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THE FUNCTION WITH PATHOLOGICAL MEAN VALUES FOR ITS ROTATIONS

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Abstract. Let $C = \{z; |z| = 1\}$ be a unit circle in the complex plane. There exists a function $f : C \to \{0,1\}$ such that, for any $z_1, ..., z_n \in C$, $\frac{1}{n}(f(z_1 z) + ... + f(z_n z))$ takes values zero and one on some open sets.

1. Introduction

For any function f on a unit circle $C = \{z; |z| = 1\}$ in the complex plane let $f(z_1 \cdot)$ be a rotation of the function $f(\cdot)$. In the paper we construct a discontinuous function $f : C \to \{0,1\}$ such that, for any $z_1, ..., z_n \in C$, $g(z) = \frac{1}{n}(f(z_1 z) + ... + f(z_n z))$ takes values 0 and 1 on some open sets. We also given an estimation of lengths of arcs which must be contained in $\{z \in C; f(z) = 0\}$, and in $\{z \in C; f(z) = 1\}$, (Corollary 5). The existence of such functions is related to some well known results of W. Sierpiński (see Historical Remarks). Our result can be added as an interesting supplement to the paper [1]. Namely, investigating the limit behavior of a random stain, E. Hensz-Chądzyńska, R. Jajte, and A. Paszkiewicz analyzed in particular the following limits. Let f be a continuous positive function on C. Then for numbers $z_1, ..., z_n$, randomly taken in the independent way,

$$(*) \qquad \frac{\sup_{z \in C} \sum_{t=0}^{T-1} [\Psi_1 f(z_1 z) + ... + \Psi_{N(t)} f(z_{N(t)} z)]}{\inf_{z \in C} \sum_{t=0}^{T-1} [\Psi_1 f(z_1 z) + ... + \Psi_{N(t)} f(z_{N(t)} z)]}$$

tends to one almost surely (compare Theorem 2 and formulas (1.4), (2.1) in [1]). Here $N(t)$ denotes a Poisson process (with intensity λ) independent with respect to random variables $z_1, z_2, ...$ The coefficients $\Psi_1, \Psi_2, ...$ can be treated as a fixed increasing sequence (this is a simplification of the formula (1.1) in [1].) The almost sure convergence of $(*)$ takes place only if the

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coefficients $\Psi_1, \Psi_2, ...$ do not increase too rapidly (compare Theorem 2 and Theorem 3 in [1]).

A. Paszkiewicz suggested that the situation can be drastically different when the function f is not continuous. This is an immediate consequence of the result of this paper. Namely, if $\frac{1}{n}(f(z_1 z) + ... + f(z_n z))$ takes values zero and one on some (open) sets then we always have:

$$\sup_{z\in C} \sum_{t=0}^{T-1} [\Psi_1 f(z_1 z) + ... + \Psi_{N(t)} f(z_{N(t)} z)] = \Psi_1 + ... + \Psi_{N(T)}$$

and

$$\inf_{z\in C} \sum_{t=0}^{T-1} [\Psi_1 f(z_1 z) + ... + \Psi_{N(t)} f(z_{N(t)} z)] = 0.$$

2. Historical Remarks

Let A be a union of arcs $a(w_i, w_i + \frac{\epsilon}{2^i}) = \{e^{2\pi it}; t \in (w_i, w_i + \frac{\epsilon}{2^i})\}$ with w_i taking all rational values in $[0,1)$. It is obvious that for any $z \in C$, the intersection $A \cap zA$ is a dense and open set in C. Moreover, for any sequence $z_1, z_2, ... \in C$ the intersection $z_1 A \cap z_2 A \cap ...$ is again a dense set, as an intersection of countably many open and dense sets $B_n = z_1 A \cap ... \cap z_n A$, $n = 1, 2, 3, ...$ Indeed B_n^C is a nowhere dense set, and $B_1 \cup B_2 \cup ...$ is a set of first category and cannot contain arcs. It was shown by Sierpiński that uncountable intersection $z_1 A \cap z_2 A \cap ...$ for any $z_1, z_2, ... \in C$ can also be obtained for a suitably constructed set A of Lebesque measure 0 (see [2],[3]).

In the paper we propose another example of a set A with such property. We also give an estimation of lengths of arcs which must be contained in $z_1 A \cap ... \cap z_n A$, $P(A) \leq \epsilon$. See Theorem 1 and Theorem 2 for details.

3. Notation

Let $C = \{z = e^{2\pi it}; t \in [0,1)\}$ and let B_C denote the Borel sets on C. Let $P(A) = \lambda(B)$ be the Lebesque measure of B when $A = \{e^{2\pi it}; t \in B\}$, $B \in B_{[0,1)}$. Denote by $a(\alpha, \beta)$ an arc $\{e^{2\pi it}; t \in [\alpha, \beta]\}$, $0 \leq \alpha < \beta \leq 1$, and by $K = \{a(\alpha,\beta);\ 0 \leq \alpha < \beta \leq 1\}$ the family of all such arcs. Finally let $zA = \{z\tilde{z}; \tilde{z} \in \mathrm{A}\}$, $z \in C, A \in B_C$, be a rotation of the set A.

Theorem 1. *For each $n \in N$ there exists $A \in B_C$ such that*

$$P(A) \leq \frac{1}{n},$$

$$\forall z \in C\ \exists a \in K: \quad a \subset A \cap zA, \text{ and } \quad P(a) \geq \frac{1}{4n^2}.$$

Proof. Let us describe the set A as the $A = A_1 \cup A_2$, where

$$A_1 = a\left(0, \frac{1}{2n} + \frac{1}{4n^2}\right),$$

$$A_2 = a\left(0, \frac{1}{4n^2}\right) \cup a\left(\frac{1}{2n}, \frac{1}{2n} + \frac{1}{4n^2}\right) \cup ... \cup a\left(\frac{2n-1}{2n}, \frac{2n-1}{2n} + \frac{1}{4n^2}\right).$$

Note that:

$$A = a\left(0, \frac{1}{2n} + \frac{1}{4n^2}\right) \cup a\left(\frac{2}{2n}, \frac{2}{2n} + \frac{1}{4n^2}\right) \cup ... \cup a\left(\frac{2n-1}{2n}, \frac{2n-1}{2n} + \frac{1}{4n^2}\right),$$

and $P(A) = \frac{1}{n} - \frac{1}{4n^2} < \frac{1}{n}$.

Let us fix any $z = e^{2\pi i\alpha}$, $\alpha \in [0, 1)$, and denote by $a\left(\frac{k_0}{2n}, \frac{k_0+1}{2n}\right)$ the element from $\left\{a\left(0, \frac{1}{2n}\right), a\left(\frac{1}{2n}, \frac{2}{2n}\right), ..., a\left(\frac{2n-1}{2n}, 1\right)\right\}$ which contains $e^{2\pi i\alpha}$. Then

$$zA_1 \supset a\left(\frac{k_0+1}{2n}, \frac{k_0+1}{2n} + \frac{1}{4n^2}\right), \qquad \text{if } 0 \le k_0 < 2n-1,$$

$$zA_1 \supset a\left(0, \frac{1}{4n^2}\right), \qquad \text{if } k_0 = 2n-1.$$

We showed that $zA_1 \cap A_2$ contains an arc a, and $P(a) = \frac{1}{4n^2}$. □

Theorem 2. *For any $\epsilon > 0, n \in N$ there exists $A \in B_C$ such that*

$$P(A) \le \epsilon$$

$$\forall z_1, ..., z_n \in C\ \exists a \in K: \quad a \subset z_1 A \cap ... \cap z_n A, \quad \text{and} \quad P(a) \ge \tfrac{\epsilon^n}{n^{2n}}.$$

Proof. Denote by

$$A_1 = a\left(0, \frac{1}{m} + \frac{1}{m^2} + ... + \frac{1}{m^n}\right),$$

$$A_2 = \bigcup_{k=0}^{m-1} a\left(\frac{k}{m}, \frac{k}{m} + \frac{1}{m^2} + ... + \frac{1}{m^n}\right),$$

$$A_3 = \bigcup_{k=0}^{m^2-1} a\left(\frac{k}{m^2}, \frac{k}{m^2} + \frac{1}{m^3} + ... + \frac{1}{m^n}\right),$$

$$\vdots$$

$$A_n = \bigcup_{k=0}^{m^{n-1}-1} a\left(\frac{k}{m^{n-1}}, \frac{k}{m^{n-1}} + \frac{1}{m^n}\right).$$

and let $A = A_1 \cup A_2 \cup ... \cup A_n$. Clearly

$$\begin{aligned} P(A) \quad &\le \quad \left(\frac{1}{m} + \frac{1}{m^2} + \cdots + \frac{1}{m^n}\right) + m\left(\frac{1}{m^2} + \frac{1}{m^3} + \cdots + \frac{1}{m^n}\right) + \\ &+ \quad m^2\left(\frac{1}{m^3} + \frac{1}{m^3} + \cdots + \frac{1}{m^n}\right) + \\ &+ \quad \dots m^{n-1}\frac{1}{m^n} = \frac{n}{m} + \frac{n-1}{m^2} + \cdots + \frac{1}{m^n} \le \frac{n^2}{m} \end{aligned}$$

and $P(A) < \epsilon$ for $m > \frac{n^2}{\epsilon}$.

We will show by induction that

$$(**) \forall z_1, ..., z_s, s < n \; \exists z \in C: \; za(0, \frac{1}{m^s} + ... + \frac{1}{m^n}) \subset z_1 A_1 \cap ... \cap z_s A_s.$$

For $s = 1$ $(**)$ is trivial.

Suppose that $(**)$ is true for some s. Let $z_{s+1}A(\frac{k_0}{m^s}, \frac{k_0+1}{m^s})$ be the arc from $\left\{z_{s+1}a\left(0, \frac{1}{m^s}\right), ..., z_{s+1}a\left(\frac{m^s-1}{m^s}, 1\right)\right\}$ which contains z. Then

$$za\left(0, \frac{1}{m^s} + ... + \frac{1}{m^n}\right) \supset z_{s+1}a\left(\frac{k_0+1}{m^s}, \frac{k_0+1}{m^s} + \frac{1}{m^{s+1}} + ... + \frac{1}{m^n}\right),$$

if $0 \le k_0 < m^s - 1$, and

$$za\left(0, \frac{1}{m^s} + ... + \frac{1}{m^n}\right) \supset z_{s+1}a\left(0, \frac{1}{m^{s+1}} + ... + \frac{1}{m^n}\right), \quad \text{if } k_0 = m^s - 1.$$

This implies that

$$z_1 A \cap .... \cap z_s A_s \supset z' a\left(0, \frac{1}{m^{s+1}} + ... + \frac{1}{m^n}\right)$$

for

$$z' = z_{s+1} \exp\left[2\pi i \frac{k_0+1}{m^s}\right], \qquad \text{if } 0 \le k_0 < m^s - 1,$$

or

$$z' = z_{s+1}, \qquad \text{if } k_0 = m^s - 1.$$

This completes the proof of the theorem. □

Corollary 1. *For any $n \in N$ there exists a Borel function $f : C \to \{0, 1\}$ such that for any $z_1, ..., z_n \in C$, $g(z) = \frac{1}{n}[f(z_1 z) + ... + f(z_n z)]$ takes values 0 and 1 on some arcs.*

Proof. Let $f = 1_A$, A as described in Theorem 2, and $P(A) < \frac{1}{n}$. From the statement of the theorem $\forall z_1, ..., z_n \in C$ there exists an arc a such that $a \in z_1 A \cap ... \cap z_n A_n$. Then $z_1^{-1} a \subset A, ..., z_n^{-1} a \subset A$, and the function g takes the value 1 on the entire arc a.

Note that, since $P(A) < \frac{1}{n}$, we have $P(z_1 A \cup ... \cup z_n A) < 1$, and for any $z_1, ..., z_n$ there exists an arc b such that $b \cap z_1 A = \emptyset, ..., b \cap z_n A = \emptyset$, and, as before, $z_1^{-1} b \cap A = \emptyset, ..., z_n^{-1} b \cap A = \emptyset$, which forces the function g to be 0 on b. □

Corollary 2. *For all $\epsilon > 0$ there exists $A \subset B_C$ such that:*
(i) A is the union of countably many closed intervals,
(ii) $P(A) < \epsilon$,
(iii) $\forall n \in N\ \forall z_1, ..., z_n \in C\ \exists a \in K: \quad \mathrm{a \subset z_1 A \cap ... \cap z_n A}$.

Proof. For every $n \in N$ there exists A_n such that A_n is the union of closed intervals, $P(A_n) < \frac{\epsilon}{2^n}$, and for any $z_1, ..., z_{2^n} \in C \quad z_1 A_n \cap ... \cap z_{2^n} A_n$ contains a closed interval a. It sufficients to set $A = A_1 \cup A_2 \cup ...$ □

Corollary 3. *There exists a Borel function $f : C \to \{0, 1\}$ such that for any $n \in N$ and any $z_1, ..., z_n \in C, \quad g(z) = \frac{1}{n}[f(z_1 z) + ... + f(z_n z)]$ takes values 0 and 1 on some sets.*

Proof. Let $A_1 = C$, then for any $z_1 \in C$ $a_1 = C$, we have $a_1 \subset z_1 A$, and $P(a_1) = \epsilon_1 = 1$. Let ϵ be a positive number and let A_2 be the union of finitely many intervals such that $P(A_2) < \frac{\epsilon_1}{4 \cdot 2}$, and for any $z_1, z_2 \in C$ there exists a_2, such that $a_2 \subset z_1 A_2 \cap z_2 A_2$, $P(a_2) = \epsilon_2 > 0$. Suppose that we have described the sets $A_1, ..., A_n$, and $\epsilon_1, ..., \epsilon_n > 0$, such that

$$(\alpha) \qquad P(A_i) < \frac{\epsilon_{i-1}}{4i}, \quad 1 \le i \le n,$$

and for any $z_1, ..., z_i$ there exists $a_i \in K$ such that $a_i \subset z_1 A_i \cap ... \cap z_i A_i$, and $P(a_i) = \epsilon_i$.

By Theorem 2 there exist A_{n+1}, $P(A_{n+1}) \le \frac{\epsilon_n}{4(n+1)}$, and $\epsilon_{n+1} > 0$, $\epsilon_{n+1} < \epsilon_n$, such that for any $z_1, ..., z_{n+1} \in C$ there exists a_{n+1} such that $a_{n+1} \subset z_1 A_{n+1} \cap ... \cap z_{n+1} A_{n+1}$ and $P(a_{n+1}) = \epsilon_{n+1}$. Then (α) is true for $n+1$, and by induction there exist $A_1, A_2, ..., \epsilon_1, \epsilon_2, ...$ such that (α) is true for every $n \in N$. Let us define:

$$f_1(z) = 1_{A_1},$$

$$f_2(z) = \begin{cases} f_1(z) & \text{for } z \notin A_2, \\ 0 & \text{for } z \in A_2, \end{cases}$$

$$f_3(z) = \begin{cases} f_2(z) & \text{for } z \notin A_3, \\ 1 & \text{for } z \in A_3, \end{cases}$$

Assuming that f_{2k+1} is defined, let us put

$$f_{2k+2}(z) = \begin{cases} f_{2k+1}(z) & \text{for } z \notin A_{2k+2}, \\ 0 & \text{for } z \in A_{2k+2}, \end{cases}$$

$$f_{2k+3}(z) = \begin{cases} f_{2k+2}(z) & \text{for } z \notin A_{2k+3}, \\ 1 & \text{for } z \in A_{2k+3}. \end{cases}$$

Finally let $f(z) = \lim_{n\to\infty} f_n(z)$, if the limit exists, and $f(z) = 0$, if the limit does not exist.

For any $n = 2k+1$ let us fix $z_1, z_2, ..., z_{2k+1} \in C$. By construction of A_{2k+1} there exists an interval a_{2k+1} such that $a_{2k+1} \subset z_1 A_{2k+1} \cap ... \cap z_{2k+1} A_{2k+1}$, and $P(a_{2k+1}) = \epsilon_{2k+1}$.
Let

$$g_i(z) = \frac{1}{2k+1}[f_i(z_1 z) + ... + f_i(z_{2k+1} z)] \quad \text{for } i = 2k+1, 2k+2, ...$$

and

$$g(z) = \frac{1}{2k+1}[f(z_1 z) + ... + f(z_{2k+1} z)].$$

We have $g_{2k+1} = 1$ on the entire arc a_{2k+1}. Also

$$P(g_{2k+1} \neq g) \leq P(g_{2k+1} \neq g_{2k+2}) + P(g_{2k+2} \neq g_{2k+3}) + ...$$

and

$$P(g_{j-1} \neq g_j) \leq \frac{\epsilon_{2k+1} \cdot (2k+1) \cdot (2k+1)!}{4^{j-2k-1} \cdot j!} \leq \frac{\epsilon_{2k+1}}{4^j},$$

for $j = 2k+2, 2k+3, ...$

In effect $P(g_{2k+1} \neq g) < \frac{\epsilon_{2k+1}}{3}$, and $g = 1$ on a set with measure greater or equal to $\epsilon_{2k+1} - \frac{\epsilon_{2k+1}}{3} = \frac{2\cdot\epsilon_{2k+1}}{3}$. Similarly we show that for any $n = 2k$, and for any $z_1, ..., z_n \in C$, $g(z) = \frac{1}{n}[f(z_1 z) + ... + f(z_n z)]$ takes value 0.

By the proof of Theorem 2 for $n = 2k+1$ the function $g = \frac{1}{2k+1}[f(z_1 z) + ... + f(z_{2k+1} z)]$ takes value 0, and for $n = 2k$ $g = \frac{1}{2k}[f(z_1 z) + ... + f(z_{2k} z)]$ takes value 1. This is true because there exists an arc

$$a'_{2k+2} \subset z_1 A_{2k+2} \cap ... \cap z_{2k+1} A_{2k+2},$$

on which

$$g = \frac{1}{2k+1}[f(z_1 z) + ... + f(z_{2k+1} z)],$$

takes value 0 with $P(a'_{2k+2}) \geq \frac{2\cdot\epsilon_{2k+2}}{3}$,
and there exists an arc

$$a'_{2k+1} \subset z_1 A_{2k+1} \cap ... \cap z_{2k} A_{2k+1},$$

on which
$$g = \frac{1}{2k}[f(z_1 z) + ... + f(z_{2k} z)],$$
takes value 1, $P(a'_{2k+1}) \geq \frac{2 \cdot \epsilon_{2k+1}}{3}$. This completes the proof of the corollary. □

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