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A NOTE ON $\mathcal{I}$-DENSITY CONTINUOUS FUNCTIONS

GRAŻYNA HORBACZEWSKA‡

Abstract. Any function from $\mathbb{R}^2$ to $\mathbb{R}^2$, which is continuous with respect the $\mathcal{I}$-density topology on the domain and on the range is a function of the Baire*1 class.

The terminology and notation we use is standard. In particular $\mathbb{R}$ denotes the real line, $\mathbb{R}^2$ – the plane, and $\mathbb{N}$ – the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$ – the set of integers.

For $(x, y) \in \mathbb{R}^2$, $r \in \mathbb{R}$, $r > 0$, $K((x, y), r)$ stands for the open disk centered at a point (x, y) with radius r and $S((x, y), r)$ – for the circle.

We denote by $\mathcal{I}$ the σ-ideal of meager sets on the plane, by $\mathcal{B}$ – the family of plane sets having the Baire property. Any set $A \in \mathcal{B}$ can be represented uniquely in the form $A = G \triangle P$, where G is a regular open set (e.g. it is equal to the interior of its closure). We denote this „regular open part" of A by $\tilde{A}$.

We recall the notion of an $\mathcal{I}$-density point on the plane ([3]). If $A \subset \mathbb{R}^2$ we denote $(n, m) \cdot A = \{(nx, my) : (x, y) \in A\}$. The point $(0, 0)$ is an $\mathcal{I}$-density point of a set $A \subset \mathbb{R}^2$, $A \in \mathcal{B}$, if $\chi_{((n,n)\cdot A)\cap[-1,1]^2}$ converges to $\chi_{[-1,1]^2}$ with respect to $\mathcal{I}$ (i.e. for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ for which $\chi_{((n_{m_p},n_{m_p})\cdot A)\cap[-1,1]^2}$ converges to $\chi_{[-1,1]^2}$ except on a set belonging to $\mathcal{I}$), where χ_S is the characteristic function of a set S.

In the definition of $\mathcal{I}$-density point we can replace an arbitrary increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers by an arbitrary increasing sequence of positive numbers tending to infinity ([7]).

A point (x_0, y_0) is an $\mathcal{I}$-density point of $A \subset \mathbb{R}^2$, $A \in \mathcal{B}$, if $(0, 0)$ is an $\mathcal{I}$-density point of the set $A - (x_0, y_0) = \{(x - x_0, y - y_0) : (x, y) \in A\}$.

The set of all $\mathcal{I}$-density points of A is written $\Phi_{\mathcal{I}}(A)$.

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‡*Faculty of Mathematics, University of Łódź*, Banacha 22 St., 90–238 Łódź, Poland. E-mail: grhorb@math.uni.lodz.pl.

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A point (x_0, y_0) is an $\mathcal{I}$-dispersion point of A, if $(x_0, y_0) \in \Phi_{\mathcal{I}}(A^c)$, where A^c denotes the complement of A.

It is easily seen that $(x_0, y_0) \in \Phi_{\mathcal{I}}(A)$, for $A \in \mathcal{B}$, if and only if for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\limsup_p (n_{m_p}, n_{m_p}) \cdot (A^c - (x_0, y_0)) \cap [-1, 1]^2 \in \mathcal{I}$$

or, equivalently

$$\limsup_p (n_{m_p}, n_{m_p}) \cdot (A^c - (x_0, y_0)) \cap K((0, 0), 1) \in \mathcal{I}.$$

The $\mathcal{I}$-density topology on the plane is defined as $d_{\mathcal{I}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}(A)\}$ ([8]).

A function from $\mathbb{R}^2$ to $\mathbb{R}^2$, which is continuous with respect to $d_{\mathcal{I}}$ on the domain and the range is called $\mathcal{I}$-density continuous ([4]). A class of $\mathcal{I}$-density continuous functions will be denoted by $C_{\mathcal{I}\mathcal{I}}$.

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is in the class Baire*1 ($f \in B^*1$) if for each perfect set $P \subset \mathbb{R}^2$, there is a portion Q of P ($Q = K \cap P \neq \emptyset$, where K is an open disk) such that $f|Q$ is continuous ([6]). It is easily seen that every continuous function (with respect to the ordinary topology on the domain and on the range) belongs to the Baire*1 class and this class is essentially smaller than Baire 1 class ($B^*1 \subset B1$).

For a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a set $A \subset \mathbb{R}^2$ we define

$$\omega(f, A) = \text{diam } f(A)$$

and, for a point $(x_0, y_0) \in \mathbb{R}^2$:

$$\omega(f, (x_0, y_0)) = \lim_{r\to 0^+} \omega(f, K((x_0, y_0), r)).$$

It is easily seen that a function f is continuous at a point (x_0, y_0) if and only if $\omega(f, (x_0, y_0)) = 0$.

We will need the following lemma (compare [8], Theorem 2, for one-dimensional case).

Lemma 1. *There exists an open set* $E = \bigcup_{n=1}^{\infty} P(a_n, b_n)$*, where* $P(a_n, b_n) = K((0, 0), b_n) - \overline{K((0, 0), a_n)}$*,* $\{b_n\}_{n\in\mathbb{N}}$ *tends to zero,* $a_{n+1} < b_{n+1} < a_n$ *for all* $n \in \mathbb{N}$*, such that* $(0, 0)$ *is an* $\mathcal{I}$*-dispersion point of* E*.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be two decreasing sequences tending to zero and such that

(i) $a_{n+1} < b_{n+1} < a_n$ for $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = 0$,

(iii) $\lim_{n\to\infty} \frac{a_n - b_{n+1}}{a_n} = 1$.

(It suffices to take for example an arbitrary $b_1 > 0$ and then $a_n = \frac{n}{n+1} b_n$, $b_{n+1} = \frac{1}{n} a_n$:

(i) is obvious,

(ii) $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$,

(iii) $\lim_{n\to\infty} \frac{b_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0$).

Put $E = \bigcup_{n=1}^{\infty} P(a_n, b_n)$. Let $\{n_m\}_{m\in\mathbb{N}}$ be an arbitrary increasing sequence of natural numbers. We want to show that there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\chi_{((n_{m_p}, n_{m_p})\cdot E)\cap K((0,0),1)} \underset{p\to\infty}{\longrightarrow} 0,$$

$\mathcal{I}$-a.e. on $K((0,0),1)$. For each natural number m, let

$$k_m = \min\{i \in \mathbb{N} : (n_m, n_m) \cdot P(a_i, b_i) \subset K((0,0),1)\}.$$

The sequence $\{n_m \cdot b_{k_m}\}_{m\in\mathbb{N}}$ is bounded (by 1), so it contains a convergent subsequence $\{n_{m_p} \cdot b_{k_{m_p}}\}_{p\in\mathbb{N}}$. Let $b = \lim_{p\to\infty} n_{m_p} \cdot b_{k_{m_p}}$. Two cases are possible.

1° $b = 0$. Then observe that $\liminf_{p\to\infty} n_{m_p} a_{k_{m_p}-1} \geq 1$. Indeed, if

$$\liminf_{p\to\infty} n_{m_p} a_{k_{m_p}-1} < 1,$$

then from (ii) it follows that for infinitely many p's we should have $n_{m_p} \cdot b_{k_{m_p}-1} < 1$. This is a contradiction with the definition of k_{m_p} as the smallest natural number i such that $n_{m_p} \cdot b_i < 1$. So $\limsup_{p\to\infty}((n_{m_p}, n_{m_p}) \cdot E) \cap K((0,0),1) = \emptyset \in \mathcal{I}$ i.e.,

$$\bigcap_{p=1}^{\infty} \bigcup_{r=p}^{\infty} ((n_{m_r}, n_{m_r}) \cdot E) \cap K((0,0),1) = \emptyset \in \mathcal{I},$$

because for every $p \in \mathbb{N}$, $r \geq p$:

$$\begin{aligned} &((n_{m_r}, n_{m_r}) \cdot E) \cap K((0,0),1) = \\ &= ((n_{m_r}, n_{m_r}) \cdot \bigcup_{i=k_{m_r}-1}^{\infty} P(a_i, b_i)) \cap K((0,0),1) = \\ &= ((n_{m_r}, n_{m_r}) \cdot \bigcup_{i=k_{m_r}}^{\infty} P(a_i, b_i)) \cup P(n_{m_r} \cdot a_{k_{m_r}-1}, 1). \end{aligned}$$

Thus,

$$\{\chi_{((n_{m_p}, n_{m_p})\cdot E)\cap K((0,0),1)}\}_{p\in\mathbb{N}}$$

converges to 0 at every point of $K((0,0),1)$.

2° $b > 0$. From (iii) it follows that $\lim_{n\to\infty} \frac{a_{n-1}}{b_n} = +\infty$. Hence, in this case:

$$\lim_{p\to\infty} n_{m_p} a_{k_{m_p}-1} = +\infty,$$

and

$$\lim_{p\to\infty} n_{m_p} a_{k_{m_p}} = b \qquad \text{(by (ii))},$$

$$\lim_{p\to\infty} n_{m_p} b_{k_{m_p}+1} = 0 \qquad \text{(by (iii))}.$$

Thus

$$\limsup_{p\to\infty} ((n_{m_p}, n_{m_p}) \cdot E) \cap K((0,0),1) \subset S((0,0),b) \in \mathcal{I},$$

so $(0,0)$ is an $\mathcal{I}$-dispersion point of E. □

For a set A and $n \in \mathbb{N}$ define

$$[A]^n = \{B \subset A;\ \text{card } B = n\}.$$

Lemma 2. (Ramsey's Theorem ([KM])) *If $n, k \in \mathbb{N}$ then for every $F : [\mathbb{N}]^n \to \{1, 2, \ldots, k\}$ there exists an infinite set $B \subset \mathbb{N}$ such that F is constant on $[B]^n$.*

The main result of this paper is

Theorem 1. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathcal{I}$-density continuous, then f is in Baire^{*1} class.*

The same result (it means belonging to Baire^{*1} class) was obtained for the case of a density continuous function and an $\mathcal{I}$-density continuous function $f : \mathbb{R} \to \mathbb{R}$ ([2], [1]). The idea of our proof comes from [1].

Proof. Assume to the contrary that for some perfect set P, $P \neq \emptyset$, the set

$$Z = \{x \in P : \quad f|P \text{ is not continuous at } x\}$$

is dense in P.

We will show there exists a point $x \in P$ such that f is not $\mathcal{I}$-density continuous at x.

We will construct sequences: $\{x_n\}_{n\in\mathbb{N}}$ of points of P, $\{K(a_n, r_n)\}_{n\in\mathbb{N}}$ of open disks, $\{I_n\}_{n\in\mathbb{N}}$ of open disks, $\{J_n\}_{n\in\mathbb{N}}$ of closed disks having the same midpoint as the corresponding I_n, and containing that corresponding I_n. The construction is inductive, and aimed at having all the objects obtained satisfy the conditions (a) through (f) listed below.

We start by choosing $x_0 \in Z$, $K(a_0, r_0) = K(x_0, 1)$, $I_0 = J_0 = \emptyset$. Assume that for all $i \in \mathbb{N}$, $i \leq n$, where n is arbitrarily chosen, the following conditions hold:

(a) $f(x_i) \in I_i \subset J_i$ $(i \neq 0)$;

(b) $J_{i-1} \cap J_i = \emptyset$ and

$$\text{diam } J_i \leq \frac{1}{3} \min\{\text{dist}(J_k, J_{k+1}) : k \in \mathbb{N},\ k < i-1\}$$

for $i > 2$;

(c) diam $J_i < \omega(f|P, x_i)$ and diam $I_i < \frac{\text{diam } J_i}{2^i}$;

(d) $x_i \in K(a_i, r_i) \cap Z \subset \overline{K(a_i, r_i)} \subset K(a_{i-1}, r_{i-1})$ and $r_i < \frac{1}{2^i}$;

(e) for every pair (k, l) from the set

$$A_i = \{(k,l) : k, l \in \mathbb{Z} \wedge [((-4^i \leq k < -2^i \vee 2^i \leq k < 4^i) \wedge (-4^i \leq l < 4^i)) \\ \vee((-2^i \leq k < 2^i) \wedge (-4^i \leq l < -2^i \vee 2^i \leq l < 4^i))]\},$$

we have

$$\left(\left(\frac{1}{r_i}, \frac{1}{r_i}\right) \cdot \left(\widetilde{f^{-1}(I_n)} - x_i\right)\right) \cap \left[\left(\frac{k}{4^i}, \frac{k+1}{4^i}\right) \times \left(\frac{l}{4^i}, \frac{l+1}{4^i}\right)\right] \neq \emptyset;$$

(f) for every $x \in \overline{K(a_i, r_i)}$ and every pair $(k,l) \in A_{i-1}$ we have

$$\left(\left(\frac{1}{r_{i-1}}, \frac{1}{r_{i-1}}\right) \cdot \left(\widetilde{f^{-1}(I_{n-1})} - x\right)\right) \cap \left[\left(\frac{k}{4^{i-1}}, \frac{k+1}{4^{i-1}}\right) \times \left(\frac{l}{4^{i-1}}, \frac{l+1}{4^{i-1}}\right)\right] \neq \emptyset.$$

We will show that we can choose such $x_{n+1}, I_{n+1}, J_{n+1}, a_{n+1}, r_{n+1}$ that the conditions (a)–(f) will be satisfied for $i = n+1$.

If $n + 1 \neq 1$, the set

$$\begin{aligned} \mathcal{U}_{(k,l)} \quad = \quad & \left\{x : \left(\left(\frac{1}{r_n}, \frac{1}{r_n}\right) \cdot \left(\widetilde{f^{-1}(I_n)} - x\right)\right) \cap \right. \\ \cap \quad & \left. \left[\left(\frac{k}{4^n}, \frac{k+1}{4^n}\right) \times \left(\frac{l}{4^n}, \frac{l+1}{4^n}\right)\right] \neq \emptyset \right\} \end{aligned}$$

contains x_n for every $(k, l) \in A_n$ (by (e)).

Observe that the sets $\mathcal{U}_{(k,l)}$ are open. Let $x \in \mathcal{U}_{(k,l)}$. It is easy to see that only the first factor of the intersection depends on x, and as both of the sets are open then if we transform the first of them slightly (it means choosing another point z suffciently near by x) by a displacement and a homothety we still stay at the same interval, so z will belong to $\mathcal{U}_{(k,l)}$. It means x is contained in $\mathcal{U}_{(k,l)}$ with a certain ball.

The set

$$\mathcal{U} = \bigcap_{(k,l) \in A_n} \mathcal{U}_{(k,l)}$$

is also open (the intersection is finite) and contains x_n. The condition (f) is satisfied for $x \in \mathcal{U}$.

If $n + 1 = 1$, (f) is void because A_0 is empty and we ignore it by defining $\mathcal{U} = \mathbb{R}^2$.

Observe that there exists a point $y \in \mathbb{R}^2$ such that

$$y \in P \cap f^{-1}(J_n^c) \cap (K(a_n, r_n) \cap \mathcal{U}),$$

it means $y \in P \cap K(a_n, r_n) \cap \mathcal{U}$ and $f(y) \notin J_n$.

Indeed (d) implies that

$$x_n \in K(a_n, r_n) \cap Z \subset K(a_n, r_n) \cap P,$$

because $Z \subset P$. We have already noticed that $x_n \in \mathcal{U}$. As $\mathcal{U} \cap K(a_n, r_n)$ is open, x_n belongs to this set with a certain neighbourhood. Because, by (c), diam $J_n < \omega(f|P, x_n)$ then there exists a point of the set P in this neighbourhood such that the value of the function at this point falls out of the set J_n. This is the searched point y.

If $y \in Z$, let $x_{n+1} = y$. Otherwise $f|P$ is continuous at y. In this case, as $K(a_n, r_n) \cap \mathcal{U}$ is open, there exists a neighbourhood of y contained in this set, where values of the function fall out of J_n. The fact that Z is dense in P guarantees the existence, in that neighbourhood, of points from the set

$$P \cap f^{-1}(J_n^c) \cap (K(a_n, r_n) \cap \mathcal{U}) \cap Z.$$

Let x_{n+1} be such a point.

Since $f(x_{n+1}) \notin J_n$, we can easily find a closed disk J_{n+1} as small as we wish, centered at $f(x_{n+1})$ and separated from J_n. The fact that $x_{n+1} \in Z$ guarantees that $f|P$ is not continuous at x_{n+1}, and, consequently, $\omega(f|P, x_{n+1}) > 0$.

So J_{n+1} can satisfy the conditions:

(b) $J_{n+1} \cap J_n = \emptyset$,

$$\text{diam } J_{n+1} \leq \frac{1}{3} \min\{\text{dist}(J_k, J_{k+1}) : k \in \mathbb{N}, k < n\};$$

(c) diam $J_{n+1} < \omega(f|P, x_{n+1})$.

Putting now as I_{n+1} a ball centered at $f(x_{n+1})$ satisfying the condition: diam $I_{n+1} = \frac{\text{diam } J_{n+1}}{2^{n+2}}$ guarantees

(a) $f(x_{n+1}) \in I_{n+1} \subset J_{n+1}$,

(c) $0 <$ diam $I_{n+1} < \frac{\text{diam } J_{n+1}}{2^{n+1}}$.

Define $K(a'_{n+1}, r'_{n+1})$ to have

$$\overline{K(a'_{n+1}, r'_{n+1})} \subset K(a_n, r_n) \cap \mathcal{U}$$

(it is possible as this is an open and nonempty set) and

$$x_{n+1} \in K(a'_{n+1}, r'_{n+1}), \qquad r'_{n+1} < \frac{1}{2^{n+1}}.$$

Now, the conditions (d) and (f) are satisfied for $K(a'_{n+1}, r'_{n+1})$ (because $\overline{K(a'_{n+1}, r'_n)} \subset \mathcal{U}$.) However, we still need to make certain that the condition (e) is satisfied. We will do it by diminution the $K(a'_{n+1}, r'_{n+1})$ eventually.

The point x_{n+1} is an $\mathcal{I}$-density point of $f^{-1}(I_{n+1})$, so $(0,0)$ is an $\mathcal{I}$-density point of $\widetilde{f^{-1}(I_{n+1})} - x_{n+1}$. Therefore, there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that the set

$$S = \liminf_{i\to\infty}(n_i, n_i)\left(\widetilde{f^{-1}(I_{n+1})} - x_{n+1}\right) \cap (-1,1)^2$$

is residual in $(-1,1)^2$. Define

$$W_i = (n_i, n_i)\left(\widetilde{f^{-1}(I_{n+1})} - x_{n+1}\right).$$

The set

$$\liminf_{i\to\infty} W_i = \bigcup_{r=1}^{\infty}\bigcap_{i\geq r} W_i$$

is residual in $(-1,1)^2$. In particular, for every $k,l\in\mathbb{Z}$, $(k,l)\in A_{n+1}$, we have

$$\left(\bigcup_{r=1}^{\infty}\bigcap_{i\geq r} W_i\right) \cap \left[\left(\frac{k}{4^{n+1}}, \frac{k+1}{4^{n+1}}\right) \times \left(\frac{l}{4^{n+1}}, \frac{l+1}{4^{n+1}}\right)\right] \neq \emptyset.$$

The sequence $\{\bigcap_{i\geq r} W_i\}_{r\in\mathbb{N}}$ is increasing. Thus, there is an $r_0\in\mathbb{N}$ such that

$$W_i \cap \left[\left(\frac{k}{4^{n+1}}, \frac{k+1}{4^{n+1}}\right) \times \left(\frac{l}{4^{n+1}}, \frac{l+1}{4^{n+1}}\right)\right] \neq \emptyset$$

for every $i\geq r_0$, $k,l\in\mathbb{Z}$, $(k,l)\in A_{n+1}$. But

$$W_i = (n_i,n_i)\cdot\left(\widetilde{f^{-1}(I_{n+1})} - x_{n+1}\right) = \left(\frac{1}{\frac{1}{n_i}}, \frac{1}{\frac{1}{n_i}}\right)\cdot\left(\widetilde{f^{-1}(I_{n+1})} - x_{n+1}\right).$$

Define $K(a_{n+1}, r_{n+1})$ as $K\left(x_{n+1}, \frac{1}{n_i}\right)$, where $i\geq r_0$ and such that $K(a_{n+1}, r_{n+1}) \subset K\left(a'_{n+1}, r'_{n+1}\right)$. The desired condition (e) is satisfied. This ends the inductive construction.

Let

$$\{x\} = \bigcap_{n\in\mathbb{N}} \overline{K(a_n, r_n)} = \bigcap_{n\in\mathbb{N}} (\overline{K(a_n, r_n)} \cap P) \neq \emptyset$$

(by Cantor's theorem and condition (d)).

We will show that f is not $\mathcal{I}$-density continuous at x. To be more specific, we will find a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that:

(1) $f(x)$ is an $\mathcal{I}$-dispersion point of $\bigcup_{i\in\mathbb{N}} I_{n_i}$, and

(2) x is not an $\mathcal{I}$-dispersion point of $f^{-1}(\bigcup_{i\in\mathbb{N}} I_{n_i})$.

We will first show x is not an $\mathcal{I}$-dispersion point of $f^{-1}(\bigcup_{i\in\mathbb{N}} I_{n_i})$ for every increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers.

Let $\{n_i\}_{i\in\mathbb{N}}$ be any increasing sequence of natural numbers. The condition (f) implies

$$\left(\left(\frac{1}{r_n},\frac{1}{r_n}\right)\cdot\left(\widetilde{f^{-1}(I_n)}-x\right)\right)\cap\left[\left(\frac{k}{4^n},\frac{k+1}{4^n}\right)\times\left(\frac{l}{4^n},\frac{l+1}{4^n}\right)\right]\neq\emptyset$$

for every $(k,l)\in A_n$. Thus, the open set

$$\mathcal{U}_n=\left(\frac{1}{r_n},\frac{1}{r_n}\right)\cdot\left(\widetilde{f^{-1}(I_n)}-x\right)$$

intersects every interval $\left(\frac{k}{4^n},\frac{k+1}{4^n}\right)\times\left(\frac{l}{4^n},\frac{l+1}{4^n}\right)\subset[-1,1]^2\setminus\left(-\frac{1}{2},\frac{1}{2}\right)^2$. This implies that for every increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers and for every $s\in\mathbb{N}$ the set $\bigcup_{i\geq s}\mathcal{U}_{n_i}$ is dense in $[-1,1]^2\setminus(-\frac{1}{2},\frac{1}{2})^2$. Let $\{n_{i_j}\}_{j\in\mathbb{N}}$ be an arbitrary subsequence of $\{n_i\}_{i\in\mathbb{N}}$. Hence

$$\limsup_{j\to\infty}\left(\frac{1}{r_{n_{i_j}}},\frac{1}{r_{n_{i_j}}}\right)\cdot\left(f^{-1}\widetilde{\left(\bigcup_{i\in\mathbb{N}}I_{n_i}\right)}-x\right)\supset\limsup_{j\to\infty}\mathcal{U}_{n_{i_j}}\notin\mathcal{I}$$

(because $\left(\frac{1}{n_{i_j}},\frac{1}{n_{i_j}}\right)\cdot\left(f^{-1}\widetilde{(\bigcup_{i\in\mathbb{N}}I_{n_i})}-x\right)\supset\mathcal{U}_{n_{i_j}}$) for every subsequence $\{n_{i_j}\}_{j\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ chosen at the begining. Thus x is not an $\mathcal{I}$-dispersion point of $f^{-1}(\bigcup_{i\in\mathbb{N}}I_{n_i})$.

We want to give a proof of (1). We need to find an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $f(x)$ is an $\mathcal{I}$-dispersion point of $\bigcup_{i\in\mathbb{N}}I_{n_i}$. For the sake of simplicity let us assume that $f(x)=(0,0)$.

We will consider two cases.

1° There exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that the sets J_{n_i}, $i\in\mathbb{N}$, are pairwise disjoint.

Consider the sequence $\{y_{n_i}\}_{i\in\mathbb{N}}$ of the midpoints of these disks ($f(x_n)=y_n$). If the sequence $\{d_{n_i}\}_{i\in\mathbb{N}}$ of their distances from $(0,0)$ is unbounded then we can choose a subsequence $\{d_{n_{i_m}}\}_{m\in\mathbb{N}}$ diverging to infinity. At the corresponding sequence of disks $\{I_{n_{i_m}}\}_{m\in\mathbb{N}}$ from a certain place there are disks such that their distances from $(0,0)$ are positive, so $(0,0)$ is an $\mathcal{I}$-dispersion point of their union. If the sequence $\{d_{n_i}\}_{i\in\mathbb{N}}$ is bounded then we can take a convergent subsequence. If its limit is different from 0 then the situation is similar to the above one. There is only one case more: $\lim_{m\to\infty}d_{n_{i_m}}=0$, it means: $\lim_{m\to\infty}y_{n_{i_m}}=(0,0)$. To make the notation simpler let $\{n_i\}_{i\in\mathbb{N}}$ be a subsequence with this property.

Define

$$c_{n_i}=\operatorname{dist}(J_{n_i},(0,0)),$$
$$d_{n_i}=c_{n_i}+\operatorname{diam}J_{n_i}.$$

We will take a subsequence by induction. Let $J_{n_{i_1}} = J_{n_1}$. We assume we have already chosen terms of the subsequence up to the number n_{i_k}. As $J_{n_{i_{k+1}}}$ we take any closed disk J_{n_i}, but contained in $K((0,0), \frac{1}{n_{i_k}} c_{n_{i_k}})$, it means $d_{n_{i_{k+1}}} < \frac{1}{n_{i_k}} c_{n_{i_k}}$. Clearly, this is possible because the midpoints of these disks converge to $(0,0)$, and they are pairwise disjoint.

Thus, we have the sequence $\{J_{n_{i_k}}\}_{k\in\mathbb{N}}$ of closed disks. We take the sequence $\{I_{n_{i_k}}\}_{k\in\mathbb{N}}$ of open disks corresponding to it.

Let

$$\begin{aligned} \alpha_{n_{i_k}} &= \operatorname{dist}\left(I_{n_{i_k}}, (0,0)\right), \\ \beta_{n_{i_k}} &= \alpha_{n_{i_k}} + \operatorname{diam} I_{n_{i_k}}, \\ z_{n_{i_k}} &= \frac{\alpha_{n_{i_k}} + \beta_{n_{i_k}}}{2}. \end{aligned}$$

Define

$$E = \bigcup_{k\in\mathbb{N}} P\left(\alpha_{n_{i_k}}, \beta_{n_{i_k}}\right),$$

where

$$P\left(\alpha_{n_{i_k}}, \beta_{n_{i_k}}\right) = K\left((0,0), \beta_{n_{i_k}}\right) - \overline{K((0,0), \alpha_{n_{i_k}})}.$$

We will show the assumptions of Lemma 1 hold:

(i) $\alpha_{n_{i_{k+1}}} < \beta_{n_{i_{k+1}}} < \alpha_{n_{i_k}}$

for $k \in \mathbb{N}$, by the construction of the set E.

(ii) $0 \le \frac{\beta_{n_{i_k}} - \alpha_{n_{i_k}}}{\beta_{n_{i_k}}} \le \frac{\beta_{n_{i_k}} - \alpha_{n_{i_k}}}{z_{n_{i_k}}} \le \frac{\beta_{n_{i_k}} - \alpha_{n_{i_k}}}{z_{n_{i_k}} - c_{n_{i_k}}} = 2\frac{\beta_{n_{i_k}} - \alpha_{n_{i_k}}}{d_{n_{i_k}} - c_{n_{i_k}}} =$
$= 2\frac{\operatorname{diam} I_{n_{i_k}}}{\operatorname{diam} J_{n_{i_k}}} \underset{k\to\infty}{\longrightarrow} 0$ (by (c)), so $\lim_{k\to\infty} \frac{\beta_{n_{i_k}} - \alpha_{n_{i_k}}}{\beta_{n_{i_k}}} = 0$, what is equivalent to $\lim_{k\to\infty} \frac{\beta_{n_{i_k}} - \alpha_{n_{i_k}}}{\alpha_{n_{i_k}}} = 0$,

(iii) $0 \le \frac{\beta_{n_{i_{k+1}}}}{\alpha_{n_{i_k}}} \le \frac{d_{n_{i_{k+1}}}}{\alpha_{n_{i_k}}} \le \frac{\frac{1}{n_{i_k}} \cdot c_{n_{i_k}}}{\alpha_{n_{i_k}}} \le \frac{1}{n_{i_k}} \frac{c_{n_{i_k}}}{c_{n_{i_k}}} \underset{k\to\infty}{\longrightarrow} 0$, so $\lim_{k\to\infty} \frac{\beta_{n_{i_{k+1}}}}{\alpha_{n_{i_k}}} = 0$.

Hence, $(0,0)$ is an $\mathcal{I}$-dispersion point of the set E and of $\bigcup_{k\in\mathbb{N}} I_{n_{i_k}}$, since $\bigcup_{k\in\mathbb{N}} I_{n_{i_k}} \subset E$.

2° There is no increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that the sets $\{J_{n_i}\}$, $i \in \mathbb{N}$, are pairwise disjoint.

Let us first consider the subsequence $\{J_{2n+1}\}_{n\in\mathbb{N}}$, indexed by the odd numbers, of the sequence $\{J_n\}_{n\in\mathbb{N}}$. Define a partition function $F : [\mathbb{N}]^2 \to \{0,1\}$ by

$$F(\{n,m\}) = 1 \iff J_{2n+1} \cap J_{2m+1} \neq \emptyset.$$

By Ramsey's Theorem, there exists an infinite subset $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$; i.e., the sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that for some $k \in \{0,1\}$, $F(\{n_i, n_j\}) = k$ for all positive integers $i \neq j$. But $k = 0$ would contradict the definition of the case 2°, which is currently being considered. Thus $k = 1$; i.e.,

$$J_{2n_i+1} \cap J_{2n_j+1} \neq \emptyset$$

for all nonnegative integers $i \neq j$.

The same type argument can be repeated for the subsequence $\{J_{2n_i}\}_{i\in\mathbb{N}}$. There exists a subsequence $\{n_{i_s}\}_{s\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that

$$J_{2n_{i_s}} \cap J_{2n_{j_t}} \neq \emptyset,$$

and

$$J_{2n_{i_s}+1} \cap J_{2n_{j_t}+1} \neq \emptyset$$

for $s \neq t$, $s,t \in \mathbb{N}$. Define $\epsilon = \operatorname{dist}(J_{2n_{i_0}}, J_{2n_{i_0}+1})$. By (b), $\epsilon > 0$. Moreover, if $x \in J_{2n_{i_s}}$ then

$$\operatorname{dist}(x, J_{2n_{i_0}}) < \operatorname{diam} J_{2n_{i_s}} \overset{(b)}{<} \frac{\epsilon}{3}.$$

Analogously, for $x \in J_{2n_{i_s}+1}$.

Put

$$B_0 = \bigcup_{s\in\mathbb{N}} J_{2n_{i_s}},$$

and

$$B_1 = \bigcup_{s\in\mathbb{N}} J_{2n_{i_s}+1}.$$

Then

$$B_0 \subset \left\{x :\ \operatorname{dist}(x, J_{2n_{i_0}}) < \frac{\epsilon}{3}\right\},$$

and

$$B_1 \subset \left\{x :\ \operatorname{dist}(x, J_{2n_{i_0}+1}) < \frac{\epsilon}{3}\right\}.$$

Hence

$$\operatorname{dist}(B_0, B_1) \geq \frac{\epsilon}{3} > 0.$$

Note that

$$S_0 = \bigcup_{n\in\mathbb{N}} I_{2n_{i_s}} \subset B_0,$$

as $I_{2n_{i_s}} \subset J_{2n_{i_s}}$ for all $s \in \mathbb{N}$, and

$$S_1 = \bigcup_{n\in\mathbb{N}} I_{2n_{i_s+1}} \subset B_1,$$

as $I_{2n_{i_s}+1} \subset J_{2n_{i_s}+1}$ for all $s \in \mathbb{N}$. Thus $\mathrm{dist}(S_0, S_1) > 0$, which implies that either $\mathrm{dist}(f(x), S_0) > 0$ or $\mathrm{dist}(f(x), S_1) > 0$. This clearly means that $f(x)$ is an $\mathcal{I}$-dispersion point of either S_0 or S_1. □

Corollary 1. *If $f \in C_{\mathcal{I}\mathcal{I}}$ then f is continuous on a dense open set.*

References

[1] K. Ciesielski, L. Larson, *Baire classification of $\mathcal{I}$-density and $\mathcal{I}$-approximately continuous functions*, Forum Math. **7** (1995), pp. 405–417.

[2] K. Ciesielski, L. Larson, K. Ostaszewski, *Density continuity versus continuity*, Forum Math. **2** (1990), pp. 265–275.

[3] R. Carrese, W. Wilczyński, *I-density points of plane sets*, Ricerche di Matematica **34**, **1** (1985), pp. 147–157.

[4] G. Horbaczewska, W. Wilczyński, *I-density continuous transformations on* $\mathbb{R}^2$, Atti Sem. Mat. Fis. Univ. Modena **XLII** (1994), pp. 279–287.

[5] K. Kuratowski, A. Mostowski, *Teoria mnogości*, PWN 1978.

[6] R.J. O'Malley, Baire ^{*1} *Darboux functions*, Proc. Amer. Math. Soc. **60** (1976), pp. 187–192.

[7] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *Remarks on I-density and I-approximately continuous functions*, Comm. Math. Univ. Carolinae **26**, 3 (1985), pp. 553–563.

[8] W. Wilczyński, *A category analogue of the density topology, approximate continuity and the approximate derivative*, Real Analysis Exchange **10**, 2 (1984-85), pp. 241–265.