big[k_2^1 \leftarrow \widehat{N}]\cdots[k_2^{r_2} \leftarrow \widehat{N}]\big].\end{aligned}$$

Because no k_1^i, k_2^j are comparable, so

$$\begin{aligned}\widehat{\mathrm{Ind}_{T_2}(m)} = \quad & \widehat{A}\big[k_3^1 \leftarrow \widehat{B}[k_2^1 \leftarrow \widehat{N}] \cdots [k_2^{r_2} \leftarrow \widehat{N}]\big] \cdots \\ & \cdots \big[k_3^{r_3} \leftarrow \widehat{B}[k_2^1 \leftarrow \widehat{N}] \cdots [k_2^{r_2} \leftarrow \widehat{N}]\big][k_1^1 \leftarrow \widehat{N}] \cdots [k_1^{r_1} \leftarrow \widehat{N}].\end{aligned}$$

From the other hand $\mathrm{Ind}_{T_1'}((L,L) \bullet m) = \widehat{A}[k_3^1 \to \widehat{B}] \cdots [k_3^{r_3} \to \widehat{B}]$. Let us denote $K_4 = \{\, k \in \{\, L, P \,\}^*; \mathrm{Ind}_{\mathrm{Ind}_{T_1'}((L,L)\bullet m)}(k) = (x, \emptyset) \,\} = \{\, k_4^1, \ldots, k_4^{r_4} \,\}$. Because $x \neq y$ then $K_4 = K_1 \cup \{\, k_2 + k_3; k_2 \in K_2, k_3 \in K_3 \,\}$. Also we have

$$\widehat{\mathrm{Ind}_{T_2'}(m)} = \widehat{A}[k_3^1 \to \widehat{B}] \cdots [k_3^{r_3} \to \widehat{B}][k_4^1 \to \widehat{N}] \cdots [k_4^{r_4} \to \widehat{N}],$$

that completes proof. $\square$

Proof. For a net like $(a, \emptyset)$, where $a \in \mathrm{AL}$ Proposition is obvious, if we set $m = 0$.

Suppose that Proposition is true for some $M \in \mathcal{N}N$. Let $A = (M, x)$ be a neural network, where $x \in \mathrm{AL}$. Let $\{\, (((M_i, x_i), N_i), n_i); i \in I \,\}$ be any subset of $\breve{A}$. Then for every $i \in I$ is $(((M_i, x_i), N_i), n_i) = (((M_i, x_i), N_i), (L) \bullet n_i')$. More $\{\, (((M_i, x_i), N_i), n_i'); i \in I \,\} \subset \breve{M}$. Thus, if $A \longrightarrow^{\mathrm{II}} S$, (A, S) is a contraction of $T \subset \breve{A}$ then $S = (S', x)$, $M \longrightarrow^{\mathrm{II}} S'$ and (M, S') is a contraction of some $T' \in \breve{M}$. Chosing m such that every sequence of contractions $\{\, (((M_i, x_i), N_i), n_i'); i \in I \,\}$ of M vanishes, we have thesis for (M, x).

Suppose that Proposition is true for some $F, X \in \mathcal{N}N$. Let $A = (F, X)$ and let $\{\, (((M_i, x_i), N_i), n_i); i \in I \,\} \subset \breve{A}$, and $\{\, (((M_i, x_i), N_i), n_i); i \in I \,\} \cap \{\, (A, \lambda) \,\} = \emptyset$. Then, for every $i \in I$ we have that $(((M_i, x_i), N_i), n_i) = (((M_i, x_i), N_i), (L) \bullet n_i')$, or $(((M_i, x_i), N_i), n_i) = (((M_i, x_i), N_i), (P) \bullet n_i'')$. Thus $I = I' \cup I''$ and we have that $\{\, (((M_i, x_i), N_i), n_i); i \in I \,\}$ is equal to $\{\, (((M_i, x_i), N_i), (L) \bullet n_i'); i \in I' \,\} \cup \{\, (((M_i, x_i), N_i), (P) \bullet n_i''); i \in I'' \,\}$ such that $\{\, (((M_i, x_i), N_i), (L) \bullet n_i'); i \in I' \,\} \subset \breve{F}$ and $\{\, (((M_i, x_i), N_i), (P) \bullet n_i''); i \in I'' \,\} \subset \breve{X}$. Because every contraction of trace of $\{\, (((M_i, x_i), N_i), n_i); i \in I \,\}$ is a contraction of some of traces of $\{\, (((M_i, x_i), N_i), (L) \bullet n_i'); i \in I' \,\}$ and $\{\, (((M_i, x_i), N_i), (P) \bullet n_i''); i \in I'' \,\}$ then putting $m_F + m_X$, where m_F is the right number for F and the set $\{\, (((M_i, x_i), N_i), (L) \bullet n_i'); i \in I' \,\}$; m_X is a right number for a net X and the set $\{\, (((M_i, x_i), N_i), (P) \bullet n_i''); i \in I'' \,\}$ Proposition is proved for $(F, X) = A$.

Suppose that Proposition is proved for $M, N \in \mathcal{N}N$. Let $A = ((M, x), N)$ and suppose that there exists $i_0 \in I$ such that $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0}) = (A, \lambda)$. Let $I = \{\, i_0 \,\} \cup I_M \cup I_N$ such that for $i \in I_M$ is $n_i \leq (L, L)$, where $i \in I_N$ $n_i \leq (P)$. Then

$$\Big\{\, (((M_i, x_i), N_i), n_i); i \in I_M \,\Big\} \quad = \Big\{\, (((M_i, x_i), N_i), (L, L) \bullet n_i'); i \in I_M \,\Big\}$$

and

$$\Big\{ \big(((M_i, x_i), N_i), n_i\big); i \in I_N \Big\} \qquad = \Big\{ \big(((M_i, x_i), N_i), (P) \bullet n_i'\big); i \in I_M \Big\}$$

By hypothesis there exists $a \in \mathbb{N} \cup \{ 0 \}$ such that any sequence of contractions of $\{ (((M_i, x_i), N_i), n_i'); i \in I_M \} \subset \check{M}$ vanishes after at most a contractions and b such that a sequence of contractions $\{ (((M_i, x_i), N_i), n_i); i \in I_N \}$ vanishes after at most b contractions. Further, if we conside a net M and vanished sequence of contractions $\{ (((M_i, x_i), N_i), n_i'); i \in I_M \}$ its result is the net M' (unique within to relation $\longrightarrow^{\mathrm{convI}}$), consisting exact c free occurences of the variable x_{i_0}. More precisely, if $F(M') \cap B(M') = \emptyset$ then $\#J = \#\{ (\mathrm{Ind}_{M'}(j), j); j \in J \} = c$, where $\mathrm{Ind}_{M'}(j) = (x_{i_0}, \emptyset)$ for $j \in J$.

One of the way to perfoming vanished sequence of contractions is the following: First we act a vanishes sequence of contractions of $\{(((M_i, x_i), N_i), n_i) : i \in I_M \}$. By hypothesis, this sequence has length less or equal to a. Because the set $\{ (L, L), (L, P), (P) \}$ generates $\widehat{A}$, so this sequence of contractions there is associated some sequence of contractions $\{ (((M_i, x_i), N_i), n_i) : i \in I \}$. As a result, we have a nerual network $A' = ((M, t), N_p)$. The trace $\{ (((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0}) \}$ has only one element, equal to $\{ A' \}$. Making operation $A' \longrightarrow^{\mathrm{II}} M_t^{N_{i_0}}$, possible, where $(A', M_t^{N_{i_0}})$ is a contraction of the trace $\{ (((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0}) \}$ let us consider the contraction of ${M'}_t^{N_{i_0}}$, where $M \longrightarrow^{\mathrm{convI}} M'$ and $F(M') \cap B(M') = \emptyset$. Let $J = \{ k \in \{ L, P \}^*; \mathrm{Ind}_{M'}(k) = (t, \emptyset) \}$. Then $\mathrm{Ind}_{{M'}_t^{N_{i_0}}}(k) = N_{i_0}$ dla $k \in J$ (see Proposition 12) and $\#J = c$. Because for $k, k' \in J$, $k \neq k'$ we have $\sim(k \le k' \vee k' \le k)$, so $\Delta k \cap \Delta k' = \emptyset$. For every (N_{i_0}, k), where $k \in J$ let us consider vanished sequence of contractions $\{ (((M_i, x_i), N_i), k \bullet n_i); i \in I_N \}$. This way, we get vanished sequence of contractions $\{ (((M_i, x_i), N_i), n_i); i \in I \}$, consisting of at least $a + cb + 1$ contractoins—we shall call it to be a special sequence.

Let us consider μ —any sequence of contractions $\{ (((M_i, x_i), N_i), n_i); i \in I \}$. The trace of $\{ (((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0}) \}$ has only one element, up to the point that contraction of trace of $(\mathrm{Ind}(n_{i_0}), n_{i_0}) = (((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$ occurs; and thereafter will be empty. Furthermore, vanished sequence of contractions $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$ consist of at least $a + b + 1$ contractions.

We can show μ as $\varphi \bullet \beta_0 \bullet \nu$, where φ is a sequence of contractions of traces of $\{ (((M_i, x_i), N_i), n_i) : i \in I \setminus \{i_0\} \}$, β_0 is a contraction of trace $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$, ν is a sequence of remaining contractions.

We have that $\Delta(L, L) \cap \Delta(P) = \emptyset$, $\mathrm{Ind}_{((M,x),N)}(L, L) = M$ and $\mathrm{Ind}_{((M,x),N)}(P) = N$, so if $\varphi = (\varphi^1, \ldots, \varphi^k)$ and $(\varphi^{j-1}, \varphi^j)$ is a contraction of

$$(((A_1, x_1), B_1), m_1) \in \mathrm{Tr}^{(\varphi^1, \ldots, \varphi^{j-1})} \Big\{ \{ (((M_i, x_i), N_i), n_i); i \in I_M \} \Big\}$$

and $(\varphi^j, \varphi^{j+1})$ is a contraction of

$$(((A_2, x_2), B_2), m_2) \in \mathrm{Tr}^{(\varphi^1,\ldots,\varphi^j)}\Big\{ \{(((M_i, x_i), N_i), n_i); i \in I_N\} \Big\},$$

then $(\varphi^{j-1}, \varphi^{j+1})$ is a contraction of $(((A_2, x_2), B_2), m_2)$ and $(\varphi^{j+1}, \varphi^j)$ is a contraction $(((A_1, x_1), B_1), m_1)$. Also we can replace φ by $\alpha_0 \bullet \eta$, where α_0 is a sequence of contractions of traces of$\{ (((M_i, x_i), N_i), n_i); i \in I_M \}$ and η is a sequence of contractions of traces of $\{ (((M_i, x_i), N_i), n_i); i \in I_N \}$ and $\alpha_0 \bullet \eta$ has the same length that φ and make contractions of the same nets.

In a similar way $\eta \bullet \beta_0$ can by replaced by β' being a contraction of the trace of $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$, after which follows sequence of contractions η acting on every N_{i_0} in the net ${P'}_y^{N_{i_0}}$, where $P \longrightarrow^{\mathrm{convI}} P'$. The sequence $\eta \bullet \beta_0$ can have length less than β', but both sequences make contcractions of the same traces and their first and last elements are the same.

By this means, the sequence μ have been replaced by $\mu' = \alpha_0 \bullet \beta' \bullet \eta^*$, where

α_0 is a sequence of contractions of traces of$\big\{ (((M_i, x_i), N_i), n_i); i \in I_M \big\}$

β' is a contraction of the trace $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$

η^* are remaining contractions.

Moreover, μ and μ' has the same elements: first and last, and make contractions of the same traces.

Let now ξ be a part of μ' consisting of β' and contractions following β' up to and including first contraction of trace of $\{ (((M_i, x_i), N_i), n_i); i \in I_N \}$.

Denoting the net ξ acts on by $((P, y), N_{i_0})$, denoting $J = \{ j \in \{L, P\}^* : \mathrm{Ind}_P(j) = (y, \emptyset) \}$ for $F(P) \cap B(P) = \emptyset$ we have that ξ can be consider as $P[j_1 \leftarrow N_1] \cdots [j_k \leftarrow N_k]$ (where J is equal to $\{ j_1, \ldots, j_k \}$ and $(N_{i_0}, F_{l,1}, \ldots, F_{l,a_l}, N_l)$ is a sequence of I,II–actions (possible that one–element) for $l \in \{ 1, \ldots, k \}$); with following after its contraction $(((R, z), S), n) \in \mathrm{Tr}\{ ((M_q, x_q), N_q), n_q \}$, where $q \in I_M$.

Lemma 2 gives us that the last element ξ is similar to net A_2 made as follows: Let $((P, y), N_{i_0}) \longrightarrow A_1$, and let $(((P, y), N_{i_0}), A_1)$ be a contraction of trace of $(((M_q, x_q), N_q), n_q)$, and let $A_1 \longrightarrow A_2$ and let (A_1, A_2) be a contraction of trace of $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$ and some finite sequence of contractions of traces of $\{ (((M_i, x_i), N_i), n_i); i \in I_N \}$.

If μ' is altered by replacing ξ as above, then the result μ'' will have the same first and last net, will contract the same traces, the number of contractions will not be less and such that after contraction of the trace of $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$ (contraction is unique, because the trace has only one element or is empty) there exists one contraction of trace of $\{(((M_i, x_i), N_i), n_i) : i \in I_M\}$ more.

From hyphotheis we have that every sequence of contractions of traces of set $\{ (((M_i, x_i), N_i), n_i); i \in I_M \}$ is finite, so repetiting this construction we get a sequence of contractions on the form $\alpha \bullet \beta \bullet \gamma$, where

α is a vanishing sequence of contractions of traces $\big\{(((M_i, x_i),\ N_i), n_i) : i \in I_M\big\}$
β is a contraction of trace $(((M_{i_0}, x_{i_0}), N_{i_0}), n_{i_0})$
γ is a vanishing sequence of contraction of traces $\big\{(((M_i, x_i), N_i), n_i) : i \in I_N\big\}$

Moreover, the number of contractions in $\alpha \bullet \beta \bullet \gamma$ is greater or equal to a number of contractions in μ. Hence μ has at most $a + cb + 1$ contractions.

By a virtue of a Proposition 1 we have proved that for every nerual network A and every subset $B = \{ (((M_i, x_i), N_i), n_i); i \in I \}$ of $\breve{A}$ there exists $m \in \mathbb{N}$ such that every sequence of contracions B vanishes after at most m contractions.

Let now ν be a vanished sequence of contractions. Then $\alpha \bullet \beta \bullet \gamma$ as has been made is either special sequence of contractions or can became special after evident chages of elements of α and γ. By hypothesis applied to M_{i_0} and N_{i_0} (equal M and N) the result of the special sequence of contractions is unique to within relation $\longrightarrow^{\text{convI}}$. Also result of any, vanished sequence of contractions is unique to within relation $\longrightarrow^{\text{convI}}$. □

Proposition 17. *Let $A, B \in \mathcal{NN}$, let $A \longrightarrow^{\text{II}} B$, let (A, B) be a contraction of $(((M, x), N), n) \in \breve{A}$ and $A_k, B_k \in \mathcal{NN}$ for $k \in \mathbb{N}$, and $A_1 \longrightarrow^{\text{red}} A_2 \longrightarrow^{\text{red}} A_3 \longrightarrow^{\text{red}} \cdots$; let for $k \in \mathbb{N}$ B_k be a result of vanished sequence of contractions of $\text{Tr}^{(A_1,\ldots,A_k)}\{ (((M, x), N), n) \}$. Then*

i) $B_1 = B$
ii) $\forall_{k \in \mathbb{N}} (B_k \longrightarrow^{\text{convI,II}} B_{k+1})$
iii) *even, if the sequence $(A_1, A_2, \ldots)$ can by continued to infinity, there exists u_m, depended of A, $(((M, x), N), n)$ and some number m such that in the sequence $(B_m, B_{m+1}, \ldots, B_{m+u_m})$ there ist at least one contraction (more precisly, there exists $j \in \{ m, m+1, \ldots, m+u_m-1 \}$ such that (B_j, B_{j+1}) is a contraction).*

Proof. i) is obvious
ii) let $\text{Tr}^{(A_1,\ldots,A_k)}\{ (((M, x), N), n) \} = \{ (((P_i, y_i), Q_i), n_i); i \in I \} \subset \breve{A}_i$ and let $A_k \longrightarrow A_{k+1}$ be a contraction of $(((R, z), S), m)$, (more precisly, if $A_k \longrightarrow^{\text{red}} A_{k+1}$ then $A_k \longrightarrow^{\text{I}} C_{1,k} \longrightarrow^{\text{I}} \cdots \longrightarrow^{\text{I}} C_{i_k,k} \longrightarrow^{\text{II}} C_{i_k+1,k} \longrightarrow^{\text{I}} \cdots \longrightarrow^{\text{I}} A_{k+1}$ and $(C_{i_k,k}, C_{i_k+1,k})$ is a contraction of trace of $\{ (((R, z), S), m) \}$, and we have that $\#\, \text{Tr}^{(A_k, C_{1,k},\ldots,C_{i_k,k})}\{(((R, z), S), m)\} = 1$). From Theorem 3 we have that B_{k+1} is equal (with respect to relation

$\longrightarrow^{\text{convI}}$) to vanished sequence of contraction of $\text{Tr}^{(A_1,\dots,A_k)}\{(((M,x),N), n)\} \cup \{(((R,z),S),m)\}$.

If we have $(((R,z),S),m) \in \{\, (((P_i,y_i),Q_i),n_i); i \in I \,\}$, then from Theorem 3 we have that $B_k \longrightarrow^{\text{convI}} B_{k+1}$; else B_{k+1} can be made from B_k by contraction of traces of $\{\, (((R,z),S),m) \,\}$, also $B_k \longrightarrow^{\text{nred}} B_{k+1}$.

iii) from ii) we have that $B_k \longrightarrow^{\text{nred}} B_{k+1}$, unless $A_k \longrightarrow^{\text{red}} A_{k+1}$ is a contraction of traces of $(((M,x),N),n)$. But, starting from some A_k, trace of $(((M,x),N),n)$, can by contracted only a finite number of times. More precisly, let us define u_m as follows:
Consider all sequenced of reductions of letgth m starting from A. Wihtin possible relations $\longrightarrow^{\text{convI}}$ we have only a finite number of them. On every of the net, on which these sequences ended let us consider, from Theorem 3 the longest sequence of contractions of traces of $(((M,x),N),n)$ and let u_m be a largest ot these. □

Proposition 18. *Let $A, B \in \mathcal{NN}$. If $A \longrightarrow B$ then there exists a sequence of nets, which starts from A, end on B and in which no expansion precedes any reduction (more precisely, if $S_i \longrightarrow^{\text{II}} S_{i+1}$ and $S_j \longrightarrow^{\text{III}} S_{j+1}$ then $i+1 \leq j$).*

Proof. Let $A \longrightarrow^{\text{B}}$ and the last expansion, which precedes reduction and let be $B_1 \longrightarrow^{\text{exp}} A_1$. After this expansion there exist a sequence of reductions $A_1 \longrightarrow^{\text{red}} A_1 \longrightarrow^{\text{red}} \cdots \longrightarrow^{\text{red}} A_n$ and $A_n \longrightarrow^{\text{convI,III}} B$. From definition, if $B_1 \longrightarrow^{\text{exp}} A_1$ then $A_1 \longrightarrow^{\text{red}} B_1$. From Proposition 13, there exists $(((M,x),N),n) \in \breve{A}_1$ such that $A_1 \longrightarrow^{\text{red}} B_1$ is a contraction of $(((M,x),N),n)$. Let B_k (for $k \in \{2,3\dots\}$) be a result of vanished sequence of contractions $(((M,x),N),n)$ from A_k. By virtue of Proposition 17 we have $B_1 \longrightarrow^{\text{convI,II}} B_2$, $B_2 \longrightarrow^{\text{convI,II}} B_3$, $\dots$, $B_{n-1} \longrightarrow^{\text{convI,II}} B_n$, $B_n \longrightarrow^{\text{convI,III}} A_n$, $A_n \longrightarrow^{\text{convI,III}} B$, what is conversion from B_1 to B, in which no expansion procedes any reducion.

By repetitions of this process, we can in a finite number of steps replace conversion $A \longrightarrow B$ by conversion, which starts from A, ends on B and no expansion procedes any reducion. □

Proposition 19. *If B is a normal form of A then $A \longrightarrow^{\text{convI,II}} B$.*

Proof. From Proposition 18 and from that, so if B is a normal form, there doesn't exists C such that $B \longrightarrow^{\text{red}} C$, we have thesis. □

Proposition 20. *If B_1, B_2 are normal forms of A then $B_1 \longrightarrow^{\text{convI}} B_2$.*

Proof. Let B_1, B_2 be normal forms of A. Also B_2 is a normal form of B_1, so from Proposition 19, $B_1 \longrightarrow^{\text{convI,II}} B_2$, hence, because B_1 is a normal form, $B_1 \longrightarrow^{\text{convI}} B_2$. □

Proposition 21. *If A has a normal form, it has a unique principle normal form.* □

Proposition 22. *If B is a normal form of A then there exists a number m such, that every sequence of reductions starting from A ends on B unique within to relation $\longrightarrow^{\text{convI}}$ after at most m reductions.*

Lemma 3. *If B is a normal form of A and there exists a sequence of reductions letgth n starting from A, ending on B then there exists a number $v_{A,n}$ such that every sequence of reductions starting from A, ends on a normal form of A after at most $v_{A,n}$ reducitons*

Proof. Induction on n. 1° If $n = 0$, we put $V_{A,0} = 0$.
2° Suppose by induction that lemma is true for $n = k$. Also let

$$A \xrightarrow{\text{red}} C \xrightarrow{\text{red}} C_1 \xrightarrow{\text{red}} \cdots \xrightarrow{\text{red}} C_{k-1} \xrightarrow{\text{red}} B$$

Let $A_1 = A$ and

$$A_1 \xrightarrow{\text{red}} A_2 \xrightarrow{\text{red}} A_3 \xrightarrow{\text{red}} \cdots$$

From Proposition 17 we have that there exists a sequence $(D_1 = C)$, $D_1 \longrightarrow^{\text{convI,II}} D_2$, $D_2 \longrightarrow^{\text{convI,II}} D_3$, ...such that $A_j \longrightarrow^{\text{convI,II}} D_j$ for all these j, for which there exists A_j and if reduction from A to C is a contraction of $(((M, x), N), n)$ then in the sequence $(D_m, D_{m+1}, \ldots D_{m+u_m})$ there exist at least one reduction.
Because a sequence of reductions $(C, C_1, C_2, \ldots)$ ends on B after k reducitons, so from hypothesis there exists $v_{C,k}$ such that any sequence of reductions starting from C ends after at most $v_{C,k}$ reductions. Hence a sequence $(D_1, D_2, \ldots)$ consists at most $v_{C,k}$ reductionss and have to end after at most $f(v_{C,k})$ steps, where f is defined as follows:

$$\begin{aligned} f(0) &= u_1 \\ f(x+1) &= f(x) + \max\{\, u_1, \ldots u_{f(x)+1} \,\} + 1. \end{aligned}$$

Since the sequence (D_j) has the same length as (A_j), so the sequence (A_j) vahishes after at most $f(v_{C,k})$ steps that means, there are no further reductions possible, also the sequence (A_j) ends on normal form.
If now $\mathfrak{A}$ is a set of all nets C such that $A \longrightarrow^{\text{red}} C$, unique within to relation $\longrightarrow^{\text{convI}}$ then because $\#\mathfrak{A} < \aleph_0$, putting $v_{A,k+1} = \max\{\, f(v_{C,k}); C \in \mathfrak{A} \,\}$ we have the right number. □

Proof. From Proposition 19, if B is a normal form of A then $A \longrightarrow^{\text{convI,II}} B$, also lemma implies thesis. □

Proposition 23. *If $A \in \mathcal{NN}$ has a normal form and $(B, m) \in \breve{A}$ then B has a normal form.*

Proof. If there exists a sequece of reductions of B, which does not terminate then there is a sequence of reductions of A, which does not terminate, what is an absurd because of Proposition 22 □

3. Net–definablity

Proposition 24. *Let S, S' be neural networks, $\widehat{S} = (T, \leq_T, f_T)$, $\widehat{S'} = (T', \leq_{T'}, f_{T'})$. If $S \longrightarrow^{\mathrm{convI}} S'$ then $(T, \leq_T) = (T', \leq_{T'})$.* □

Definition 34. *Let us denote a function $[\,]\colon \mathbb{N} \to \mathcal{N}N$ as follows:*

$$\begin{aligned}
[1] &= ((((a,\emptyset),(b,\emptyset),b),a)\\
[2] &= ((((a,\emptyset),((a,\emptyset),(b,\emptyset))),b),a)\\
[3] &= ((((a,\emptyset),((a,\emptyset),((a,\emptyset),(b,\emptyset)))),b),a)\\
&\vdots
\end{aligned}$$

Definition 35. *Let us define a function $S\colon \mathcal{N}N \to \mathcal{N}N$ by*

$$S(T) = ((((((b,\emptyset),((((a,\emptyset),(b,\emptyset)),(c,\emptyset)))),c),b),a),T)$$

Proposition 25. *For any $n \in \mathbb{N}$ we have $S([n]) \longrightarrow^{\mathrm{convI,II}} [n+1]$* □

Definition 36. *Let M, N be neural networks. Let us define: $+, \cdot, pow\colon \mathcal{N}N \times \mathcal{N}N \to \mathcal{N}N$ by*

$$\begin{aligned}
M + N &= ((((M,(a,\emptyset)),((N,(a,\emptyset)),(b,\emptyset))),b),a)\\
M \cdot N &= ((M,(N,(a,\emptyset))),a)\\
M pow N &= (N, M).
\end{aligned}$$

Instead of write $MpowN$ we shall write M^N.

Proposition 26. *For any $m, n \in \mathbb{N}$ we have:*

i) $[m] + [n] \longrightarrow [m+n]$
ii) $[m] \cdot [n] \longrightarrow [m \cdot n]$
iii) $[m]^{[n]\cdot[k]} \longrightarrow [m]^{[n]} + [m]^{[n]}$
iv) $[m]^{[n]} \longrightarrow [m^n]$

□

Proposition 27. *Let $M, N, K \in \mathcal{N}N$ and let $M \longrightarrow N$. We have:*

i) $M + K \longrightarrow N + K$
ii) $K + M \longrightarrow K + N$
iii) $M \cdot K \longrightarrow N \cdot K$
iv) $K \cdot M \longrightarrow K \cdot N$
v) $M^K \longrightarrow N^K$
vi) $K^M \longrightarrow K^N$

□

Proposition 28. *Let $m, n, k \in \mathbb{N}$. We have:*

i) $([m] + [n]) + [k] \longrightarrow [m] + ([n] + [k])$
ii) $([m] \cdot [n]) \cdot [k] \longrightarrow [m] \cdot ([n] \cdot [k])$
iii) $([m] + [n]) \cdot [k] \longrightarrow ([m] \cdot [k]) + ([n] \cdot [k])$
iv) $[m]^{([n] \cdot [k])} \longrightarrow ([m]^{[n]})^{[k]}$
v) $S([m]) \longrightarrow [m] + [1]$

□

Definition 37. *Let f be a function $f \colon A \to \mathbb{N}$, where $A \subset N$. We shall call f to be net–definable if there exists $[f] \in \mathcal{N}N$ such that for every $x \in A$ is $([f], [x]) \longrightarrow [f(x)]$ and for every $x \notin A$ $([f], [x])$ has no normal form.*

Remark 4. *If $f \colon A \to \mathbb{N}$, where $A \subset \mathbb{N}$ is net–definable and for some $x, y \in \mathbb{N}$ is $([f], [x]) \longrightarrow [y]$ then because for every $a \in \mathbb{N}$ is $[a] \in F\mathcal{N}N$, from Proposition 21 we have that $f(x) = y$.*

Definition 38. *Let $f \colon A \to \mathbb{N}$, where $A \subset \mathbb{N} \times \mathbb{N}$. We shall call f to be net–definable if there exists $[f] \in \mathcal{N}N$ such that for every $(x, y) \in A$ is $(([f], [x]), [y]) \longrightarrow [f(x, y)]$ and for every $(x, y) \notin A$ $(([f], [x]), [y])$ has no normal form.*

Definition 39. *Let us define $I \in \mathcal{N}N$ by $I = ((a, \emptyset), a)$.*

Proposition 29. *For every $M \in \mathcal{N}N$ is $(I, M) \longrightarrow M$ and for every $n \in \mathbb{N}$ is $([n], I) \longrightarrow I$.* □

Definition 40. *Let $L, M, N \in \mathcal{N}N$. We shall denote:*

i) $\langle M, N \rangle = ((((a, \emptyset), M), N), a)$
ii) $\langle L, M, N \rangle = (((((a, \emptyset), L), M), N), a)$
iii) $2_1 = (((a, \emptyset), (((((c, \emptyset), I), (c, \emptyset)), c), b)), a)$
iv) $2_2 = (((a, \emptyset), (((((b, \emptyset), I), (c, \emptyset)), c), b)), a)$
v) $3_1 = (((a, \emptyset), ((((((((c, \emptyset), I), (d, \emptyset)), I), (b, \emptyset)), d), c), b)), a)$
vi) $3_2 = (((a, \emptyset), ((((((((b, \emptyset), I), (b, \emptyset)), K), (c, \emptyset)), d), c), b)), a)$
vii) $3_3 = (((a, \emptyset), ((((((((b, \emptyset), I), (c, \emptyset)), I), (d, \emptyset)), d), c), b)), a)$.

Proposition 30. *Let $l, m, n \in \mathbb{N}$. Then*

i) $(2_1, \langle [m], [n] \rangle) \longrightarrow [m]$
ii) $(2_2, \langle [m], [n] \rangle) \longrightarrow [n]$
iii) $(3_1, \langle [l], [m], [n] \rangle) \longrightarrow [l]$
iv) $(3_1, \langle [l], [m], [n] \rangle) \longrightarrow [m]$
v) $(3_1, \langle [l], [m], [n] \rangle) \longrightarrow [n]$

□

Definition 41. *Let us denote a neural network P by*

$$P = ((3_3, (((a, \emptyset), (\langle S((3_1, b)), (3_1, b), (3_2, b)\rangle, b)), \langle [1], [1], [1]\rangle)), a).$$

Proposition 31. *We have: $(P, [1]) \longrightarrow [1]$ and $(P, S([n])) \longrightarrow [n]$ for $n \in \mathbb{N}$, so if $P\colon \mathbb{N} \to \mathbb{N}$ is a function 'previous' then for every $n \in \mathbb{N}$ is $(P, [n]) \longrightarrow [P(n)]$.*

Proof. It is easy to check that for $k, l, m \in \mathbb{N}$ is:

$$((\langle S((3_1, b)), (3_1, b), (3_2, b)\rangle, b), \langle [k], [l], [m]\rangle) \longrightarrow \langle S(([k])), [k], [l]\rangle \longrightarrow \langle [k+1], [k], [l]\rangle .$$

Similary, we check that $(P, [1]) \longrightarrow [1]$ and $(P, [2]) \longrightarrow [1]$. Let us denote by $A = (\langle S((3_1, b), (3_1, b), (3_2, b)\rangle, b)$. For every $a \in \mathbb{N}$, if for every $n \in \mathbb{N}$ is

$$(([n], A), \langle [1], [1], [1]\rangle) \longrightarrow \langle [a], [b], [c]\rangle ,$$

then

$$(([n+1], A), \langle [1], [1], [1]\rangle) \longrightarrow \langle [a+1], [a], [b]\rangle ,$$

where $a, b, c \in \mathbb{N}$. So $(P, [1]) \longrightarrow [1]$, $(P, [2]) \longrightarrow [1]$, $(P, [3]) \longrightarrow [3]. \ldots$ □

Remark 5. *A. Church [1] writes that this method is due to S. C. Kleene [3].*

Definition 42. *Let us define a function $\dot{-}\colon \mathcal{N}N \times \mathcal{N}N \to \mathcal{N}N$ by*

$$M \dot{-} N = ((N, P), M).$$

Proposition 32. *For any $m, n \in \mathbb{N}$ is:*

$$[m] \dot{-} [n] = \begin{cases} [1], & \text{if } m \leq n; \\ [m-n], & \text{if } m > n. \end{cases}$$

□

Definition 43. *Let us define* $\min \in \mathcal{N}N$ *and* $\max \in \mathcal{N}N$ *as follows:*

$$\begin{aligned} \min &= (((S((b, \emptyset)) \dot{-} (S((b, \emptyset)) \dot{-} (a, \emptyset))), b), a) \\ \max &= (((a + b \dot{-} ((\min, (a, \emptyset)), (b, \emptyset))), b), a). \end{aligned}$$

Proposition 33. *For any $m, n \in \mathbb{N}$ is:*

i) $((\min, [m]), [n]) \longrightarrow [\min(m, n)]$
ii) $((\max, [m]), [n]) \longrightarrow [\max(m, n)]$.

□

Definition 44. *Let us define* $\mathrm{par} \in \mathcal{N}N$ *by*

$$\mathrm{par} = ((((a, \emptyset), ((3 \dot{-} (b, \emptyset)), b)), [2]), a).$$

Proposition 34. *For any $m \in \mathbb{N}$ we have:*

$$(\mathrm{par}, [m]) \longrightarrow \begin{cases} [1], & \text{if } m \text{ is odd;} \\ [2], & \text{if } m \text{ is even.} \end{cases}$$

□

Definition 45. *Let us define a neural network $H \in \mathcal{NN}$ by*

$$H = ((P, (2_1, (((a, \emptyset), (< (P, ((2_1, b) + (2_2, b))), [3] \dot{-} (2_2, b) >, b)), < [1], [2] >))), a).$$

Proposition 35. *Let $m \in \mathbb{N}$ and p be the least number not less than $m/2$. Then*

$$(H, [m]) \longrightarrow [p].$$

□

Definition 46. *Let us denote nerual networks $\alpha, U, Z, Z' \in \mathcal{NN}$ by*

$$\begin{array}{ll} \alpha & = (((b, \emptyset), ((\langle A, B\rangle, d), c)), b), \quad \text{where} \\ A & = (((((d, \emptyset), P), (c, \emptyset), ((((e, \emptyset), [1]), I), e)), (c, \emptyset)), \\ B & = ((((((d, \emptyset), P), (c, \emptyset)), (((((h, \emptyset), [1]), I), S), h)), \\ & \quad (((((((k, \emptyset), (i, \emptyset)), (j, \emptyset)), (((l, \emptyset), [1]), l)), k), j), i)), (d, \emptyset)) \\ U & = ((((a, \emptyset), \alpha), \langle [1], [1] \rangle), a) \\ Z & = ((2_2, (U, (a, \emptyset))), a) \\ Z' & = ((U, (a, \emptyset)), ((b \dot{-} c, c), b)), a) \end{array}$$

Proposition 36. *For any $m, n \in \mathbb{N}$ is:*

$$(\alpha, \langle [m], [n] \rangle) \longrightarrow \begin{cases} \langle S([m]), [1] \rangle, & \text{if } m \dot{-} n = 1; \\ \langle [m], [S(n)] \rangle, & \text{if } m \dot{-} n > 1; \end{cases}$$

□

Proposition 37. *Let $(U, [k]) \longrightarrow \langle [x], [y] \rangle$. Then*

$$(U, [k+1]) \longrightarrow \begin{cases} \langle [x], [y+1] \rangle, & \text{if } x \dot{-} y > 1; \\ \langle [x+1], [1] \rangle, & \text{if } x \dot{-} y = 1. \end{cases}$$

and

$$(Z, [k]) \longrightarrow [x],$$
$$(Z', [k]) \longrightarrow [x \dot{-} y]$$

for $k, x, y \in \mathbb{N}$. □

Proposition 38. *A sequance $(a_i)_{i=1}^{\infty}$, where*

$$a_1 = \big\langle (Z, [i]), (Z', [i])) \big\rangle$$

contains all ordered pairs without repetitions. □

Definition 47. *Let us define a nerual network* $nr \in \mathcal{NN}$ *by*

$$nr = (((S,(H,(((a,\emptyset)+(b,\emptyset))\cdot(P,((a,\emptyset)+(b,\emptyset)))))) \dot{-} b),a).$$

Proposition 39. *For any* $m,n \in \mathbb{N}$ *is*

$$\begin{aligned}(Z,((\mathrm{nr},[m]),[n])) &\longrightarrow [m],\\ (Z',((\mathrm{nr},[m]),[n])) &\longrightarrow [n].\end{aligned}$$

□

Proposition 40. *The following functions:*

i) $\mathbb{N} \ni n \mapsto S(n) \in \mathbb{N}$,
ii) $\mathbb{N}\times\mathbb{N} \ni (m,n) \mapsto m+n \in \mathbb{N}$,
iii) $\mathbb{N}\times\mathbb{N} \ni (m,n) \mapsto mn \in \mathbb{N}$,
iv) $\mathbb{N}\times\mathbb{N} \ni (m,n) \mapsto m^n \in \mathbb{N}$,
v) $\mathbb{N}\times\mathbb{N} \ni (m,n) \mapsto m \dot{-} n \in \mathbb{N}$,
vi) $\mathbb{N}\times\mathbb{N} \ni (m,n) \mapsto \mathrm{nr}(m,n) \in \mathbb{N}$,

are net–definable. □

Definition 48. *Let* f *(on domain* A*) be a propositional function. A function* $F\colon A \to \mathbb{N}$ *is said to be its characteristic function, if*

$$\forall_{a\in A}\big((F(a)=2 \iff f(a)),(F(a)=1 \iff \sim f(a))\big).$$

Definition 49. *The propositional function is said to be net–definable if its charakteristic funcion is a net–definable function.*

Proposition 41. *Propositional functions:* x *is less than* y *and* x *is equal to* y *are net–definable.*

Proof. The neural network

$$M = ((((\min,[2]),S(a)\dot{-} b),b),a)$$

defines the charakteristic function of the former and net

$$(((4 \dot{-} ((M,(a,\emptyset)),(b,\emptyset)) + ((M,(b,\emptyset)),(a,\emptyset))),b),a)$$

the latter. □

Proposition 42. *Let* R *be a net–definable propositonal function with domain contained in* $\mathbb{N}^{n+1}$*, such that*

$$\forall_{x_1,\ldots,x_n\in\mathbb{N}}\exists_y\left(R(x_1,\ldots,x_n,y),\forall_{y>z\in\mathbb{N}}\big((x_1,\ldots,x_n,z)\in A\big)\right).$$

Then function $F\colon \mathbb{N}^n \to \mathbb{N}$ *such that* $F(x_1,\ldots,x_n)=y$*, where* y *is the least number such that* $R(x_1,\ldots,x_n,y)$ *is net–definable.*

Proof. Let us denote

$$C = (((((n,\emptyset),\mathcal{R}),\mathcal{F}),X),n),$$

where

$$\begin{aligned}
\mathcal{R} \quad &= (((r,\emptyset),((((((s,\emptyset),[1]),I),I),(((((((g,\emptyset),[1]),\\
&\qquad ((t,\emptyset),(x,\emptyset))),I),(x,\emptyset)),t),g),x)),s)),r)\\
\mathcal{F} \quad &= ((((((f,\emptyset),I),[1]),I),I),f)\\
X \quad &= ((((((((g,\emptyset),((t,\emptyset),(S,(x,\emptyset)))),(S,(x,\emptyset))),(g,\emptyset)),(t,\emptyset)),t),g),x).
\end{aligned}$$

It is easy to check that

$$(C,[1]) \longrightarrow X,$$

and

$$(C,[2]) \longrightarrow ((((((((g,\emptyset,[1]),((t,\emptyset),(x,\emptyset))),I),(x,\emptyset)),t),g),x).$$

Also, if $n \in \mathbb{N}$, $T \in \mathcal{N}N$ and $(T,[n]) \longrightarrow [1]$, or $(T,[n]) \longrightarrow [2]$ then

$$((((C,[1]),[n]),C),T) \longrightarrow ((((C,(T,(S,[n]))),(S,[n])),C),T)$$

More,

$$((((C,[2]),[n]),C),T) \longrightarrow [n].$$

Also, if we put

$$p = ((((((C,((t,\emptyset),(x,\emptyset))),(x,\emptyset)),C),(t,\emptyset)),x),t),$$

then we have that

$$((p,T),[n]) \xrightarrow{\text{red}} \begin{cases} n, & \text{if } (T,[n]) \longrightarrow [2];\\ ((p,T),(S,[n])), & \text{if } (T,[n]) \longrightarrow [1]. \end{cases}$$

and from Proposition 23, if $(T,[n])$ has no normal form then also $((p,T),[n])$ has no normal form. If t is a propositional function of one natural variable, net–definable by a net T then $((p,T),[n]) \longrightarrow [y]$, where y is the least number not less than n, such that $t(y)$ (if we suppose that there exists such a y, not less than n that $t(y)$ and the set $\{\, n, n+1, \ldots, y \,\}$ is contained in domain of the function t) and in no other case $((p,T),[n])$ has a normal form. Indeed, if $\sim t(y)$ then for all y not less than n is:

$$\begin{aligned}
((p,T),[n]) \qquad &\xrightarrow{\text{red}} ((((C,(T,[n])),[n]),C),T)\\
&\xrightarrow{\text{red}} ((((C,(T,(S,[n]))),(S,[n])),C),T)\\
&\xrightarrow{\text{red}} ((((C,(T,(S,(S,[n])))),(S,(S,[n]))),C),T)\\
&\xrightarrow{\text{red}} \ldots
\end{aligned}$$

to infinity, hence $((p,T),[n])$ has no normal form.

Let $[R]$ be a net, which net–defines the propositonal function R referred to in Proposition. Then putting

$$[F] = ((p, ((([R], x_1), \dots, x_n)), [1])$$

we have thesis. □

Remark 6. *A. Church [1] says, this Proposition is from S. C. Kleene [3], and a formula p has been discovered also by S. C. Kleene.*

Definition 50. *We say, the function F of n variables to be defined by compositon of functions G and $H_1, \dots, H_m$, if*

$$F(x_1, \dots, x_n) = G(H_1(x_1, \dots, x_n), \dots, H_m(x_1, \dots, x_n)).$$

Definition 51. *We shall say, the function F of $n+1$ variables to be defined by primitive recursion of G_1 and G_2, if*

$$\begin{aligned} F(x_1, \dots, x_n, 1) &= G_1(x_1, \dots, x_n), \\ F(x_1, \dots, x_n, y+1) &= G_2(x_1, \dots, x_n, y, F(x_1, \dots, x_n, y)). \end{aligned}$$

Definition 52. *The class of primitive recursive functions is the least class of funcions of natural numbers and with natural values, consisting constant function, equal to one, the function succesor, the projection on i'th factor and such that, if functions $G, H_1, \dots, H_m$ are primitive recursive and there is possible to composite then composition is primitive recursive and if G_1 and G_2 are primitive recursive and F is defined by primitive recursion of G_1 and G_2 then F is primitive recursive.*

Remark 7. *This definition is from Church [1].*

Definition 53. *The class of computable funcions is the least class of funcitons, of natural numberes and with natural values, which satisfies following statements:*

1° *The function $S(x) = x + 1$, functions $I(x, y) = x$ and $I'(x, y) = y$ and constant function $1(x) = 1$ are computable.*

2° *If two functions f and $g_1, \dots g_m$ are computable then funciton h which is composition of f and $g_1, \dots g_m$ is computable.*

3° *If two functions f and g are computable then function h which is given by a primitive recursion schema of functions f and g is computable.*

4° *If function f is computable and there is satisfied the condition of efectivity:*

$$\forall_u \exists_x \big(f(u, x) = 1\big),$$

then function h defined by efective minimum:

$$h(u) = (\mu x)\big(f(u, x) = 2\big)$$

is computable.

Remark 8. *This definitions is a bit modified from Grzegorczyk [2], about which he writes that is from S. C. Kleene* Recursive predicates and quantifiers, *Trans. Amer. Math. Soc. 53 (1943), pp 41–73.*

Proposition 43. *Every class X of functions consisting functions $\mathbb{N} \ni x \mapsto x+1 \in \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \ni (x,y) \mapsto x \dot{-} y \in \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \ni (x,y) \mapsto x^y \in \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \ni (x,y) \mapsto \max(x,y) \in \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \ni (x,y) \mapsto \min(x,y) \in \mathbb{N}$, closed with respect to:*

a) *composition*
b) *efective minimum*

is also closed with respect to operation of primitive recursion.

Proof. First, let us define some functions and relations:

$$\begin{aligned}
1 \quad & = x \dot{-} x, \\
2 \quad & = S(1), \\
xy \quad & = (\mu z)(2^z = (2^x)^y), \\
x + y \quad & = (\mu z)(2^z = 2^x 2^y), \\
x < y \quad & \iff \min(2, S(x) \dot{-} b) = 2, \\
4 \quad & = S(S(S(1))), \\
x = y \quad & \iff 4 \dot{-} (\min(2, S(x) \dot{-} y) + \min(2, S(y) \dot{-} x)) = 2.
\end{aligned}$$

If f and g are characteristic functions of R and S then

$$\begin{aligned}
S \circ S \circ 1 \dot{-} f \quad & \text{is a characteristic function of} \quad \sim R, \\
\min(f,g) \quad & \text{is a characteristic function of} \quad R \vee S, \\
\max(f,g) \quad & \text{is a characteristic function of} \quad R, S.
\end{aligned}$$

Furthermore, following equivalence is true:

$$\forall_{x<n}\big(\sim R(x), R(n)\big) \iff n = (\mu x)\big(R(x)\big).$$

Let us denote by

$$\left[\frac{x}{y}\right] = (\mu z)(y(z+1) > x),$$

and

$$x|y \iff x\left[\frac{y}{x}\right] = y.$$

Let us put

$$\mathrm{pr}(x) \iff \Big(\sim(x=1), x = (\mu z)\big(x = 1 \vee (z > 1, z|x)\big)\Big).$$

In such a definition $\mathrm{pr}(x)$ means: x is a prime number.

Let us define a function

$$\exp(x,y) = (\mu z)\Big(y = 1 \vee \sim\big(y^{z+1}|x\big)\Big).$$

In such a definition $\exp(x,y)$ denotes the greatest number y such that $y^{\exp(x,y)}|x$.

Let us denote also

$$\mathrm{pri}(z) = \big(\mu y\big)\Bigg(\sim \mathrm{pr}(z) \vee \Bigg(y > 1, \exp(y,2) = 2,$$

$$z = (\nu v)\Bigg(\mathrm{pr}(v), \Big(y = 1 \vee \exp\big(y, (\mu u)(\mathrm{pr}(u), u > v)\big) \neq \exp(y,v) + 1\Big)\Bigg)\Bigg)\Bigg)$$

Then

$$\mathrm{pri}(y) = \begin{cases} 1, & \text{if } y \text{ is not a prime number;} \\ 2^2 3^3 5^4 \dots z^{n+1}, & \text{if } z \text{ is a } n\text{'th prime number.} \end{cases}$$

Let now

$$p_n = (\mu z)\Big(\mathrm{pr}(z), \exp\big(\mathrm{pri}(z), z\big) = n + 1\Big).$$

By virtue of the properties of a function pri, p_n is n'th prime number. Suppose that function h is defined of f and g be schema of primitive recursion:

$$\begin{aligned} h(x,1) \quad &= f(x) \\ h(x,n+1) \quad &= g(x,n,h(x,n)) \end{aligned}$$

Then $u = h(x,u)$ if and only if exists such a number m, that

1° $\exp(m,z) = f(x)$,
2° $\exp(m,p_{i+1}) = g(x,i,\exp(m,p_i))$, dla $i < n$
3° $\exp(m,p_n) = u$.

This number m is

$$m = 2^{h(x,1)} 3^{h(x,2)} \cdots p_n^{h(x,n)}.$$

So $h(x,n) = \exp(m,p_n)$, where m is the least number satifing conditions 1°–3°. Hence h can be defined as follows:

$$h(x,u) = \exp\Bigg(\big(\mu m\big)\Bigg(\exp(m,2) = f(x),$$

$$n = (\mu i)\Big(\exp(m,p_{i+1}) \neq g\big(x,i,\exp(m,p_i)\big)\Big)\Bigg), p_n\Bigg)$$

□

Remark 9. *This proof is a bit modification of Grzegorczyk [2] pp 351–355.*

Theorem 10. *Every computable function is a net–definable function.*

Proof. Proof is immediately from definition 53 and Propositions 43, 40 and 42 □

Proposition 44. *Every primitive recursive function is a net–definable function.* □

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